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Determination of Noise Threshold from Signal Histogram in the Wavelet Domain

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Abstract

Thresholding in frequency domain is a simple and effective noise reduction technique. Determination of the threshold is critical to the image quality. The optimal threshold minimizing the Mean Square Error (MSE) is chosen adaptively in the wavelet domain; we utilize an equation of the MSE for the soft-thresholded signal and the histogram of wavelet coefficients of the original image and noisy image. The histogram of the original signal is estimated through the deconvolution assuming that the probability density functions (pdfs) of the original signal and the noise are statistically independent. The proposed method is quite general in that it does not assume any prior for the source pdf.

Keywords : Image denoising, Thresholding, Wavelet thresholding

Ⅰ. INTRODUCTION

Noise is hard to separate from the original image in the image domain. However noise power is stronger than the original signal at high frequencies for natural images. Therefore noise can be reduced in the frequency domain by reducing spectral components where noise is dominant.

Denoising methods can be divided into two categories: Bayesian methods and thresholding methods. The former efficiently reduces the noise using the Bayesian estimation or the maximum a posteriori (MAP) estimation. There are various statistical approaches for denoising such as hidden Markov Modeling (HMM) method[4] and bayesian least squares-Gaussian scale mixtures (BLS-GSM) method[5]. Soft and hard thresholding are the two main approaches for denoising[1~3, 6~7]. Block Matching 3D (BM3D)[8], which is regarded as one of state of the art denoising methods, also employs a thresholding technique.

It is critical to determine the proper threshold value to achieve high performance. A small threshold will remove signals together with noise. On the other hand, a large threshold cannot eliminate noise at high frequencies. There are several methods to determine the optimal threshold. Visu Shrink is a global threshold proposed by Donoho and Johnstone[3]. The threshold is for a data of size N with Gaussian noise. The SURE Shrink is based on the Stein's Unbiased Risk Estimator, which assumes that noise follows Gaussian probability density function (pdf)[1, 9]. The Bayes Shrink determines the soft-threshold that minimizes Bayesian risk with the Generalized

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Gaussian Distribution (GGD) as the signal prior[2]. Fodor and Kamath compared various thresholding algorithms and concluded that the SURE Shrink and the Bayes Shrink outperform other wavelet-based methods[10]. The performance of these methods are almost optimal in that the Mean Square Error (MSE) is comparable to the error calculated with the optimal threshold called as "oracle threshold", when we know the original signal[3]. Therefore, it is not feasible to reduce the noise further using thresholding. However the presented method can deal with any noise pdf provided that the noise is statistically independent of signal, showing the comparable performance to SURE Shrink or Bayes Shrink which show almost optimal MSE performance. Furthermore, our method does not have any prior for the signal, since we estimate the signal pdf from the pdf of observed noisy image.

We propose a method to determine the optimal threshold in the wavelet domain using the pdf of the wavelet coefficients of the observed signal together with the noise pdf model. Our main contribution is presentation of an explicit relationship between the threshold and the histograms of the original image and additive noise. We use a closed form equation of the MSE of the soft-thresholded signal whose pdf is a delta function[3], to obtain the MSE of a thresholded image if we know the pdf of the original signal, since the MSE is the weighted sum of the MSEs of each signal component. Then the optimal threshold which results in the minimum MSE (MMSE) can be found. A recent work[6] presents the lower and upper bounds for MSE of thresholded signal using Chi-square distribution of Gaussian noise. The MSE also depends on the number of nonzero coefficients after thresholding. However, our method calculates the MSE instead of the confidence interval; furthermore it provides intuitive understanding on the performance of the thresholding. Using the calculated histogram of the original image employing the deconvolution, we can obtain the MSE for a given threshold and then the optimal threshold can be found, which minimizes the MSE. The proposed method results in highly accurate threshold as verified in the experiment section.

Ⅱ. ANALYSIS OF THE MSE Using the Signal pdf on the Wavelet Domain

The aim of this paper is to determine the optimal threshold which minimizes the MSE of the thresholded image in the wavelet domain.

In the following subsections, we will represent the MSE of the thresholded signal when the signal pdf is a delta function, and then derive the MSE for general cases when signal is composed of many frequency components. We also provide a method to recover the signal pdf from the pdf of the observed noisy signal.

