KYUNGPOOK Math. J. 54(2014), 685-697
http://dx.doi.org/10.5666/KMJ.2014.54.4.685

On Pseudo Null Bertrand Curves in Minkowski Space-time

İsmail Gök*
Department of Mathematics, Faculty of Science, Ankara University, 06100, Ankara, Turkey
e-mail : igok@science.ankara.edu.tr

Semra Kaya Nurkan
Department of Mathematics, Faculty of Science and Arts, Usak University, 64200, Usak, Turkey
e-mail : semrakaya_gs@yahoo.com

Kazim İlarslan
Department of Mathematics, Faculty of Sciences and Arts, Kırıkkale University, 71450, Kırıkkale, Turkey
e-mail : kilarslan@yahoo.com

Abstract. In this paper, we prove that there are no pseudo null Bertrand curve with curvature functions $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s)$ other than itself in Minkowski space-time $\mathbb{E}_1^4$ and by using the similar idea of *Matsuda and Yorozu* [13], we define a new kind of Bertrand curve and called it pseudo null $(1, 3)$-Bertrand curve. Also we give some characterizations and an example of pseudo null $(1, 3)$-Bertrand curves in Minkowski space-time.

1. Introduction

Many work has been studied about the general theory of curves in an Euclidean space (or more generally in a Riemannian manifold). So now, we have extensive knowledge on its local geometry as well as its global geometry. Characterization of a regular curve is one of the important and interesting problems in the theory of curves in Euclidean space. There are two ways widely used to solve these problems: to figure out the relationship between the Frenet vectors of the curves (see [11]), and to determine the shape and size of a regular curve by using its curvatures. k_1 (or $\varkappa$) and k_2 (or τ), the curvature functions of a regular curve, have an effective

* Corresponding Author.
Received September 18, 2012; revised January 10, 2013; accepted June 28, 2013.
2010 Mathematics Subject Classification: 53C40, 53C50.
Key words and phrases: Minkowski space-time, pseudo null curve, Frenet vectors, $(1, 3)$-Bertrand curve.

role. For example: if $k_1 = 0 = k_2$, the curve is a geodesic or if k_1 =constant$\neq 0$ and $k_2 = 0$, the curve is a circle with radius $(1/k_1)$, etc..This paper deals with the characterization of Bertrand curve which is one of the samples of regular curves.

In 1845, *Saint Venant* (see [17]) proposed the question whether the principal normal of a curve is the principal normal of another's on the surface generated by the principal normal of the given one. Bertrand answered this question in [3] published in 1850. He proved that a necessary and sufficient condition for the existence of such a second curve is required in fact a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by $k_1$and k_2 respectively, we have $\lambda k_1 + \mu k_2 = 1$, $\lambda, \mu \in \mathbb{R}$. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called *Conjugate Bertrand Curves*, or more commonly *Bertrand Curves* (see [11]).

There are many important papers on Bertrand curves in Euclidean space (see: [4],[6],[15]).

When we investigate the properties of Bertrand curves in Euclidean n-space, it is easy to see that either k_2 or k_3 is zero which means that Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ are degenerate curves (see [15]). This result is restated by *Matsuda and Yorozu* [13]. They proved that *there was not any special Bertrand curves in* $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1],[2],[7],[9],[10],[18]) as well as in Euclidean space.

In this paper, we prove that there is no any pseudo null Bertrand curve with nonzero curvature function $(k_2(s))$ other than itself in Minkowski space-time $\mathbb{E}_1^4$ and define a new kind of Bertrand curves in $\mathbb{E}_1^4$ calling it as pseudo null $(1,3)$-Bertrand curve. It also gives some characterizations and an example of pseudo null $(1,3)$-Bertrand curves in Minkowski space-time $\mathbb{E}_1^4$.

2. Preliminaries

The Minkowski space-time $\mathbb{E}_1^4$ is the real vector space $\mathbb{R}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + \sum_{i=2}^{4} dx_i^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{R}^4$. Recall that a vector $v \in \mathbb{E}_1^4 \backslash \{0\}$ can be *spacelike* if $g(v,v) > 0$, *timelike* if $g(v,v) < 0$ and *null (lightlike)* if $g(v,v) = 0$. In particular, the vector $v = 0$ is a spacelike . The norm of a vector v is given by $||v||_L = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal, if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null. A spacelike or a timelike curve $\alpha(s)$ has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([14]). We assume that each null curve α in this paper is parametrized by a special parameter p such that $g(\alpha''(p), \alpha''(p)) = 1$, which called the *distinguished parameter* of α.

