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Some properties of reliability, ratio, maximum and minimum in a bivariate exponential distribution with a dependence parameter†

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Abstract

In this paper, we derived estimators of reliability $P(Y < X)$ and the distribution of ratio in the bivariate exponential density. We also considered the means and variances of $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$. We finally presented how $E(M)$, $E(m)$, $Var(M)$ and $Var(m)$ are varied with respect to the ones in the bivariate exponential density.

Keywords: Bivariate exponential density, dependence parameter, maximum, minimum, reliability.

1. Introduction

In reliability studies of mechanical components, dependence between two components occurs quite often as in Saunders (2007). Among two identical component functions, if some system at least has one of those, then that system has a functional correlation between those system components. Initially, we assume the marginal life distribution of the two components has the independent exponential function, $\text{Exp}(1/\lambda)$ with mean parameter $1/\lambda$. Failure of one changes the life distribution of the other, $\text{Exp}(1/\lambda\theta)$.

Especially if $\theta = 1$, the two components in function are independent. For $\theta > 1$, if the workload of one component is increased, then the mean life of the other will be decreased.

Let (X, Y) denote the life times of the two components having a bivariate exponential model. Then, the joint probability function of (X, Y) can be expressed as,

$$f_{X,Y}(x, y) = 2\theta\lambda^2 \exp(-2\lambda x - \lambda\theta y) \text{ for } x, y > 0, \tag{1.1}$$

where $\theta > 0$ and $\lambda > 0$. Here θ is called the dependence parameter in Cho *et al.* (2006).

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The most well known bivariate exponential distribution was derived by Marshall *et al.* (1967) by considering shock models. Iyer *et al.* (2002) presented a bivariate exponential distribution derived from packet communication networks.

In the case X and Y belong to the different distribution families, then Ali *et al.* (2010a, 2010b) studied the ratio of two independent exponential Pareto variables, and considered the estimation of $P(Y < X)$. Hakamipour *et al.* (2011) and Kim (2012) considered extremes, moments of ratio, and the approximate the maximum likelihood estimator (MLE) of parameters having bivariate Pareto distribution. Ker (2005) studied the maximum of bivariate normal random variables.

Therefore, we have some motivations of considering the reliability estimation of $P(Y < X)$, the distribution of the ratio $Y/(X+Y)$, and maximum and minimum properties having the bivariate exponential density (1.1).

In this paper, we shall derive estimators of the reliability $P(Y < X)$, and ratio distribution in the bivariate exponential density (1.1). By providing the numerical values of mean, variance, and coefficients of skewness, we observe the several trends for the density of the ratio R. We also consider the means and variances of $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$. We finally present how $E(M)$, $E(m)$, $Var(M)$ and $Var(m)$ are varied with respect to the ones in the bivariate exponential density (1.1).

2. Estimation of reliability $P(Y < X)$

For life times of the two components (X, Y) having the bivariate exponential density (1.1), reliability $P(Y < X)$ is considered as follows.

Proposition 2.1 Let (X, Y) be two components of life times having bivariate exponential density (1.1). Then the reliability $R(\rho) \equiv P(Y < X) = 1/(1+\rho)$ is a monotone decreasing function of $\rho = 2/\theta$.

From Proposition 2.1, inference on $R(\rho)$ is equivalent to one on ρ, since $R(\rho)$ is a monotone function of ρ (McCool, 1991). Hence it is sufficient to consider an estimation of ρ instead of estimating $R(\rho)$.

Let $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ be random samples in the bivariate exponential density (1.1). Then MLE $\hat{\rho}$ of ρ is

$$\widehat{\rho} = \sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} X_i. \tag{2.1}$$

In order to find mean and variance of $\hat{\rho}$, we can show the as follows.

Lemma 2.1 Let X be a gamma random variable having mean $\alpha\beta$ and variance $\alpha\beta^2$. If $\alpha > 2$, then $E(1/X) = 1/\{(\alpha-1)\beta\}$ and $E(1/X^2) = 1/\left\{(\alpha-1)(\alpha-2)\beta^2\right\}$.

