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Weakly Prime Ideals in Involution po-Γ-Semigroups

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Abstract. The concept of prime and weakly prime ideal in semigroups has been introduced by G. Szasz [4]. In this paper, we define the involution in po-Γ-semigroups, then we extend some results on prime, semiprime and weakly prime ideals to the involution po-Γ-semigroup S. Also, we characterize intra-regular involution po-Γ-semigroups. We establish that in the involution po-Γ-semigroup S such that the involution preserves the order, an ideal of S is prime if and only if it is both weakly prime and semiprime and if S is commutative, then the prime and weakly prime ideals of S coincide. Finally, we prove that if S is a po-Γ-semigroup with order preserving involution, then the ideals of S are prime if and only if S is intra-regular.

1. Introduction and Preliminaries

The notion of the Γ-semigroup was introduced by M. K. Sen [6] in 1981 as a generalization of semigroups and ternary semigroups. This paper builds upon the previous publications of the authors, where the classical approaches to the prime ideals and weakly prime ideals for semigroups are developed based on the unary operation involution. Several authors extended the results of semigroups to Γ-semigroups. The concept of prime and weakly prime ideal in semigroups has been introduced by G. Szasz [4] and thereafter, M. Petrich [8] described these concepts for semigroups. Furthermore, N. Kehayopulu [12-14] introduced prime, weakly prime ideals in ordered semigroups (partially ordered semigroups) by extending the corresponding notions of ring theory that was first studied by N. H. McCoy [10] and O. Steinfeld [17].

It is widely considered that T. E. Nordahl and H. E. Scheiblich [19] introduced the involution in semigroups, but there are some other earlier authors who based their work on involution [3,5,18]. Involutions have been used in many fields of mathematics, e.g. projective, euclidian, differential, etc. They appear most generally in functional analysis and topology, and specifically in algebra. Involution, which is

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a particular type of symmetry on algebraic structures, is an antiautomorphism of the algebraic system. The main motivation for investigation of involution po-Γ-semigroups is the class of involution rings, involution algebras, po-semigroups, by considering only the multiplicative structure and leaving out the additive structure. Recently, C. Y. Wu [2] extended the involution to ordered semigroups under the natural hypothesis that the involution preserves the ordering so that ideals remain ideals after being operated by the involution. Involution po-Γ-semigroups generalizes groups, semigroups and more generally, inverse semigroups. As a matter of fact, po-Γ-semigroups are a generalization of semigroups and ordered semigroups. Consequently, the analogous results of po-semigroups (semigroups) can be extracted from po-Γ-semigroups (po-semigroups). In other words, the results of semigroups can be extracted from po-semigroups, Γ-semigroups and po-Γ-semigroups. Due to these facts, we are tempted to impose the involution that preserves the order on po-Γ-semigroups. In this paper we basically extend and generalize to involution po-Γ-semigroups these results for ordered semigroups described in [4,12-14]. We obtain some important classical properties of involution po-Γ-semigroups and characterize intra-regular involution po-Γ-semigroups. In fact the class of weakly prime ideals are a generalization of the class of prime ideals as every weakly prime ideal is a prime ideal.

We now recall below some relevant concepts that will be needed throughout the paper.

We follow the definition of the Γ-semigroup by M. K. Sen and N. K. Saha [7] given in 1986 as follows:

Definition 1.1. Let S and Γ be two nonempty sets. Then a triple of the form $(S, \Gamma, \cdot)$ is called a Γ-semigroup, where $\cdot$ is a ternary operation $S \times \Gamma \times S \to S$ such that $(x \cdot \alpha \cdot y) \cdot \beta \cdot z = x \cdot \alpha \cdot (y \cdot \beta \cdot z)$ for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$. Let T be a nonempty subset of $(S, \Gamma, \cdot)$. Then T is called a sub-Γ-semigroup of $(S, \Gamma, \cdot)$ if $a \cdot \gamma \cdot b \in T$ for all $a, b \in T$ and $\gamma \in \Gamma$. Furthermore, a Γ-semigroup S is said to be commutative if $a \cdot \gamma \cdot b = b \cdot \gamma \cdot a$ for all $a, b \in S$ and $\gamma \in \Gamma$.

Example 1.2. Suppose S is a semigroup and Γ is any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a \cdot \gamma \cdot b = a \cdot b$ for all $a, b \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semigroup.

Example 1.3. Suppose S is the set of all negative rational numbers and $\Gamma = \{-\frac{1}{p}$: p is prime $\}$. Define $a \cdot \alpha \cdot b =$ the usual product of rational numbers a, α, b for all $a, b \in S$ and $\alpha \in \Gamma$. Then it is easy to verify that S is a Γ-semigroup.

The above examples show that every semigroup is a Γ-semigroup and thus Γ-semigroups generalize semigroups. For other examples of Γ-semigroups, one can refer [6-7,11].