In a semi-Euclidean space, some normal vectors of a regular curve may be null vectors. In Minkowski space-time, such curves are necessarily spacelike. If its principal normal is null , such curves are called *pseudo null curves,* if its second normal is null, the curve is called a *partially null* curve. Curves with null normals have at most two curvatures. These curves were defined and studied by W. B. Bonnor in [5] (see also [16]).

Let $\{T, N_1, N_2, N_3\}$ be the moving Frenet frame along a pseudo null curve α in $\mathbb{E}_1^4$. If α is a pseudo null curve , the Frenet equations are given by ([5]):

$$\begin{aligned} T' &= k_1 N_1, \\ N_1' &= k_2 N_2 \\ N_2' &= k_3 N_1 - k_2 N_3, \\ N_3' &= -k_1 T - k_3 N_2 \end{aligned} \tag{2.1}$$

where the first curvature $k_1(s) = 0$, if α is a straight line, or $k_1(s) = 1$ in all other cases. Such curve has two curvatures $k_2(s)$ and $k_3(s)$. Moreover, the Frenet vectors of a pseudo null curve α satisfy the following conditions:

$$\begin{gathered} g(T,T) = g(N_2,N_2) = g(N_1,N_3) = 1, \\ g(N_1,N_1) = g(N_3,N_3) = 0 \\ g(T,N_1) = g(T,N_2) = g(T,N_3) = g(N_1,N_2) = g(N_2,N_3) = 0. \end{gathered} \tag{2.2}$$

In this study we consider that the curve α is not a straight line, that is, the first curvature of α, $k_1(s) = 1$.

3. On Pseudo Null Bertrand Curves in Minkowski Space-time

In this section, by the following theorem, we prove that there is no any pseudo null Bertrand curves with curvature functions $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s)$ in Minkowski space-time $\mathbb{E}_1^4$ other than itself.

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a pseudo null curve with curvature functions $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s)$. Then, there is no any Bertrand mate of α in Minkowski space-time $\mathbb{E}_1^4$ other than itself.*

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a pseudo null Bertrand curve in $\mathbb{E}_1^4$ and $\beta : \overline{I} \subset \mathbb{R} \to \mathbb{E}_1^4$ be a pseudo null Bertrand mate of α. We assume that β is different from α. Let the pairs of $\alpha(s)$ and $\beta(\overline{s}) = \beta(\varphi(s))$ (where $\varphi : I \to \overline{I}, \overline{s} = \varphi(s)$ is a regular $C^\infty-$function) be corresponding points of α and β. Then we can write,

$$\beta(\overline{s}) = \beta(\varphi(s)) = \alpha(s) + \lambda(s) N_1(s) \tag{3.1}$$

where λ is a C^∞-function on I. Differentiating Eq. (3.1) with respect to s and by using Frenet formulas given in Eq. (2.1), we get

$$\varphi'(s) \overline{T}(\varphi(s)) = T(s) + \lambda'(s) N_1(s) + \lambda(s) k_2(s) N_2(s). \tag{3.2}$$

and differentiating Eq. (3.2) with respect to s, we have

$$\begin{aligned}&\varphi''(s)\,\overline{T}(\varphi(s)) + \left(\varphi'(s)\right)^2 \overline{k}_1(\varphi(s))\,\overline{N}_1(\varphi(s))\\ &= \left\{k_1(s) + \lambda''(s) + \lambda(s)\,k_2(s)\,k_3(s)\right\} N_1(s) + \left\{2\lambda'(s)\,k_2(s) + \lambda(s)\,k_2'(s)\right\} N_2(s)\\ &+ \left\{-\lambda(s)\,k_2^2(s)\right\} N_3(s)\end{aligned}$$

If we take the inner product with $\overline{N}_1(\varphi(s))$ on both sides of the last equation, we have

$$\begin{aligned}&g(\varphi''(s)\,\overline{T}(\varphi(s)), \overline{N}_1(\varphi(s))) + g\left(\varphi'(s)\right)^2 \overline{k}_1(\varphi(s))\,\overline{N}_1(\varphi(s)), \overline{N}_1(\varphi(s))\\ &= g\left(\left\{k_1(s) + \lambda''(s) + \lambda(s)\,k_2(s)\,k_3(s)\right\} N_1(s), \overline{N}_1(\varphi(s))\right)\\ &+ g\left(\left\{2\lambda'(s)\,k_2(s) + \lambda(s)\,k_2'(s)\right\} N_2(s),\ \overline{N}_1(\varphi(s))\right)\\ &- g\left(\lambda(s)\,k_2^2(s)\,N_3(s), \overline{N}_1(\varphi(s))\ \right)\end{aligned}$$