From Lemma 2.1 and (2.1), it is possible to derivate as follows.

$$E(\widehat{\rho}) = \frac{n}{n-1}\rho. \tag{2.2}$$

From (2.2), an unbiased estimator $\tilde{\rho}$ of ρ is given by as follows.

$$\widetilde{\rho} = \frac{n-1}{n}\left(\sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} X_i\right). \tag{2.3}$$

If we choose a constant "$c = c_0$" satisfying the following,

$$\min_c E\left(\left[c\sum_{i=1}^{n} Y_i/\sum_{i=1}^{n} X_i - \rho\right]^2\right) = E\left(\left[c_0\sum_{i=1}^{n} Y_i/\sum_{i=1}^{n} X_i - \rho\right]^2\right),$$

then from Lemma 2.1, constant c_0 is derivated as follows.

$$c_0 = (n-2)/(n+1).$$

If we define $\bar{\rho}$ as follows,

$$\bar{\rho} = \frac{n-2}{n+1}\left(\sum_{i=1}^{n} Y_i/\sum_{i=1}^{n} X_i\right)$$

then $\bar{\rho}$ has minimum mean squared error among estimators in $\{c\sum_{i=1}^{n} Y_i/\sum_{i=1}^{n} X_i|c>0\}$. Since inference on $R(\rho)$ is equivalent to inference on ρ (McCool, 1991), we obtain the proposition as follows.

Proposition 2.2 Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ be random samples in the bivariate exponential density (1.1). Then the reliability estimator $R(\bar{\rho})$ performs better than the other two reliability estimators $R(\hat{\rho})$ and $R(\tilde{\rho})$ in the sense of MSE.

Since $2(2\lambda\sum_{i=1}^{n} X_i)$ and $2(\lambda\theta\sum_{i=1}^{n} Y_i)$ having the independent χ^2 -distribution with degree of freedom (d.f.) $2n$ respectively, then Q= $\rho(\sum_{i=1}^{n} Y_i/\sum_{i=1}^{n} X_i)$ is a pivot quantity having a F-distribution with d.f. $(2n, 2n)$.

From the monotone decreasing property of $R(\rho)$ in Proposition 2.1, a $(1-\alpha)100\%$ confidence interval of $R(\rho)$ is given by as follows.

$$\left(R(F_{\alpha/2}(2n,2n)\left(\sum_{i=1}^{n} Y_i/\sum_{i=1}^{n} X_i\right)\right),\ R\left(\frac{1}{F_{\alpha/2}(2n,2n)}\left(\sum_{i=1}^{n} Y_i/\sum_{i=1}^{n} X_i\right)\right)\right)$$

where $F_{\alpha/2}(2n, 2n)$ is the upper $\alpha/2$ percent of F-distribution with d.f. $(2n, 2n)$.

3. Distribution of ratio $Y/(X+Y)$

In this section, we consider the distribution of ratio $R = Y/(X+Y)$ for (X, Y), which it's the two components of life times having the bivariate exponential density (1.1).

First, from the quotient density in Rohatgi (1976) the density of $W = X/Y$ is derived as follows.

$$f_W(w) = \frac{1}{\rho}\left(w + \frac{1}{\rho}\right)^{-2},\ w > 0. \tag{3.1}$$

Since the ratio is defined as follows

$$R = Y/(X+Y) = 1/(1+W),$$

then the density of ratio R is given by as follows.

$$f_R(r) = \frac{1}{\rho}\left[\left(\frac{1}{\rho} - 1\right) r + 1\right]^{-2}, \ 0 < r < 1, \tag{3.2}$$

where $w + \frac{1}{\rho} > 0$ iff $(\frac{1}{\rho} - 1)r + 1 > 0$.

From the density (3.1) and formula 2.29 in Oberhettinger (1974), *k-th* moment of the ratio R is obtained as follows.