The concept of the po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee [20] in 1996. A po-Γ-semigroup is an ordered set $(S, \leq)$ at the same time a

Γ-semigroup $(S, \Gamma, \cdot)$ such that $a \leq b \Rightarrow a \cdot \alpha \cdot x \leq b \cdot \alpha \cdot x$ and $x \cdot \beta \cdot a \leq x \cdot \beta \cdot b$ for all $a, b, x \in S$ and $\alpha, \beta \in \Gamma$.

Notation 1: For subsets A, B of a po-Γ-semigroup S, the product set $A \cdot B$ of the pair (A, B) relative to S is defined as $A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as $A^2 = A \cdot A = A \cdot \Gamma \cdot A$.

Notation 2: For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m \text{ for some } m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Definition 1.4. A po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ with a unary operation $\star : S \longrightarrow S$ is called an *involution po-Γ-semigroup* if (i) $(x^\star)^\star = x$ and $(x \cdot \alpha \cdot y)^\star = y^\star \cdot \alpha \cdot x^\star$ for all $x, y \in S$ and $\alpha \in \Gamma$. The unary operation $\star$ is called an involution. Furthermore, if for all $a, b \in S$ with $a \geq b \Rightarrow a^\star \geq b^\star$, then we call $\star$ an order preserving involution.

If there is no likelihood of confusion, we identify the involution po-Γ-semigroup $(S, \Gamma, \cdot, \leq, \star)$ by S. Throughout this paper, for the sake of smoothness, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$.

Example 1.5. Let S be the set of all $m \times n$ matrices and Γ be the set of $n \times m$ matrices, where m, n are positive integers. Furthermore, define $P \leq Q \Leftrightarrow P \subseteq Q$ for all $P, Q \subseteq S$, then S is a po-Γ-semigroup under the usual matrix multiplication.

Example 1.6.([14]) Let $S = \{a, b, c, d, e\}$. Define a mapping $S \times \Gamma \times S \to S$ by $a \cdot \gamma \cdot b = a \cdot b$ for all $a, b \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semigroup. Define the involution $\star$ by $a^\star = e$ (and so $e^\star = a$), $b^\star = c$ and $d^\star = d$. It is easy to verify that S is an involution po-Γ-semigroup such that the involution $\star$ preserves the order where the order and multiplication on S is respectively given by

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, b), (d, c), (d, d), (e, c), (e, e)\} \text{ and}$$

.	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Example 1.7. Let $P(S)$ be the power set of any nonempty set S and Γ a topology on S. If we define $LMN = L \cap M \cap N$ and $L \leq N \Leftrightarrow L \subseteq N$ for all $L, N \in P(S)$ and $M \in \Gamma$, then $P(S)$ is a po-Γ-semigroup.

For some properties of po-Γ-semigroups, readers can see [1 9,16,21-23].

Suppose S is a po-Γ-semigroup and I is a nonempty subset of S. Then I is called a right (resp. left) ideal of S if

(i) $I\Gamma S \subseteq I$(rep. SΓI $\subseteq$ I),

(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$.

Equivalent definition:

(i) $I\Gamma S \subseteq I$ (resp. $S\Gamma I \subseteq I$).

(ii) $(I] = I$.

An ideal I of S is both a right and left ideal of a po-Γ-semigroup S. A right, left or ideal I of S is called proper if $I \neq S$. We denote by $L(s), R(s)$ and $I(s)$ the left ideal, right ideal and the ideal generated by s. Obviously, $L(s) = (s \cup S\Gamma s]$, $R(s) = (s \cup s\Gamma S]$, $I(s) = (s \cup S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S]$.

Definition 1.8. Suppose S is a po-Γ-semigroup with involution and $P \subseteq S$. Then P is called *prime* if $A, B \subseteq S$, $A\Gamma B \subseteq P$ implies $A^\star \subseteq P$ or $B^\star \subseteq P$.

Equivalent definition: $x, y \in S$, $x\alpha y \in P$, then $x^\star \in P$ or $y^\star \in P$, where $\alpha \in \Gamma$.

Definition 1.9. Suppose S is a po-Γ-semigroup with involution and $P \subseteq S$. Then P is called *weakly prime* if for ideals A, B of S such that $A\Gamma B \subseteq P$ implies $A^\star \subseteq P$ or $B^\star \subseteq P$.

Definition 1.10. Suppose S is a po-Γ-semigroup with involution and $P \subseteq S$. Then P is called *semiprime* if for any subset A of S, $A\Gamma A \subseteq P$ implies $A^\star \subseteq P$.

Equivalent definition: $x \in S$, $x\alpha x \in P$, then $x^\star \in P$, where $\alpha \in \Gamma$.

2. Main Results

Our starting point is the following lemma analogous to [15, Lemma 1] which we can easily prove.

Lemma 2.1. Suppose S is a po-Γ-semigroup with involution. Then we have the following results.