2.1. The MSE of a Thresholded Image with a Single Component

First, we analyze the MSE when the pdf of the signal has only one component, i.e. the signal pdf is a delta function $p_X(\boldsymbol{x}) = \delta(\boldsymbol{x}-c)$ where X represents the signal and c is the magnitude of the signal. The pdf of the noise N with standard deviation σ is assumed to be Gaussian, therefore the noise pdf is given as

$$p_N(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}. \tag{3}$$

Then the pdf of the observed signal Y is the convolution of the pdfs of the original signal and the noise assuming that noise is statistically independent of the signal. Therefore the pdf of the observed signal is

$$p_Y(\boldsymbol{x}) = p_X(\boldsymbol{x}) * p_N(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-c)^2}{2\sigma^2}}, \tag{4}$$

which is also a shifted form of the Gaussian pdf. Therefore the pdf of the soft-thresholded signal, $\hat{\boldsymbol{x}} = (\boldsymbol{y} - \mathrm{sgn}(\boldsymbol{y}) \cdot \lambda) \cdot \mathrm{I}(|\,\boldsymbol{y}| > \lambda)$, is as follows.

$$p_{\hat{X}}(x) = \begin{cases} p_Y(x-\lambda), & x<0 \\ \delta(x)\int_{-\lambda}^{\lambda} p_Y(x)dx, & x=0 \\ p_Y(x+\lambda), & x>0 \end{cases} \quad (5)$$

The MSE of the thresholded signal from the original signal is

$$\begin{aligned}\varepsilon &= \|(\hat{x}-x)\|_2^2 \\ &= \frac{\sigma^3}{\sqrt{2\pi}}\left[\left(\frac{\lambda+c}{\sigma^2}\right)e^{-\frac{(\lambda+c)^2}{2\sigma^2}} + \left(\frac{\lambda-c}{\sigma^2}\right)e^{-\frac{(\lambda-c)^2}{2\sigma^2}}\right] - \sqrt{\frac{2}{\pi}}\sigma\lambda\left(e^{-\frac{(\lambda+c)^2}{2\sigma^2}} + e^{-\frac{(\lambda-c)^2}{2\sigma^2}}\right) \\ &\quad + \left(\frac{c^2-\sigma^2-\lambda^2}{2}\right)\left[erf\left(\frac{\lambda+c}{\sqrt{2}\sigma}\right) + erf\left(\frac{\lambda-c}{\sqrt{2}\sigma}\right)\right] + \sigma^2 + \lambda^2,\end{aligned} \quad (6)$$

where the *erf* (error function) is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \quad (7)$$

Fig. 1 shows a plot of Equation (6). When the threshold is zero, the observed wavelet coefficients are not affected at all by the soft-thresholding. Therefore the MSE is equal to the noise variance σ^2 irrespective of the signal. On the other hand, as the threshold λ increases to infinity, all the wavelet coefficients become zero after thresholding, therefore the MSE converges to c^2. If $c=0$, the MSE decreases with the increase of the threshold as

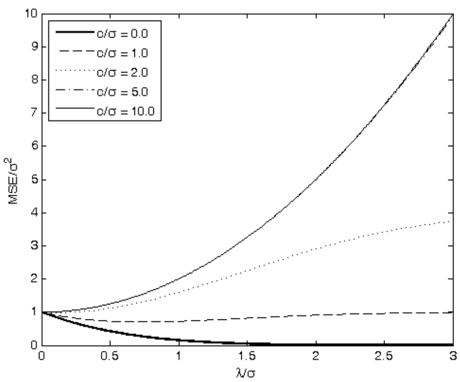

Fig. 1. MSE of a single component signal whose magnitude is c. Soft-threshold level is λ. The threshold is normalized with respect to the noise standard deviation σ.

shown in Fig. 1, since the large threshold makes most data zero which is the true value. Therefore thresholding is effective for sparse signals where most of the coefficients are zero. If the signal amplitude is less than the noise standard deviation, i.e., $c/\sigma<1$, then the noise is reduced after soft-thresholding; otherwise the MSE increases. Consequently, we can reduce the noise through soft-thresholding when most of the frequency domain coefficients of the original signal are less than the standard deviation of the noise. It is obvious that the performance of the thresholding depends on the pdf of the signal coefficients.

In Fig. 1, we note that when the signal is much larger than the noise, for example $c/\sigma \ge 5$, the MSE curves are almost the same regardless of the signal magnitude c. The value of the MSE can be approximated as $\sigma^2+\lambda^2$ in this case. Therefore, the tail distribution, where magnitude of the signal is large, does not influence the optimal threshold value either as observed in [2]. We will analyze MSE for general cases when the signal is composed of many components with different magnitudes.

2.2 The MSE of a Thresholded Image

The MSE of a thresholded image is the sum of the MSEs (ε_j) of each signal component c_j weighted by its probability $p_j = p_X(X=c_j)$. In this process, the probability distribution of the signal, p_j, is obtained through the deconvolution using the pdfs of the noisy image and the noise. In other words, the prior signal pdf is not required. The deconvolution algorithm is represented in Section 3.3. We use the discrete probability distribution of the signal instead of the continuous one. Then the MSE can be represented as

$$\varepsilon = \sum_j \varepsilon_j p_j, \quad (8)$$

where ε_j is the MSE of a signal component with unit magnitude whose amplitude is c_j, which is calculated from Equation (6). Since ε_j depends on the noise pdf, the MSE of the thresholded image depends on the source pdf as well as the noise pdf. If we have the pdfs of the source and noise, then we can find the optimal threshold minimizing the MSE using (8).