For $k = 1, 2, 3. \cdots$,

$$E(R^k) = \begin{cases} \dfrac{\rho}{k+1} {}_2F_1(2, 1; k+2; 1-\rho), \ \text{if } 0 < \rho < 1 \\ \dfrac{1}{k+1} {}_2F_1(k, 1; k+2; 1-1/\rho), \ \text{if } \rho \geq 1. \end{cases} \tag{3.3}$$

where, ${}_2F_1(a, b; c; x)$ is the hypergeometric function.

From the recursion formulas 15.2.13 and 15.2.25 of hypergeometric function, the formula 15.1.8 in Abramowitz and Stegun (1970), and *k-th* moment of the ratio R in (3.3), Table 3.1 provides numerical values of mean, variance, and coefficients of skewness for the density (3.2) of the ratio R when ρ =1/8(2)8.

Table 3.1 Mean, variance, and coefficients of skewness for the density (3.2) of the ratio R when ρ=1/8(2)8

ρ	mean	variance	skewness
1/8	.19664	.04801	1.60988
1/4	.28280	.06483	1.00876
1/2	.38629	.07819	0.48613
1	.5	.08333	0
2	.61371	.07819	-0.48613
4	.71720	.06483	-1.00876
8	.80336	.04801	-1.60988

From Table 3.1 we observe the trends of the density (3.2) of the ratio R as follows.

Fact 3.1 Let (X, Y) be two components of life times having the bivariate exponential density (1.1) and $\rho = 2/\theta$. Then

(a) the density (3.2) of the ratio R is symmetric at $\rho = 1$.

(b) it's right-skewed when $\rho < 1$, but left-skewed when $\rho > 1$.

4. Maximum and minimum of (X, Y)

In this section, we consider $k-th$ moments of $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$ when (X, Y) is a pair of two components of life times having the bivariate exponential density (1.1).

Proposition 4.1 Let (X, Y) be two components of life times having the bivariate exponential density (1.1), and $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$. Then for $k = 1, 2, 3. \cdots$,

(a) $E(m^k) = \Gamma(k+1)\left[\left(\frac{1}{2\lambda}\right)^k\left(1+\frac{\theta}{2}\right)^{-k-1} + \left(\frac{1}{\lambda\theta}\right)^k\left(1+\frac{2}{\theta}\right)^{-k-1}\right],$

where $\Gamma(a)$ is the gamma function.

(b) $E(M^k) = \Gamma(k+1)\left[\left(\frac{1}{2\lambda}\right)^k + \left(\frac{1}{\lambda\theta}\right)^k - \left(\left(\frac{1}{2\lambda}\right)^k\left(1+\frac{\theta}{2}\right)^{-k-1} + \left(\frac{1}{\lambda\theta}\right)^k\left(1+\frac{2}{\theta}\right)^{-k-1}\right)\right]$

$= \Gamma(k+1)\left[\left(\frac{1}{2\lambda}\right)^k + \left(\frac{1}{\lambda\theta}\right)^k\right] - E(m^k).$

Proof

(a) It follows from the formula 3.381(1) in Gradshteyn and Ryzhik (1965), and the formula 17.89 in Oberhettinger and Badii (1973) by integral calculations.

(b) It follows from the formula 3.381(3) in Gradshteyn and Ryzhik (1965), and the formula 17.90 in Oberhettinger and Badii (1973) by integral calculations. □

From the density (1.1), we obtain as follows.

$$\mu_1 \equiv E(X) = \frac{1}{2\lambda},\ \sigma_1^2 \equiv Var(X) = \frac{1}{4\lambda^2} = \mu_1^2$$

and

$$\mu_2 \equiv E(Y) = \frac{1}{\lambda\theta},\ \sigma_2^2 \equiv Var(Y) = \frac{1}{\lambda^2\theta^2} = \mu_2^2. \tag{4.1}$$

As putting k =1 and 2 in Proposition 4.1 and applying the relations in (4.1), we obtain the proposition as follows.