(i) $A \subseteq (A]$ for any $A \subseteq S$.

(ii) $(A] \subseteq (B]$ for any $A \subseteq B \subseteq S$.

(iii) $(A]\Gamma(B] \subseteq (A\Gamma B]$ for all $A, B \subseteq S$.

(iv) $((A]] \subseteq (A]$ for all $A \subseteq S$.

(v) For any right (left, two-sided) ideal I of S, $(I] = I$.

(vi) If I and J are ideals of S, then $(I\Gamma J]$ and $I \cap J$ are also ideals of S.

(vii) For any $s \in S$, $(S\Gamma s\Gamma S]$ is an ideal of S.

Lemma 2.2. Suppose S is an involution po-Γ-semigroup such that the involution $\star$ admits order. Then we have

(i) $(b\Gamma S\Gamma a]^\star = (a^\star\Gamma S\Gamma b^\star]$ for any $a, b \in S$.

(ii) $(S\Gamma a\Gamma S]^\star = (S\Gamma a^\star\Gamma S]$ for any $a \in S$.

(iii) $I^\star$ is an ideal of S for any ideal I of S.

Proof. (i) Suppose $x \in (b\Gamma S\Gamma a]^\star$. As $x^\star \in (b\Gamma S\Gamma a]$, $x^\star \leq b\alpha s\beta a$ for $s \in S$ and $\alpha, \beta \in \Gamma$. Then $x \leq (b\alpha s\beta a)^\star = a^\star\beta s^\star\alpha b^\star \in a^\star\Gamma S\Gamma b^\star$ since $\star$ is an order preserving involution. So $x \in (a^\star\Gamma S\Gamma b^\star]$ and therefore we obtain $(b\Gamma S\Gamma a]^\star \subseteq (a^\star\Gamma S\Gamma b^\star]$. Furthermore, if $x \in (a^\star\Gamma S\Gamma b^\star]$, then $x \leq a^\star\alpha s\beta b^\star$ for some $s \in S$ and $\alpha, \beta \in \Gamma$. So $x^\star \leq b\alpha s^\star\beta a \in b\Gamma S\Gamma a$ since $a^\star\alpha s\beta b^\star = (b\gamma s^\star\delta a)^\star$ for $\alpha, \beta, \gamma, \delta \in \Gamma$. This shows that $x^\star \in (b\Gamma S\Gamma a]$ and $x \in (b\Gamma S\Gamma a]^\star$. So $(a^\star\Gamma S\Gamma b^\star] \subseteq (b\Gamma S\Gamma a]^\star$. Hence $(b\Gamma S\Gamma a]^\star = (a^\star\Gamma S\Gamma b^\star]$. (ii) The proof is similar to (i). (iii) Suppose I is an ideal of S. As $S\Gamma I \subseteq I$, we obtain $(S\Gamma I)^\star \subseteq (I)^\star$. So $I^\star\Gamma S^* \subseteq I^\star$. As $\star$ is an involution on S, $(s^\star)^\star = s$ for every $s \in S$, and so $S^\star = S$. Therefore $I^\star\Gamma S \subseteq I^\star$. In the same way as $I\Gamma S \subseteq I$, we obtain $S\Gamma I^\star \subseteq I^\star$. Suppose $a \in I^\star$, and $b \leq a$, then $b^\star \leq a^\star$. Since $a^\star \in I$ and I is an ideal. Therefore $b^\star \in I$, and so $b \in I^\star$ and hence $I^\star$ is an ideal of S. □

Theorem 2.3. *Suppose S is a po Γ-semigroup such that S admits an order preserving involution $\star$. An ideal of S is prime if and only if it is both weakly prime and semiprime. Furthermore, if S is commutative, then the prime and weakly prime ideals coincide.*

Proof. Let I be a prime ideal of S. Then it is obviously weakly prime and semiprime. Conversely, let P be an ideal of S which is weakly prime and semiprime. Suppose $a\alpha b \in P$ for $\alpha \in \Gamma$, we need to prove that $a^\star \in P$ or $b^\star \in P$. By Lemma 2.1, $(b\Gamma S\Gamma a]\Gamma(b\Gamma S\Gamma a] \subseteq (S\Gamma a\Gamma b\Gamma S] \subseteq (S\Gamma P\Gamma S] \subseteq (P] = P$. So P is semiprime and it follows that $(b\Gamma S\Gamma a]^\star \subseteq P$. Now we have

$$\begin{aligned}
(S\Gamma a^\star\Gamma S]\Gamma(S\Gamma b^\star\Gamma S] &\subseteq (S\Gamma a^\star\Gamma S\Gamma S\Gamma b^\star\Gamma S] \\
&\subseteq (S\Gamma(a^\star\Gamma S\Gamma b^\star)\Gamma S] \\
&= (S\Gamma((S\Gamma b^\star)^\star\Gamma a)^\star\Gamma S] \\
&= (S\Gamma(b\Gamma S\Gamma a)^\star\Gamma S] \\
&\subseteq (S\Gamma(b\Gamma S\Gamma a]^\star\Gamma S] \\
&\subseteq (S\Gamma P\Gamma S] \\
&\subseteq P.
\end{aligned}$$