In most cases, noise can be assumed to be statistically independent of the signal, hence, the pdf of the noisy signal is the convolution of the image and noise pdfs. Therefore, we can estimate the pdf of the original signal by applying deconvolution to the noisy pdf using the noise pdf model. Since deconvolution is an ill-posed problem, we cannot recover the signal pdf exactly. However, the MSE estimation is not sensitive to minor errors in the signal pdf. Therefore the threshold value estimated using this proposed method is quite stable as verified from experiments. In this paper, the Richardson-Lucy deconvolution algorithm is adopted for its accuracy and efficiency.

Ⅲ. EXPERIMENTAL RESULTS

We used “Barbara”, “Lena”, “Pepper” and “Cameraman” gray scale images with Gaussian noise having standard deviations σ=5 and 20. ‘Daubechies 8’ wavelet transform is applied to two levels. The noise standard deviation is estimated using robust MAD (Median Absolute Deviation)[1~2]. The noise standard deviation is estimated as

$$\hat{\sigma} = \frac{1}{0.6745} median|subband \text{ HH}|, \quad (9)$$

where *subband* HH denotes the wavelet coefficients in the diagonal detail (HH) of the highest scale.

The MSE of the proposed method is compared with the Bayes Shrink and the SURE Shrink, and we observe that the three methods yield comparable MSEs approaching the MMSE denoted as a ground truth. The results are shown in Table 1. The MSE of “Barbara” image is larger than that of the others. This is because the image has high frequency components more than the other images. Therefore the noise reduction for “Barbara” image is not successful as other noisy images due to the elimination of high frequency components by soft-thresholding. Fig. 2 shows a visual comparison of soft-thresholding using various shrinkage methods applied to “Cameraman” image. Visual image quality as well as MSE is almost the same among Bayes, Sure shrink, and the proposed method.

Table 1. MSEs of soft-thresholded images comparing ground truth, proposed Method, Bayes Shrink, and SURE Shrink.

Images	Standard deviation	Noisy image	Ground truth	Proposed method	Bayes Shrink	SURE Shrink
Pepper	σ = 5	24.97	14.74	20.21	17.80	20.28
	σ = 20	399.59	73.69	74.21	79.21	74.36
Lena	σ = 5	24.97	12.80	13.90	13.78	14.03
	σ = 20	399.59	68.33	68.74	69.81	68.51
Barbara	σ = 5	24.97	16.01	18.70	17.02	18.72
	σ = 20	399.59	123.28	125.80	126.44	125.40
Cameraman	σ = 5	24.97	9.02	9.04	9.40	9.05
	σ = 20	399.59	59.88	60.18	61.46	60.14

Ⅳ. CONCLUSTION

We apply the soft-thresholding method for denoising. We can calculate the MSE of a thresholded noisy image given the histogram of the original image in the wavelet domain; therefore we can find the optimal threshold that minimizes the MSE. While Bayes Shrink adopts GGD model for signal pdf prior, we estimate the pdf of the original signal from that of the observed signal and the noise pdf, since the pdf of the observed noisy signal is the convolution of the signal and noise pdfs assuming statistical independence between them. The proposed method can be applied to any noise pdf, while SURE Shrink assumes Gaussian noise. The performance of the proposed method is comparable to other methods

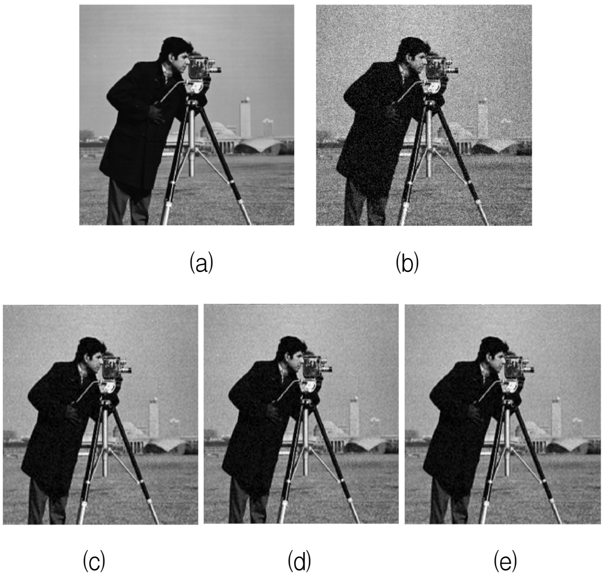

Fig. 2. Images after thresholding. (a) Original image, (b) Noisy image, and soft-thresholded images using thresholds calculated by (c) SURE Shrink, (d) Bayes Shrink and (e) Proposed method. Noise sigma is σ =20.

such as SURE Shrink or Bayes Shrink. Our method also shows the limit on the performance of a thresholding method: it depends on the portion of the original signal components whose absolute magnitude is less than the noise standard deviation.

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