KYUNGPOOK Math. J. 54(2014), 619-627
http://dx.doi.org/10.5666/KMJ.2014.54.4.619

Characterizations of Zero-Term Rank Preservers of Matrices over Semirings

Kyung-Tae Kang and Seok-Zun Song*
Department of Mathematics, Jeju National University, Jeju 690-756, Korea
e-mail : kangkt@jejunu.ac.kr *and* szsong@jejunu.ac.kr

LeRoy B. Beasley
Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322-3900, USA
e-mail : leroy.b.beasley@aggiemail.usu.edu

Luis Hernandez Encinas
Department of Information Processing and Cryptography, Institute of Physical and Information Technologies, Spanish National Research Council, C / Serrano 144, 28006-Madrid, Spain
e-mail : luis@iec.csic.es

Abstract. Let $\mathcal{M}(S)$ denote the set of all $m\times n$ matrices over a semiring S. For $A \in \mathcal{M}(S)$, *zero-term rank* of A is the minimal number of lines (rows or columns) needed to cover all zero entries in A. In [5], the authors obtained that a linear operator on $\mathcal{M}(S)$ preserves zero-term rank if and only if it preserves zero-term ranks 0 and 1. In this paper, we obtain new characterizations of linear operators on $\mathcal{M}(S)$ that preserve zero-term rank. Consequently we obtain that a linear operator on $\mathcal{M}(S)$ preserves zero-term rank if and only if it preserves two consecutive zero-term ranks k and $k+1$, where $0 \le k \le \min\{m, n\} - 1$ if and only if it strongly preserves zero-term rank h, where $1 \le h \le \min\{m, n\}$.

1. Introduction and Preliminaries

A *semiring* ([2]) is a set S equipped with two binary operations $+$ and $\cdot$ such that $(S, +)$ is a commutative monoid with identity element 0 and $(S, \cdot)$ is a monoid with identity element 1. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates S. Thus all rings with identity are semirings.

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* Corresponding Author.
Received October 20, 2013; accepted April 9, 2014.
2010 Mathematics Subject Classification: 15A03, 15A04, 15A86.
Key words and phrases: Semiring, zero-term rank, linear operator, (strongly) preserve.
This work was supported under the framework of international cooperation program managed by National Research Foundation of Korea (NRF-2013K2A1A2053670).

A semiring S is *commutative* if $(S,\cdot)$ is Abelian; S is *antinegative* if 0 is the only element to have an additive inverse. Thus, no ring is antinegative semiring except $\{0\}$. The following are some examples of semirings which occur in combinatorics. Let $\mathbb{B}=\{0,1\}$. Then $(\mathbb{B},+,\cdot)$ is a semiring (the *binary Boolean semiring*) if arithmetic in $\mathbb{B}$ follows the usual rules except that $1+1=1$. If $\mathbb{F}$ is the real interval $[0,1]$, then $(\mathbb{F},+,\cdot)=(\mathbb{F},\max,\min)$ is a semiring (the *fuzzy semiring*). If $\mathbb{P}$ is any subring with identity, of $\mathbb{R}$, the reals (under real addition and multiplication), and $\mathbb{P}_+$ denotes the nonnegative part of $\mathbb{P}$, then $\mathbb{P}_+$ is a semiring. In particular $\mathbb{Z}_+$, the nonnegative integers, is a semiring. These are all commutative and antinegative semirings.

Hereafter, S will denote an arbitrary commutative and antinegative semiring. Let $\mathcal{M}(S)$ be the set of all $m\times n$ matrices with entries in a semiring S. The matrix $O_{m,n}$ is the $m\times n$ zero matrix and the matrix $J_{m,n}$ is the $m\times n$ matrix all of whose entries are 1. We will suppress the subscripts on these matrices when the orders are evident from the context and we write O and J, respectively. Algebraic operations on $\mathcal{M}(S)$ are defined as if the underlying scalars were in a field.

The *zero-term rank*, $z(A)$, of $A\in\mathcal{M}(S)$ is the minimal number k of lines (rows or columns) needed to cover all zero entries in A. That is, $z(A)$ is the minimal number k such that all zero entries of A are contained in r rows and $k-r$ columns. The *term rank*, $t(A)$, of A is the minimal number k of lines (rows or columns) needed to cover all nonzero entries in A.