We note that $(S\Gamma a^\star\Gamma S]$, $(S\Gamma b^\star\Gamma S]$ are ideals, and P is weakly prime. So $(S\Gamma a^\star\Gamma S]^\star \subseteq P$ or $(S\Gamma b^\star\Gamma S]^\star \subseteq P$. Hence by Lemma 2.2, $(S\Gamma a\Gamma S] \subseteq P$

or $(S\Gamma b\Gamma S] \subseteq P$. Now to show that P is prime, we simply need to prove that if $(S\Gamma a\Gamma S] \subseteq P$ then $a^\star \in P$. The other statement can be proved similarly. If $(S\Gamma a\Gamma S] \subseteq P$ then $I(a)\Gamma I(a)\Gamma I(a) = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]^3 \subseteq ((a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S)^3] \subseteq (S\Gamma(a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S)\Gamma S] \subseteq (S\Gamma a\Gamma S] \subseteq P$. So $I(a)\Gamma(I(a)\Gamma I(a)] = (I(a)]\Gamma(I(a)\Gamma I(a)] \subseteq ((I(a))^3] \subseteq (P] = P$ by Lemma 2.2. We know that P is weakly prime and $I(a)$, $(I(a)\Gamma I(a)]$ are ideals. This implies that $(I(a))^\star \subseteq P$ or $(I(a)\Gamma I(a)]^\star \subseteq P$. Let $(I(a))^\star \subseteq P$. Therefore $a^\star \in (I(a))^\star \subseteq P$. Again let $(I(a)\Gamma I(a)]^\star \subseteq P$. So $a^\star\gamma a^\star \in (I(a)\Gamma I(a))^\star \subseteq (I(a)\Gamma I(a)]^\star \subseteq P$ for $\gamma \in \Gamma$ since $a\gamma a \in I(a)\Gamma I(a)$ and so $a = (a^\star)^\star \in P$ since P is semiprime. Now P is an ideal shows that $a\gamma a \in P$, therefore $a^\star \in P$ as P is semiprime. Now we prove the last statement. Suppose P is an ideal of S. If P is prime then clearly P is weakly prime. Conversely, Suppose P is weakly prime. Let $a\gamma b \in P$ for $\gamma \in \Gamma$. As S is commutative, we obtain $I(a)\Gamma I(b) = (a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]\Gamma(b \cup S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S] \subseteq ((a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S]\Gamma(b \cup S\Gamma b \cup b\Gamma S \cup S\Gamma b\Gamma S)] \subseteq (a\alpha b \cup S\Gamma a\beta b]$ for $\alpha, \beta \in \Gamma$. We note that $(a\alpha b \cup S\Gamma a\beta b] \subseteq (P] = P$ for $\alpha, \beta \in \Gamma$. Therefore $I(a)\Gamma I(b) \subseteq P$, and so we obtain $(I(a))^\star \in P$ or $(I(b))^\star \in P$ since P is weakly prime. Hence $a^\star \in P$ or $b^\star \in P$ and it follows that P is prime. □

Proposition 2.4. Suppose S is a po-Γ-semigroup with order preserving involution $\star$. Then the following statements are equivalent.

(i) $(A^\star\Gamma A^\star] = A$ for any ideal A of S.

(ii) $A^\star \cap B^\star = (A\Gamma B]$ for any ideals A, B of S.

(iii) $I(a) \cap I(b) = ((I(a))^\star\Gamma(I(b))^\star]$ for any $a, b \in S$.

(iv) $I(a) = (I(a^\star)\Gamma I(a^\star)]$ for any $a \in S$.

(v) $a \in (S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]$ for any $a \in S$.