or if we consider that $\overline{N}_1(\varphi(s))$ is parallel to $N_1(s)$, that is, $\overline{N}_1(\varphi(s)) = cN_1(s)$ where $c \in \mathbb{R} - \{0\}$ then the above equality

$$\begin{aligned}&g(\varphi''(s)\,\overline{T}(\varphi(s)), \overline{N}_1(\varphi(s))) + g\left(\varphi'(s)\right)^2 \overline{k}_1(\varphi(s))\,\overline{N}_1(\varphi(s)), \overline{N}_1(\varphi(s))\\ &= g\left(\left\{k_1(s) + \lambda''(s) + \lambda(s)\,k_2(s)\,k_3(s)\right\} N_1(s), cN_1(s)\right)\\ &+ g\left(\left\{2\lambda'(s)\,k_2(s) + \lambda(s)\,k_2'(s)\right\} N_2(s),\ cN_1(s)\right)\\ &- g\left(\lambda(s)\,k_2^2(s)\,N_3(s), cN_1(s)\ \right)\end{aligned}$$

holds. By using Frenet formulas for α and β given in Eq. (2.1), we get

$$c\lambda(s)\,k_2^2(s) = 0$$

for all $s \in I$. Thus, since $c \in \mathbb{R} - \{0\}$ and $k_2(s) \neq 0$, we get $\lambda(s) = 0$. In this case, we can rewrite Eq.(3.1) as follows

$$\beta(\overline{s}) = \beta(\varphi(s)) = \alpha(s), \tag{3.3}$$

Thus, there is no any Bertrand mate of α in Minkowski space-time $\mathbb{E}_1^4$ other than itself.

□

As a result of Theorem 3.1, we give the following corollary without proof.

Corollary 3.2. A pseudo null curve α with the curvature function $k_1(s)$ is a pseudo-null Bertrand curve if and only if α is a degenerate plane curve.

4. On Pseudo Null $(1,3)$-Bertrand Curves in Minkowski Space-time

In this section, firstly we will define pseudo null $(1,3)$- Bertrand curves with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s)$ in Minkowski space-time $\mathbb{E}_1^4$ and some characterizations of the curves will given.

Definition 4.1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^4$ and $\beta : \overline{I} \subset \mathbb{R} \to \mathbb{E}_1^4$ be pseudo null curves with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, $k_3(s)$ and $\overline{k_1}(\varphi(s))$, $\overline{k_2}(\varphi(s))$, $\overline{k_3}(\varphi(s))$, respectively, where $\varphi : I \to \overline{I}, \overline{s} = \varphi(s)$ is a regular $C^\infty-$function such that each point $\alpha(s)$ of α corresponds to the point $\beta(\overline{s}) = \beta(\varphi(s))$ of β for all $s \in I$. If the Frenet $(1,3)$-normal plane at each point $\alpha(s)$ of α coincides with the Frenet $(1,3)$-normal plane at corresponding point $\beta(\overline{s}) = \beta(\varphi(s))$ of β for all $s \in I$, α is called a pseudo null $(1,3)$-Bertrand curve in $\mathbb{E}_1^4$ and β is called a pseudo null $(1,3)$-Bertrand mate of α.

Theorem 4.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a pseudo null curve with curvature functions $k_1(s) = 1$, $k_2(s) \neq 0$ and $k_3(s)$. Then, α is a pseudo null $(1,3)$-Bertrand curve if and only if there exist constant real numbers λ, μ, γ, satisfying the followings:*

$$\lambda k_2(s) - \mu k_3(s) \neq 0, \ \lambda \neq 0, \ \mu \neq 0 \tag{4.1-a}$$

$$\gamma \left[\lambda k_2(s) - \mu k_3(s)\right] + \mu = 1, \tag{4.1-b}$$

$$\gamma + k_3(s) = 0 \tag{4.1-c}$$

$$\mu \neq 1, \tag{4.1-d}$$

for all $s \in I$.

Proof. We assume that α is a pseudo null $(1,3)$-Bertrand curve parametrized by arclenght s. The pseudo null $(1,3)$-Bertrand mate β is given by arc-lenght $\overline{s}$. Then, we can write

$$\beta\left(\overline{s}\right) = \beta\left(\varphi\left(s\right)\right) = \alpha(s) + \lambda(s)N_1(s) + \mu(s)N_3(s) \tag{4.2}$$

for all $s \in I$, where $\lambda(s)$ and $\mu(s)$ are $C^\infty-$functions on I. Differentiating Eq. (4.2) with respect to s, and by using the Frenet equations given in Eq. (2.1), we have

$$\begin{aligned} \overline{T}\left(\varphi\left(s\right)\right)\varphi'\left(s\right) = \ & \left[1-\mu(s)\right]T\left(s\right) + \lambda'(s)N_1(s) \\ & + \left[\lambda(s)k_2(s) - \mu(s)k_3(s)\right]N_2(s) + \mu'(s)N_3(s) \end{aligned} \tag{4.3}$$

for all $s \in I$.