Proposition 4.2 Let (X, Y) be two components of life times having bivariate exponential density (1.1), and $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$. Then

(a) $E(m) = \mu_1(1+\mu_1/\mu_2)^{-1}$,

$E(m^2) = 2[\mu_1^2(1+\mu_1/\mu_2)^{-3} + \mu_2^2(1+\mu_2/\mu_1)^{-3}]$,

and

$Var(m) = E(m^2) - E^2(m)$.

(b) $E(M) = \mu_1 + \mu_2 - E(m)$, $E(M^2) = 2(\mu_1^2 + \mu_2^2) - E(m^2)$,

and

$Var(M) = 2(\mu_1^2 + \mu_2^2) - E(m^2)$ - $[\mu_1 + \mu_2 - E(m)]^2$.

Now we consider how $E(M)$, $E(m)$, $Var(M)$ and $Var(m)$ vary with respect to means μ_1 and μ_2 in the bivariate exponential density (1.1).

Proposition 4.3 Let (X, Y) be two components of life times having bivariate exponential density (1.1), and $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$. Then for $\mu_1 = E(X)$ and $\mu_2 = E(Y)$,

(a) $E(M)$ and $E(m)$ are monotone increasing functions of μ_i for i =1 and 2.

(b) $Var(m)$ is a monotone increasing function of μ_i for i=1 and 2.

(c) $Var(M)$ is a monotone increasing function of μ_1 $(> \mu_2)$.

(d) $Var(M)$ is a monotone increasing function of μ_2 $(> \mu_1)$.

Proof (a) It is possible to derivate as follows.

$$\frac{\partial E(m)}{\partial \mu_1} = (1 + \mu_1/\mu_2)^{-2} > 0,$$
$$\frac{\partial E(M)}{\partial \mu_1} = 1 - \frac{\partial E(m)}{\partial \mu_1} = 1 - (1 + \mu_1/\mu_2)^{-2} > 0,$$
$$\frac{\partial E(m)}{\partial \mu_2} = (1 + \mu_2/\mu_1)^{-2} > 0,$$

and

$$\frac{\partial E(M)}{\partial \mu_2} = 1 - \frac{\partial E(m)}{\partial \mu_2} = 1 - (1 + \mu_2/\mu_1)^{-2} > 0.$$

(b) Since

$$\frac{\partial E(m^2)}{\partial \mu_1} = 4\mu_1(1 + \mu_1/\mu_2)^{-3} \quad \text{and} \quad \frac{\partial Var(m)}{\partial \mu_1} = \frac{\partial E(m^2)}{\partial \mu_1} - 2E(m)\frac{\partial E(m)}{\partial \mu_1},$$

from

$$\frac{\partial E(m)}{\partial \mu_1} = (1 + \mu_1/\mu_2)^{-2} \quad \text{and} \quad E(m) = \mu_1(1 + \mu_1/\mu_2)^{-2} + \mu_2(1 + \mu_2/\mu_1)^{-2},$$

we obtain as follows.

$$\frac{\partial Var(m)}{\partial \mu_1} = 2\mu_1(1 + \mu_1/\mu_2)^{-3} > 0.$$

And, since

$$\frac{\partial E(m^2)}{\partial \mu_2} = 4\frac{\mu_1^3}{\mu_2^2}(1 + \mu_1/\mu_2)^{-3} \text{ and } \frac{\partial Var(m)}{\partial \mu_2} = \frac{\partial E(m^2)}{\partial \mu_2} - 2E(m)\frac{\partial E(m)}{\partial \mu_2},$$

from

$$\frac{\partial E(m)}{\partial \mu_2} = \frac{\mu_1^2}{\mu_2^2}(1+\mu_1/\mu_2)^{-2} \text{ and } E(m) = \mu_1(1+\mu_1/\mu_2)^{-2} + \frac{\mu_1^2}{\mu_2}(1+\mu_1/\mu_2)^{-2},$$

we obtain as follows.