Proof. (i) $\Rightarrow$ (ii). As $A^\star$, $B^\star$ are ideals, by our assumption and Lemma 2.1 we obtain$(A\Gamma B] \subseteq (A\Gamma S] \subseteq (A] = ((A^\star\Gamma A^\star]] = (A^\star\Gamma A^\star] \subseteq (A^\star] = A^\star$. In a similar fashion, we have $(A\Gamma B] \subseteq (S\Gamma B] \subseteq (B] = ((B^\star\Gamma B^\star]] = (B^\star\Gamma B^\star] \subseteq (B^\star] = B^\star$. So $(A\Gamma B] \subseteq A^\star \cap B^\star$. Moreover, $A^\star \cap B^\star$ is an ideal shows that $A^\star \cap B^\star = ((A^\star \cap B^\star)^\star\Gamma(A^\star \cap B^\star)^\star] = ((A \cap B)\Gamma(A \cap B)] \subseteq (A\Gamma B]$. Thus we obtain $(A\Gamma B] \subseteq A^\star \cap B^\star$ and $A^\star \cap B^\star \subseteq (A\Gamma B]$. Hence $A^\star \cap B^\star = (A\Gamma B]$. (ii) $\Rightarrow$ (iii). By Lemma 2.2, we have $(I(a))^\star$ and $(I(b))^\star$ are ideals. Hence follows the result. (iii) $\Rightarrow$ (iv). As $I(a) = ((I(a))^\star\Gamma(I(a))^\star]$ by our assumption, we simply need to show that $(I(a))^\star = I(a^\star)$. Obviously $a^\star \in (I(a))^\star$. Therefore $I(a^\star) \subseteq (I(a))^\star$ since $(I(a))^\star$ is an ideal. Now suppose $x \in (I(a))^\star$. We have $x^\star \in I(a) = (a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S]$. This shows that $x^\star \leq a$ or $x^\star \leq a\alpha v$ or $x^\star \leq v\alpha a$ or $x^\star \leq v\alpha a\beta w$ for some $v, w \in S$ and $\alpha, \beta \in \Gamma$. So $x \leq a^\star$ or $x \leq v^\star\alpha a^\star \in S\Gamma a^\star$ or $x \leq a^\star\alpha v^\star \in a^\star\Gamma S$ or $x^\star \leq w^\star\alpha a^\star\beta v^\star \in S\Gamma a^\star\Gamma S$ for some $v^\star, w^\star \in S$ and

$\alpha, \beta \in \Gamma$, and so $x \in (a^\star]$ or $x \in (S\Gamma a^\star]$ or $x \in (a^\star\Gamma S]$ or $x \in (S\Gamma a^\star\Gamma S]$. So $x \in (a^\star] \cup (S\Gamma a^\star] \cup (a^\star\Gamma S] \cup (S\Gamma a^\star\Gamma S] \subseteq (a^\star \cup S\Gamma a^\star \cup a^\star\Gamma S \cup S\Gamma a^\star\Gamma S] = I(a^\star)$. This implies $(I(a))^\star \subseteq I(a^\star)$. Hence $(I(a))^\star = I(a^\star)$. (iv) $\Rightarrow$ (v). For this, we show (1) $I(a) = ((I(a^\star)^6\Gamma I(a)]$ and (2) $((I(a^\star))^6\ \Gamma I(a)] \subseteq (S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]$. This will imply that $a \in I(a) \subseteq (S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]$.

(1) By Lemma 2.1 and our assumption, we obtain $I(a) = (I(a^\star)\Gamma I(a^\star)] = ((I(a)\Gamma I(a)]\Gamma(I(a)\Gamma I(a)]] \subseteq ((I(a)\Gamma I(a)\Gamma I(a)\Gamma I(a)]] = (I(a)\Gamma I(a)\Gamma I(a)\Gamma I(a)]$. Moreover,

$$\begin{aligned}
&(I(a)\Gamma I(a)\Gamma I(a)\Gamma I(a)]\\
&\quad = ((I(a^\star)\Gamma I(a^\star)]\Gamma(I(a^\star)\Gamma I(a^\star)]\Gamma(I(a^\star)\Gamma I(a^\star)]\Gamma(I(a)]\\
&\quad \subseteq ((I(a^\star))^6\Gamma I(a)]\\
&\quad \subseteq (S\Gamma I(a)]\Gamma I(a) \subseteq (I(a)]\\
&\quad = I(a) \text{ such that } I(a) \subseteq ((I(a^\star))^6\Gamma I(a)] \subseteq I(a).
\end{aligned}$$

So $I(a) = ((I(a^\star))^6\Gamma I(a)]$.

(2) As $(I(a))^3 \subseteq (S\Gamma a\Gamma]$ by Theorem 2.3, we obtain $(I(a))^5 = (I(a))^3\Gamma I(a)\Gamma I(a) \subseteq (S\Gamma a\Gamma S]\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S]\Gamma(S] \subseteq (S\Gamma a\Gamma S\Gamma(a \cup a\Gamma S \cup S\Gamma a \cup S\Gamma a\Gamma S)\Gamma S]$. Obviously $S\Gamma(a\cup a\Gamma S\cup S\Gamma a\cup S\Gamma a\Gamma S)\Gamma S \subseteq S\Gamma a\Gamma S$, and so $(S\Gamma a\Gamma S\Gamma(a\cup a\Gamma S\cup S\Gamma a\cup S\Gamma a\Gamma S)\Gamma S] \subseteq (S\Gamma a\Gamma S\Gamma S\Gamma a\Gamma S] \subseteq (S\Gamma a\Gamma S\Gamma a\Gamma S]$. So $(I(a))^5 \subseteq (S\Gamma a\Gamma S\Gamma a\Gamma S]$ and therefore $(I(a^\star))^5 \subseteq (S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]$. We have

$$\begin{aligned}
((I(a^\star))^6\Gamma I(a)] &\subseteq ((S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]\Gamma I(a^\star)\Gamma I(a)]\\
&\subseteq ((S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]\Gamma(S]]\\
&\subseteq (S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S\Gamma S]\\
&\subseteq (S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]
\end{aligned}$$