Since the plane spanned by $N_1(s)$ and $N_3(s)$ coincides with the plane spanned by $\overline{N}_1(\varphi\left(s\right))$ and $\overline{N}_3(\varphi\left(s\right))$, we can write

$$\overline{N}_1(\varphi\left(s\right)) = a\left(s\right)N_1(s) + b\left(s\right)N_3(s), \tag{4.4}$$

$$\overline{N}_3(\varphi(s)) = c(s)\, N_1(s) + d(s)N_3(s) \tag{4.5}$$

and by using Eq. (4.4) and Eq. (4.5) we can easily see that

$$\lambda'(s) = 0,\ \mu'(s) = 0,$$

that is, λ and μ are constant functions on I.

So, we can rewrite Eq. (4.2) and Eq. (4.3) for all $s \in I$, respectively as follows

$$\beta(\overline{s}) = \beta(\varphi(s)) = \alpha(s) + \lambda N_1(s) + \mu N_3(s) \tag{4.6}$$

and

$$\overline{T}(\varphi(s))\, \varphi'(s) = [1-\mu]\, T(s) + [\lambda k_2(s) - \mu k_3(s)]\, N_2(s). \tag{4.7}$$

Here notice that

$$\left(\varphi'(s)\right)^2 = [1-\mu]^2 + [\lambda k_2(s) - \mu k_3(s)]^2 \neq 0 \tag{4.8}$$

for all $s \in I$. If we consider

$$u(s) = \left[\frac{1-\mu}{\varphi'(s)}\right],\ v(s) = \left[\frac{\lambda k_2(s) - \mu k_3(s)}{\varphi'(s)}\right], \tag{4.9}$$

it is easy to obtain

$$\overline{T}(\varphi(s)) = u(s)T(s) + v(s)N_2(s) \tag{4.10}$$

where $u(s)$ and $v(s)$ are C^∞-functions on I. Differentiating Eq. (4.10) with respect to s and using the Frenet equations, we obtain

$$\begin{aligned} \overline{N}_1(\varphi(s))\, \varphi'(s) &= u'(s)T(s) + [u(s) + v(s)k_3(s)]\, N_1(s) \\ &\quad + v'(s)N_2(s) - v(s)k_2(s)N_3(s). \end{aligned} \tag{4.11}$$

Since $\overline{N}_1(\varphi(s))$ is expressed by linear combination of $N_1(s)$ and $N_3(s)$,

$$u'(s) = 0,\ v'(s) = 0,$$

that is, u and v are constant functions on I. So, we can rewrite Eq. (4.11) as follows

$$\overline{N}_1(\varphi(s))\, \varphi'(s) = [u + v k_3(s)]\, N_1(s) - v k_2(s) N_3(s). \tag{4.12}$$

By using Eq. (4.9), we can show that

$$v(1-\mu) = u(\lambda k_2(s) - \mu k_3(s)), \tag{4.13}$$

where v must be non-zero. If we take $v = 0$ in the Eq. (4.12), we get

$$\overline{N}_1(\varphi(s))\, \varphi'(s) = u N_1(s)$$

thus we obtain $\overline{N}_1(\varphi(s)) = \pm N_1(s)$ for all $s \in I$. This is a contradiction according to the Theorem (3.1). Thus we must consider only the case of $v \neq 0$, and then it is easy to see that

$$\lambda k_2(s) - \mu k_3(s) \neq 0 \tag{4.14}$$

for all $s \in I$. Morever, from the Theorem (3.1) we can easily see that $\lambda \neq 0$ and $\mu \neq 0$. Thus, we obtain relation (4.1-a).

If the constant γ is taken as $\gamma = \dfrac{u}{v}$ and by using Eq. (4.13) we have

$$\gamma(\lambda k_2(s) - \mu k_3(s)) + \mu = 1$$

for all $s \in I$. Thus we obtained relation (4.1-b).