$$\frac{\partial Var(m)}{\partial \mu_1} = 4\frac{\mu_1^3}{\mu_2^2}(1+\mu_1/\mu_2)^{-4} > 0.$$

(c) From Proposition 4.3(b) and $\dfrac{\partial Var(M)}{\partial \mu_1} = \dfrac{\partial E(M^2)}{\partial \mu_1} - 2E(M)\dfrac{\partial E(M)}{\partial \mu_1}$,

$$\frac{\partial Var(M)}{\partial \mu_1} = 4\mu_1 - \frac{\partial E(m^2)}{\partial \mu_1} - 2\left(1 - \frac{\partial E(m)}{\partial \mu_1}\right)(\mu_1 + \mu_2 - E(m)).$$

Since

$$\frac{\partial E(m^2)}{\partial \mu_1} = 4\mu_1(1+\mu_1/\mu_2)^{-3}, \ \frac{\partial E(m)}{\partial \mu_1} = (1+\mu_1/\mu_2)^{-2},$$

and

$$E(m) = \mu_1(1+\mu_1/\mu_2)^{-2} + \mu_2(1+\mu_2/\mu_1)^{-2},$$

$$\begin{aligned}\frac{\partial Var(M)}{\partial \mu_1} &= 2(\mu_1-\mu_2) + 4\frac{\mu_1^2}{\mu_2}(1+\mu_1/\mu_2)^{-3} + 2\mu_1(1+\mu_1/\mu_2)^{-4}(1+\mu_1\mu_2+\mu_2/\mu_1)\\ &\quad +2\frac{\mu_1^2}{\mu_2}(1+\mu_1/\mu_2)^{-4}[(1+\mu_1/\mu_2)^2-1] > 0 \ \text{ if } \ \mu_1 > \mu_2.\end{aligned}$$

(d) By the similar method in (c),

$$\begin{aligned}\frac{\partial Var(M)}{\partial \mu_2} =& 2(\mu_2-\mu_1) + 4\frac{\mu_1^2}{\mu_2}(1+\mu_1/\mu_2)^{-3} + 2\mu_1(1+\mu_1/\mu_2)^{-4}[(1+\mu_1/\mu_2)^2 - \mu_1^3/\mu_2^3]\\ &+ 2\frac{\mu_1^2}{\mu_2}(1+\mu_1/\mu_2)^{-4}[(1+\mu_1/\mu_2)^2-1] > 0 \ \text{ if } \ \mu_2 > \mu_1. \qquad \square\end{aligned}$$

Since $\sigma_i^2 = \mu_i^2$ for i=1 and 2, and $\frac{\partial E(m)}{\partial \sigma_i^2} = \frac{\partial E(m)}{\partial \mu_i}\frac{\partial \mu_i}{\partial \sigma_i^2} = \frac{1}{2}\sigma_i^{-1}\frac{\partial E(m)}{\partial \mu_i}$, signs of $\frac{\partial E(m)}{\partial \sigma_i^2}$ and $\frac{\partial E(m)}{\partial \mu_i}$ are same. By applying it by $E(M)$, $Var(m)$, and $Var(M)$, we considered how $E(M)$, $E(m)$, $Var(M)$, and $Var(m)$ are varied with respect to the variances σ_1^2 and σ_2^2 in the bivariate exponential density (1.1). Then we obtain as follows.

Corollary 4.1 Let (X, Y) be two components of life times having the bivariate exponential density (1.1) and $M = \max\{X, Y\}$ and $m = \min\{X, Y\}$. Then for $\sigma_1^2 = Var(X)$ and $\sigma_2^2 = Var(Y)$,

(a) $E(M)$ and $E(m)$ are monotone increasing functions of σ_i^2 for i=1 and 2.

(b) $Var(m)$ is a monotone increasing function of σ_i^2 for i=1 and 2.

(c) $Var(M)$ is a monotone increasing function of σ_1^2 ($> \sigma_2^2$).

(d) $Var(M)$ is a monotone increasing function of σ_2^2 ($> \sigma_1^2$).

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