Therefore $((I(a^\star))^6\Gamma I(a)] \subseteq (S\Gamma a^\star\Gamma S\Gamma a^\star\Gamma S]$. $(v) \Rightarrow (i)$. Let $x \in (A^\star\Gamma A^\star]$. Then $x \leq y\alpha z$ for some $y, z \in A^\star$ and $\alpha \in \Gamma$. By our assumption $y \in (S\Gamma y^\star\Gamma S\Gamma y^\star\Gamma S]$, then $y \leq u_1\alpha y^\star\beta u_2\gamma y^\star\delta u_3$ for some $u_i \in S$, $i = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. In a similar fashion, $z \leq v_1\alpha z^\star\beta v_2\gamma z^\star\delta v_3$ for some $v_i \in S$, $i = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Therefore, $y\alpha z \leq u_1\beta y^\star\gamma u_2\delta y^\star\theta u_3\lambda v_1\mu z^\star\nu v_2\gamma_2 z^\star\gamma_1 v_3 \in S\Gamma y^\star\Gamma S \subseteq S\Gamma A\Gamma S \subseteq A$ for $\alpha, \beta, \gamma, \delta, \theta, \lambda, \mu, \nu, \gamma_1, \gamma_2 \in \Gamma$. So $x \in (A]$ since $x \leq y\alpha z$, and so $(A^\star\Gamma A^\star] \subseteq (A] = A$. If $x \in A$, then we obtain $x \leq w_1\alpha x^\star\beta w_2\gamma x^\star\delta w_3$ for some $w_i \in S$, $i = 1, 2, 3$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ since $x \in (S\Gamma x^\star\Gamma S\Gamma x^\star\Gamma S]$. It is now obvious that $w_1\alpha x^\star\beta w_2 \in A^\star$ and $x^\star\alpha w_3 \in A^\star$ as $A^\star$ is an ordered Γ-ideal of S by Lemma 2.2. So $x \leq w_1\alpha x^\star\beta w_2\gamma x^\star\lambda w_3 \in A^\star\Gamma A^\star$ for $\alpha, \beta, \gamma, \lambda \in \Gamma$ and so $A \subseteq (A\Gamma A^\star]$. Hence $A = (A^\star\Gamma A^\star]$. □

Theorem 2.5. *Suppose S is a po-Γ-semigroup having order preserving involution $\star$. The ideals of S are weakly prime if and only if $A^\star = (A\Gamma A]$ for any ideal A of S and any two ideals are comparable under the inclusion relation.*

Proof. Let the ideals of S be weakly prime. Suppose A, B are any ideals of S. As

$B^\star$ is an ideal and $(A\Gamma B^\star]$ is weakly prime. Thus $A\Gamma B^\star \subseteq (A\Gamma B^\star]$ shows that $A^\star \subseteq (A\Gamma B^\star]$ or $B \subseteq (A\Gamma B^\star]$. If $A^\star \subseteq (A\Gamma B^\star]$, then $A^\star \subseteq (S\Gamma B^\star] \subseteq (B^\star] = B^\star$ and so $(A^\star)^\star \subseteq (B^\star)^\star$. This means $A \subseteq B$. If $B \subseteq (A\Gamma B^\star]$, then $B \subseteq (A\Gamma S] \subseteq (A] = A$. It now follows that A and B are comparable. We claim $A^\star = (A\Gamma A]$. As $(A\Gamma A]$ is weakly prime and $A\Gamma A \subseteq (A\Gamma A]$, we obtain $A^\star \subseteq (A\Gamma A]$. Also suppose $x \in (A\Gamma A]$. Then $x \leq a_1\alpha a_2 \in A\Gamma A$ for some $a_1, a_2 \in A$ and $\alpha \in \Gamma$. As $A^\star \subseteq (A\Gamma A]$, we obtain $a_1^\star \leq u_1\alpha v_1 \in A\Gamma A$ and $a_2^\star \leq u_2\beta v_2 \in A\Gamma A$ for some $u_1, u_2, v_1, v_2 \in A$ and $\alpha, \beta \in \Gamma$. Thus $a_1 \leq (u_1\alpha v_1)^\star$ and $a_2 \leq (u_2\beta v_2)^\star$. This shows that $x \leq a_1\alpha a_2 \leq (u_1\beta v_1)^\star\gamma(v_1\delta v_2)^\star \in (A\Gamma A)^\star\Gamma(A\Gamma A)^\star = A^\star\Gamma A^\star\Gamma A^\star\Gamma A^\star \subseteq A^\star$ since $A^\star$ is an ideal for $\alpha, \beta, \gamma, \delta \in \Gamma$. It follows that $x \in (A^\star] = A^\star$. So $(A\Gamma A] \subseteq A^\star$. Conversely, assume A, B and P are ideals of S such that $A\Gamma B \subseteq P$. As $A^\star = (A\Gamma A]$, we obtain $A^\star \cap B^\star = (A\Gamma B]$ by Proposition 2.4. As A and B are comparable, two cases arise. If $A \subseteq B$, then $A^\star \subseteq B^\star$, and so $A^\star = A^\star \cap B^\star = (A\Gamma B] \subseteq (P] = P$ by Proposition 2.4. Also if $B \subseteq A$, then $B^\star \subseteq A^\star$, and so $B^\star = A^\star \cap B^\star = (A\Gamma B] \subseteq (P] = P$. Hence P is weakly prime. □