From Eq. (4.12) we have

$$g(\overline{N}_1(\varphi(s))\varphi'(s), \overline{N}_1(\varphi(s))\varphi'(s)) = -2[u + vk_3(s)]\,vk_2(s)$$

and then

$$0 = -2[u + vk_3(s)]\,vk_2(s) \tag{4.15}$$

for all $s \in I$. Since $k_2(s) \neq 0$ and $v \neq 0$, we get

$$u + vk_3(s) = 0.$$

By using Eq. (4.9), we have

$$\begin{aligned} \left(\frac{1-\mu}{\varphi'(s)}\right) + \left(\frac{\lambda k_2(s) - \mu k_3(s)}{\varphi'(s)}\right) k_3(s) &= 0 \\ \gamma(\lambda k_2(s) - \mu k_3(s)) + (\lambda k_2(s) - \mu k_3(s))\,k_3(s) &= 0 \\ (\lambda k_2(s) - \mu k_3(s))(\gamma + k_3(s)) &= 0. \end{aligned}$$

From Eq. (4.14), it is easy to see that

$$(\gamma + k_3(s)) = 0,$$

for all $s \in I$. Thus, we obtain relation (4.1-c).

From Eq. (4.1-b) and Eq. (4.8), we get

$$(\varphi'(s))^2 = (\lambda k_2(s) - \mu k_3(s))^2 \left[\gamma^2 + 1\right]. \tag{4.16}$$

From Eq. (4.9) and Eq. (4.12), we have

$$\overline{N}_1(\varphi(s)) = \frac{(\lambda k_2(s) - \mu k_3(s))}{(\varphi'(s))^2}\left[(\gamma + k_3(s))\,N_1(s) - k_2(s)N_3(s)\right], \tag{4.17}$$

$$\overline{N}_1\left(\varphi\left(s\right)\right)=-\frac{\left(\lambda k_2(s)-\mu k_3(s)\right)}{\left(\varphi^{\prime}\left(s\right)\right)^2}k_2(s)N_3(s),$$
$$\overline{N}_1\left(\varphi\left(s\right)\right)=-\frac{k_2(s)}{\varphi^{\prime}\left(s\right)}vN_3(s).$$

Differentiating Eq. (4.17) with respect to s and by using the Frenet equations, we obtain

(4.18)
$$\overline{k}_2(\varphi(s))\overline{N}_2(\varphi(s))\varphi^{\prime}\left(s\right)=\left(-\frac{k_2(s)}{\varphi^{\prime}\left(s\right)}\right)^{\prime}vN_3(s)+\frac{k_2(s)}{\varphi^{\prime}\left(s\right)}v\left(T(s)+k_3(s)N_2\left(s\right)\right).$$

Since $\overline{N}_2(\varphi(s))\in Sp\left\{T(s),N_2\left(s\right)\right\}$, we obtain

$$\left(-\frac{k_2(s)}{\varphi^{\prime}\left(s\right)}\right)^{\prime}=0$$

that is, $\frac{k_2(s)}{\varphi^{\prime}(s)}$ is a non-zero constant.So, we can rewrite Eq. (4.18) as follows

$$\overline{k}_2(\varphi(s))\overline{N}_2(\varphi(s))\varphi^{\prime}\left(s\right)=\frac{k_2(s)}{\varphi^{\prime}\left(s\right)}v\left(T(s)+k_3(s)N_2\left(s\right)\right).$$

If we denote

$$A(s)=\gamma\left(\gamma^2+1\right)^{-1}\left(1-\mu\right)^{-1}k_2(s) \tag{4.19}$$

and

$$B(s)=\gamma\left(\gamma^2+1\right)^{-1}\left(1-\mu\right)^{-1}k_2(s)k_3(s). \tag{4.20}$$

We obtain
$$\overline{k}_2(\varphi(s))\overline{N}_2(\varphi(s))\varphi^{\prime}\left(s\right)=A(s)T(s)+B(s)N_2\left(s\right).$$

Since $\varphi^{\prime}\left(s\right)\overline{k}_2\left(\varphi\left(s\right)\right)\overline{N}_2\left(\varphi\left(s\right)\right)\neq 0$ for $\forall s\in I$, we have

$$\mu\neq 1$$

for all $s\in I$. Thus, we obtain relation (4.1-d).

Conversely, we assume that $\alpha: I\subset\mathbb{R}\to\mathbb{E}_1^4$ be a pseudo null curve with curvature functions $k_1(s)=1$, $k_2(s)\neq 0$, $k_3(s)$ satisfying the relation $(4.1-a)$, $(4.1-b)$, $(4.1-c)$ and $(4.1-d)$ for constant numbers λ, δ, γ and we define a pseudo null curve $\beta: I\subset\mathbb{R}\to\mathbb{E}_1^4$ such as

$$\beta(s)=\alpha\left(s\right)+\lambda N_1(s)+\mu N_3(s) \tag{4.21}$$

for all $s\in I$. Differentiating Eq. (4.21) with respect to s and by using the Frenet equations, we have

$$\frac{d\beta\left(s\right)}{ds}=\left(1-\mu\right)T\left(s\right)+\left(\lambda k_2(s)-\mu k_3(s)\right)N_2(s),$$

thus, by using the Eq. (4.1-b), we obtain

$$\frac{d\beta(s)}{ds} = (\lambda k_2(s) - \mu k_3(s))(\gamma T(s) + N_2(s))$$

for all $s \in I$. Also, we get

$$\left\| \frac{d\beta(s)}{ds} \right\|_L = \xi(\lambda k_2(s) - \mu k_3(s))\sqrt{(\gamma^2+1)} \tag{4.22}$$

Then we can write

$$\overline{s} = \varphi(s) = \int_0^s \left\| \frac{d\beta(t)}{dt} \right\|_L dt \qquad (\forall s \in I)$$

where $\varphi : I \to \overline{I}$ is a regular $C^\infty-$function, and we obtain

$$\varphi'(s) = \xi(\lambda k_2(s) - \mu k_3(s))\sqrt{(\gamma^2+1)},$$

for all $s \in I$. Differentiating Eq. (4.21) with respect to s, we get

$$\varphi'(s) \left. \frac{d\beta(\overline{s})}{d\overline{s}} \right|_{\overline{s}=\varphi(s)} = (\lambda k_2(s) - \mu k_3(s))\{\gamma T(s) + N_2(s)\}$$

or

$$\overline{T}(\varphi(s)) = \xi\left(\gamma^2+1\right)^{-\frac{1}{2}}(\gamma T(s) + N_2(s)) \tag{4.23}$$

for all $s \in I$. Differentiating Eq. (4.23) with respect to s, we have

$$\varphi'(s)\overline{T}'(\varphi(s)) = -\xi\left(\gamma^2+1\right)^{-\frac{1}{2}} k_2(s) N_3(s).$$

and by using the Frenet equations,

$$\overline{N}_1(\varphi(s)) = \frac{\gamma k_2(s)}{(\gamma^2+1)(1-\mu)} N_3(s). \tag{4.24}$$

Differentiating Eq. (4.23) with respect to s,

$$\overline{k}_2(\varphi(s))\overline{N}_2(\varphi(s))\varphi'(s) = \frac{\gamma}{\gamma^2+1}(1-\mu)^{-1} k_2(s)(T(s) + k_3(s) N_2(s))$$

and

$$g\left(\overline{k}_2(\varphi(s))\overline{N}_2(\varphi(s))\varphi'(s), \overline{k}_2(\varphi(s))\overline{N}_2(\varphi(s))\varphi'(s)\right) = \overline{k}_2^2(\varphi(s))\varphi'(s)^2$$

$$\overline{k}_2^2(\varphi(s))\varphi'(s)^2 = \frac{\gamma}{\gamma^2+1}(1-\mu)^{-1} k_2(s)\left(1 + k_3^2(s)\right)$$

$$\overline{k}_2(\varphi(s)) = \left(\frac{\gamma}{1-\mu}\right)^{\frac{3}{2}} \frac{\sqrt{k_2(s)\left(1+k_3^2(s)\right)}}{\gamma^2+1}.$$

Since $\overline{N}_3(\varphi(s))$ is expressed by linear combination of $N_1(s)$ and $N_3(s)$, we get

$$\overline{N}_3(\varphi(s)) = m(s)N_1(s) + n(s)N_3(s) \tag{4.25}$$

and

$$g\left(\overline{N}_3(\varphi(s)), \overline{N}_1(\varphi(s))\right) = 1.$$

Besides we can show that

$$m(s)\frac{\gamma k_2(s)}{\left(\gamma^2+1\right)(1-\mu)} = 1,$$

$$m(s) = \frac{\left(\gamma^2+1\right)(1-\mu)}{\gamma k_2(s)}.$$

Since

$$g\left(\overline{N}_3(\varphi(s)), \overline{N}_3(\varphi(s))\right) = 0,$$

we can show that

$$2m(s)n(s) = 0,$$

$$n(s) \neq 0.$$

So, we can rewrite Eq. (4.25) as follows

$$\overline{N}_3(\varphi(s)) = \frac{\left(\gamma^2+1\right)(1-\mu)}{\gamma k_2(s)} N_1(s). \tag{4.26}$$

Then from the Frenet equations for the curve β and the above equalities, we have

$$\overline{N}_3'(\varphi(s)) = -\overline{T}(\varphi(s)) - \overline{k}_3(\varphi(s))\overline{N}_2(\varphi(s)). \tag{4.27}$$

Differentiating Eq. (4.26) with respect to s, and by using the Frenet equations,

$$\overline{N}_3'(\varphi(s)) = \xi\sqrt{\left(\gamma^2+1\right)}\left(\left(\frac{1}{k_2(s)}\right)' N_1(s) + N_2(s)\right). \tag{4.28}$$