Proposition 2.6. Suppose S is an involution po-Γ-semigroup. Then S is intra-regular if and only if the ideals of S are semiprime.

Proof. Let I be an ideal of S having $s\alpha s \in I$ for some $s \in S$ and $\alpha \in \Gamma$. As S is intra regular, we obtain $s^\star \in (S\Gamma s\gamma s\Gamma S] \subseteq (S\Gamma I\Gamma S] \subseteq (I] = I$ for $\gamma \in \Gamma$ and therefore I is semiprime. Conversely, let $s \in S$. It is now obvious that $(S\Gamma s^\star\gamma s^\star\Gamma S]$ is an ideal. Therefore $(s\Gamma s^\star\gamma s^\star\Gamma S]$ is semiprime by our assumption. This shows that $s\gamma s = (s^\star\alpha s^\star)^\star \in (S\Gamma s^\star\beta s^\star\Gamma S]$ since $(s^\star\alpha s^\star)\beta(s^\star\gamma s^\star) \in S\Gamma s^\star\delta s^\star\Gamma S \subseteq (S\Gamma s^\star\lambda s^\star\Gamma S]$ for $\alpha, \beta, \gamma, \delta, \lambda \in \Gamma$. So $s^\star \in (S\Gamma s^\star\alpha s^\star\Gamma S]$ and so $s^\star\alpha s^\star \in (S\Gamma s^\star\beta s^\star\Gamma S]$ for $\alpha, \beta \in \Gamma$. Hence $s \in (S\Gamma s^\star\alpha s^\star\Gamma S]$ and it follows that S is intra-regular. □

Proposition 2.7. Suppose S is a po-Γ-semigroup with involution. If S is intra-regular, then $(S\Gamma x\alpha y\Gamma S] = (S\Gamma x^\star\beta y^\star\Gamma S]$ for some $x, y \in S$ and $\alpha, \beta \in \Gamma$.

Proof. Suppose $x, y \in S$. As S is intra-regular, it follows that $x\alpha y \in (S\Gamma(x\beta y)^\star\gamma(x\delta y)^\star\Gamma S] = (S\Gamma y^\star\gamma_1 x^\star\gamma_2 y^\star\gamma_3 x^\star\Gamma S] \subseteq (S\Gamma x^\star\alpha y^\star\Gamma S]$
for $\alpha, \beta, \gamma, \gamma_1, \gamma_2, \gamma_3 \in \Gamma$. Therefore $x\alpha y \leq u_1\beta x^\star\gamma y^\star\delta u_2$ for some $u_1, u_2 \in S$. Therefore $u_3\alpha x\beta y\gamma u_4 \leq u_3\delta u_1\theta x^\star\lambda y^\star\mu u_2\nu u_4 \in S\Gamma x^\star\alpha y^\star\Gamma S \subseteq (S\Gamma x^\star\alpha y^\star\Gamma S]$ for any $u_3, u_4 \in S$ and $\alpha, \beta, \gamma, \delta, \theta, \lambda \in \Gamma$. This shows that $S\Gamma x\alpha y\Gamma S \subseteq (S\Gamma x^\star\alpha y^\star\Gamma]$, therefore $(S\Gamma x\alpha y\Gamma S] \subseteq ((S\Gamma x^\star\alpha y^\star\Gamma S]] = (S\Gamma x^\star\alpha\Gamma S]$ by Lemma 2.1. We obtain $(S\Gamma x^\star\alpha y^\star\Gamma S] \subseteq (S\Gamma x\beta y\Gamma S]$. Hence $(S\Gamma x\alpha y\Gamma S] = (S\Gamma x^\star\beta y^\star\Gamma S]$ for $\alpha, \beta \in \Gamma$. □

Proposition 2.8. Suppose S is a po-Γ-semigroup with order preserving involution $\star$. If the ideals of S are semiprime, then

(i) $I(s) = (S\Gamma s\Gamma S]$ for any $s \in S$, and

(ii) $I(x\alpha y) = I(x) \cap I(y)$ for any $x, y \in S$ and $\alpha \in \Gamma$.

Proof. (i) Suppose $s \in S$. Recall that $(S\Gamma s\Gamma S]$ is an ideal and so is semiprime. Since

$(s\alpha s)\alpha(s\alpha s) = (s\alpha s)^2 = s^4 \in (S\Gamma s\Gamma S]$ gives $s^\star \alpha s^\star = (s\alpha s)^\star \in (S\Gamma s\Gamma S]$ for $\alpha \in \Gamma$. In a similar fashion, $s \in (S\Gamma s\Gamma S]$ so that $I(s) \subseteq (S\Gamma s\Gamma S]$. Moreover $(S\Gamma s\Gamma S] \subseteq (s\cup s\Gamma S \cup S\Gamma s \cup S\Gamma s\Gamma S] = I(s)$. Hence $I(x) = (S\Gamma x\Gamma S]$. (ii)As $x\alpha y \in I(x)\Gamma S \subseteq I(x)$, we obtain $I(x\alpha y) \subseteq I(x)$. Also $I(x\alpha y) \subseteq I(y)$ since $x\alpha y \in S\Gamma I(y) \subseteq I(y)$. So $I(x\alpha y) \subseteq I(x) \cap I(y)$. If $z \in I(x) \cap I(y)$, then $z \in (S\Gamma x\Gamma S] \cap (S\Gamma y\Gamma S]$ by (i), and so $z \leq u_1\alpha x\beta u_2$ and $z \leq v_1\alpha y\beta v_2$ for some $u_1, u_2, v_1, v_2 \in S$ and for $\alpha, \beta \in \Gamma$. Recall $(y\alpha_1 v_2\alpha_2 u_1\alpha_3 x)^2 = (y\alpha_4 v_2\alpha_5 u_1\alpha_6 x)\alpha_7(y\alpha_8 v_2\alpha_9 u_1\alpha_{10} x) \in (S\Gamma x\alpha_{11} y\Gamma S] = I(x\alpha_{12} y)$ for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12} \in \Gamma$ and that $I(x\alpha y)$ is semiprime. So $(y\alpha v_2\beta u_1\gamma x)^\star \in I(x\alpha y)$. So $z^\star\alpha z^\star \leq (u_1\alpha x\beta u_2)^\star\gamma(v_1\alpha y\beta v_2)^\star = u_2^\star\alpha(y\beta v_2\gamma u_1\delta x)^\star\theta v_1^\star \in I(x\alpha y)$, and so $z^\star\alpha z^\star \in (I(x\alpha y)] = I(x\alpha y)$ for $\alpha, \beta, \gamma, \delta, \theta \in \Gamma$. This implies that $z \in I(x\alpha y)$, then $I(x) \cap I(y) \subseteq I(x\alpha y)$. □

Theorem 2.9. *Suppose S is an involution po-Γ-semigroup such that the involution admits the order. The ideals of S are prime if and only if S is intra-regular and any two ideals are comparable under the inclusion relation.*

Proof. If the ideals are prime, then they are weakly prime and hence they are comparable by Theorem 2.5. Suppose $s \in S$. Recall that $(S\Gamma s^\star\alpha s^\star\Gamma S]$ is an ideal by Lemma 2.1 and hence prime. So $(s\alpha s)\alpha(s\alpha s) = s^4 \in (S\Gamma s^\star\alpha s^\star\Gamma S]$ since $(s^\star)^4\alpha(s^\star)^4 \in (S\Gamma s^\star\beta s^\star\Gamma S]$ for $\alpha, \beta \in \Gamma$. In a similar fashion, we have $(s^\star\alpha s^\star) = (s^\star)^2 \in (S\Gamma s^\star\alpha s^\star\Gamma S]$ and $s \in (S\Gamma s^\star\alpha s^\star\Gamma S]$. It follows that S is intra-regular. Conversely, assume that S is intra-regular and any two ideals are comparable under the inclusion relation $\subseteq$. Suppose T is any ideal of S and $a\alpha b \in T$, where $a, b \in S$ and $\alpha \in \Gamma$. Claim $a^\star \in T$ or $b^\star \in T$. By Proposition 2.6, $I(a)$ is semiprime. Thus $a\alpha a \in I(a)$ implies $a^\star \in I(a)$. We can similarly prove $b^\star \in I(b)$. By our assumption, we obtain $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$, then $a^\star \in I(a) = I(a) \cap I(b) = I(a\alpha b) \subseteq T$ by Proposition 2.8. If $I(b) \subseteq I(a)$, then we obtain $b^\star \in I(b) = I(a) \cap I(b) = I(a\alpha b) \subseteq T$. □

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