So, by using Eq. (4.27) and Eq. (4.28)

$$\overline{k}_3(\varphi(s)) = \xi k_3(s)$$

is obtained. Notice that

$$g\left(\overline{T},\overline{T}\right) = g\left(\overline{N}_2,\overline{N}_2\right) = g\left(\overline{N}_1,\overline{N}_3\right) = 1, \quad g\left(\overline{N}_1,\overline{N}_1\right) = g\left(\overline{N}_3,\overline{N}_3\right) = 0$$

and

$$g\left(\overline{T},\overline{N}_1\right)=g\left(\overline{T},\overline{N}_2\right)=g\left(\overline{T},\overline{N}_3\right)=g\left(\overline{N}_1,\overline{N}_2\right)=g\left(\overline{N}_2,\overline{N}_3\right)=0,$$

for all $s\in I$ where $\left\{\overline{T},\overline{N}_1,\overline{N}_2,\overline{N}_3\right\}$ is moving Frenet frame along pseudo null curve β in E_1^4. And it is trivial that the Frenet $(1,3)$-normal plane at each point $\alpha(s)$ of α coincides with the Frenet $(1,3)$-normal plane at corresponding point $\beta(\overline{s})$ of β. Hence α is a pseudo null $(1,3)$- Bertrand curve in $\mathbb{E}_1^4$. This completes the proof.

Corollary 4.3. Let $\alpha : I\subset\mathbb{R}\to\mathbb{E}_1^4$ be a pseudo null $(1,3)$-Bertrand curve with curvatures functions $k_1(s)=1$, $k_2(s)\neq 0$, $k_3(s)$ and β be a pseudo null $(1,3)$-Bertrand mate of α with curvatures functions $\overline{k_1}(\varphi(s))$, $\overline{k_2}(\varphi(s))$, $\overline{k_3}(\varphi(s))$. Then the relations between these curvatures functions are

$$\overline{k}_1(\varphi(s))=1,$$

$$\overline{k}_2(\varphi(s))=\left(\frac{\gamma}{1-\mu}\right)^{\frac{3}{2}}\frac{\sqrt{k_2(s)\left(1+k_3^2(s)\right)}}{\gamma^2+1},$$

$$\overline{k}_3(\varphi(s))=\xi k_3(s),$$

where

$$\xi=\begin{cases}1 & , \quad \lambda k_2(s)-\mu k_3(s)>0\\ -1 & , \quad \lambda k_2(s)-\mu k_3(s)<0\end{cases}.$$

Proof. It is obvious using the similar method in the proof of above theorem.

Corollary 4.4. Let $\alpha : I\subset\mathbb{R}\to\mathbb{E}_1^4$ be a pseudo null $(1,3)$-Bertrand curve with curvatures functions $k_1(s)=1$, $k_2(s)\neq 0$, $k_3(s)$ and β be a pseudo null $(1,3)$-Bertrand mate of the curve α and $\varphi : I\to\overline{I}, \overline{s}=\varphi(s)$ is a regular $C^\infty-$function such that each point $\alpha(s)$ of the curve α corresponds to the point $\beta(\overline{s})=\beta(\varphi(s))$ of the curve β for all $s\in I$. Then the distance between the points $\alpha(s)$ and $\beta(\overline{s})$ is constant for all $s\in I$.

Proof. Let $\alpha : I\subset\mathbb{R}\to\mathbb{E}_1^4$ be a pseudo null $(1,3)$-Bertrand curve with curvatures functions $k_1(s)=1$, $k_2(s)\neq 0$ and $k_3(s)$ and β be a pseudo null $(1,3)$-Bertrand mate of the curve α. We assume that β is different from α. Let the pairs of $\alpha(s)$ and $\beta(\overline{s})=\beta(\varphi(s))$ (where $\varphi : I\to\overline{I}, \overline{s}=\varphi(s)$ is a regular $C^\infty-$function) be of corresponding points of α and β. Then we can write,

$$\beta(\overline{s})=\beta(\varphi(s))=\alpha(s)+\lambda N_1(s)+\mu N_3(s)$$

where λ and μ are non-zero constants. Thus,

$$\beta(\overline{s})-\alpha(s)=\lambda N_1(s)+\mu N_3(s)$$

and

$$\|\beta(\overline{s})-\alpha(s)\|=\sqrt{2\lambda\mu}.$$

So, $d(\alpha(s), \beta(\bar{s}))$ =constant, which completes the proof.

Example: (The pseudo null curve equation given in [8]) Let us consider a pseudo null curve with the equation

$$\alpha(s) = \frac{3}{\sqrt{10}} \left(\frac{1}{9} \cosh(3s), \frac{1}{9} \sinh(3s), \sin(s), -\cos(s) \right).$$

The Frenet Frame of α is given by

$$\begin{aligned}
T(s) &= \frac{3}{\sqrt{10}} (\frac{1}{3} \sinh(3s), \frac{1}{3} \cosh(3s), \cos(s), \sin(s)), \\
N_1(s) &= \frac{3}{\sqrt{10}} (\cosh(3s), \sinh(3s), -\sin(s), \cos(s)), \\
N_2(s) &= \frac{1}{\sqrt{10}} (3\sinh(3s), 3\cosh(3s), -\cos(s), -\sin(s)), \\
N_3(s) &= \frac{5}{3\sqrt{10}} (-\cosh(3s), -\sinh(3s), -\sin(s), \cos(s)).
\end{aligned}$$

The curvatures of α are

$$k_1(s) = 1, \quad k_2(s) = 3, \quad k_3(s) = \frac{4}{3}.$$

We take constant λ, μ, and γ defined by

$$\lambda = -\frac{17}{18}, \quad \mu = -1, \quad \gamma = -\frac{4}{3}$$

Then, it is obvious that Eq. (4.1-a), Eq. (4.1-b), Eq. (4.1-c) and Eq. (4.1-d) are hold. Therefore, the curve α is a pseudo-null $(1,3)$-Bertrand curve in $\mathbb{E}_1^4$. In this case, by using Eq. (4.2) the pseudo-null $(1,3)$-Bertrand mate of the curve α is given as follows:

$$\beta(s) = \frac{-5}{6\sqrt{10}} (\cosh(3s), \sinh(3s), -9\sin(s), 9\cos(s)).$$

Acknowledgements. The authors would like to express their sincere gratitude to the referee for valuable suggestions to improve the paper.

References

[1] H. Balgetir, M. Bektaş and M. Ergüt, *Bertrand curves for nonnull curves in 3-dimensional Lorentzian space*, Hadronic J., **27(2)**(2004), 229-236.

[2] H. Balgetir, M. Bektaş and J. Inoguchi, *Null Bertrand curves in Minkowski 3-space and their characterizations*, Note Mat., **23(1)**(2004/05), 7-13.

[3] J. M. Bertrand, *Mémoire sur la théorie des courbes á double courbure*, Comptes Rendus, **36**(1850).

[4] Ch. Bioche, *Sur les courbes de M. Bertrand*, Bull. Soc. Math. France, **17**(1889), 109-112.

[5] W. B. Bonnor, *Curves with null normals in Minkowski space-time, A random walk in relativity and cosmology,* Wiley Easten Limitid, (1985), 33-47.

[6] J. F. Burke, *Bertrand Curves Associated with a Pair of Curves*, Mathematics Magazine, **34(1)**(1960), 60-62.

[7] N. Ekmekci and K. İlarslan, *On Bertrand curves and their characterization*, Differ. Geom. Dyn. Syst., **3(2)**(2001), 17-24.

[8] K. İlarslan and E. Nesovic, *Some Characterizations Of Pseudo And Partially Null Osculating Curves In Minkowski Space,* Int. Electron. J. Geom., **4(2)**(2011), 1-12.

[9] D. H. Jin, *Null Bertrand curves in a Lorentz manifold*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., **15(3)**(2008), 209-215.

[10] F. Kahraman, İ. Gök and K. İlarslan, *Generalized Null Bertrand Curves in Minkowski Space-Time, to appear in* Scientic Annals of "Al.I. Cuza" University of Iasi (2012).

[11] W. Kuhnel, *Differential geometry: curves-surfaces-manifolds*, Braunschweig, Wiesbaden, (1999).

[12] M. Külahcı and M. Ergüt, *Bertrand curves of AW(k)-type in Lorentzian space*, Nonlinear Analysis: Theory, Methods & Applications, **70**(2009), 1725-1731.

[13] H. Matsuda and S. Yorozu, *Notes on Bertrand curves*, Yokohama Math. J., **50(1-2)**(2003), 41-58.

[14] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

[15] L. R. Pears, *Bertrand curves in Riemannian space*, J. London Math. Soc., **1-10(2)**(1935), 180-183.

[16] B. Rouxel, *Sur certaines varietes* V^2 2 dimensionelles d'un espace-temps de Minkowski M^4, Comptes. Rendus. Acad. Sc. Paris, **274**(1972), 1750-1752.

[17] B. Saint Venant, *Mémoire sur les lignes courbes non planes*, Journal de l'Ecole Polytechnique, **18**(1845), 1-76.

[18] J. K. Whittemore, *Bertrand curves and helices*, Duke Math. J., **6**(1940), 235-245.