From now on we will assume that $2\le m\le n$ unless specified otherwise. It follows that $0\le z(A)\le m$ for all $A\in\mathcal{M}(S)$. Evidently we have that

$$z(O)=t(J)=m \quad \text{and} \quad z(J)=t(O)=0.$$

An operator $T:\mathcal{M}(S)\to\mathcal{M}(S)$ is called *linear* if $T(\alpha A+\beta B)=\alpha T(A)+\beta T(B)$ for all $A,B\in\mathcal{M}(S)$ and for all $\alpha,\beta\in S$. Let $T:\mathcal{M}(S)\to\mathcal{M}(S)$ be a linear operator. If f is a function defined on $\mathcal{M}(S)$, then T *preserves* the function f if $f(T(A))=f(A)$ for all $A\in\mathcal{M}(S)$. There are many papers on linear operators that preserve matrix functions over S(see [1]-[6] and therein). Beasley and Pullman([3]) characterized linear operators on $\mathcal{M}(S)$ that preserve term rank. Recently Beasely, Kang and Song([6]) extended their results and obtained new characterizations of linear operators on $\mathcal{M}(S)$ that preserve term rank. But there are few papers on zero-term rank preservers of matrices over S. Beasley, Song and Lee([5]) have characterized linear operators on $\mathcal{M}(S)$ that preserve zero-term rank as following:

Theorem 1.1.([5]) *For a linear operator* $T:\mathcal{M}(S)\to\mathcal{M}(S)$*, the following are equivalent*:

(i) T *preserves zero-term rank*;

(ii) T *preserves zero-term rank* 1 *and* $J\sqsubseteq T(J)$;

(iii) T *is a* (P,Q,B)*-operator.*

We note that the condition (ii) in Theorem 1.1 means that T preserves zero-term ranks 0 and 1(see Lemma 2.2).

In this paper, we generalize the conditions of Theorem 1.1 to any two consecutive zero-term rank preservers. Furthermore we obtain other characterizations of the zero-term rank preservers.

2. Preliminary

If A and B are matrices in $\mathcal{M}(S)$, we say that B *dominates* A (written $A \sqsubseteq B$ or $B \sqsupseteq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. This provides a reflexive and transitive relation on $\mathcal{M}(S)$.

Lemma 2.1. For matrices A and B in $\mathcal{M}(S)$, we have:

(i) $z(A+B) \leq z(A) + z(B)$;

(ii) if $A \sqsubseteq B$, then $z(B) \leq z(A)$;

(iii) if $T : \mathcal{M}(S) \to \mathcal{M}(S)$ is a linear operator and $A \sqsubseteq B$, then $T(A) \sqsubseteq T(B)$.

Proof. The results follow from the definitions of both zero-term rank and linear operator. □

For a linear operator $T : \mathcal{M}(S) \to \mathcal{M}(S)$ and $0 \leq k \leq m$, we say that

(1) T *preserves zero-term rank* k if $z(T(X)) = k$ whenever $z(X) = k$ for all X;

(2) T *preserves zero-term rank* if $z(T(X)) = z(X)$ for all X.

Lemma 2.2. Suppose that $T : \mathcal{M}(S) \to \mathcal{M}(S)$ is a linear operator. Then $J \sqsubseteq T(J)$ if and only if T preserves zero-term rank 0.

Proof. Suppose that $J \sqsubseteq T(J)$ and A is any matrix in $\mathcal{M}(S)$ with $z(A) = 0$. Clearly $J \sqsubseteq A$ and hence Lemma 2.1(iii) implies that $J \sqsubseteq T(J) \sqsubseteq T(A)$. That is, $z(T(A)) = 0$. Therefore T preserves zero-term rank 0. The converse is obvious. □

A matrix in $\mathcal{M}(S)$ is called a *cell* if it has exactly one 1 entry. We denote the cell whose one 1 entry is in the $(i,j)^{th}$ position by $E_{i,j}$. Further we let $\mathcal{E}_{m,n}$ be the set of all cells in $\mathcal{M}(S)$. That is, $\mathcal{E}_{m,n} = \{E_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

Let $\mathbb{B}$ be the binary Boolean semiring and $\mathcal{M}(\mathbb{B})$ be the set of all $m \times n$ Boolean matrices with entries in $\mathbb{B}$.

Lemma 2.3. ([1]) If $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is a linear operator, then T is invertible if and only if T permute $\mathcal{E}_{m,n}$.

An $m \times n$ matrix L is called a *line matrix* if $L = \sum_{l=1}^{n} E_{i,l}$ for some $i \in \{1, \ldots, m\}$ or $L = \sum_{s=1}^{m} E_{s,j}$ for some $j \in \{1, \ldots, n\}$: $R_i = \sum_{l=1}^{n} E_{i,l}$ is the ith *row matrix* and $C_j = \sum_{s=1}^{m} E_{s,j}$ is the jth *column matrix.*

For matrices A and B in $\mathcal{M}(S)$, the matrix $A \circ B$ denotes the *Hadamard* or *Schur product.* That is, the (i,j)th entry of $A \circ B$ is $a_{i,j}b_{i,j}$. A nonzero $s \in S$ is a *zero divisor* if $s's = 0$ for some nonzero $s' \in S$.

If P and Q are permutation matrices of orders m and n, respectively, and B is a matrix in $\mathcal{M}(S)$ none of whose entries is a zero divisor or zero, then an operator $T : \mathcal{M}(S) \to \mathcal{M}(S)$ is called a (P, Q, B)-*operator* if $T(X) = P(X \circ B)Q$ for all X, or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all X, where X^t denotes the transpose of X. If $B = J$ we say that T is a (P, Q)-*operator*.

The number of nonzero entries of a matrix $A \in \mathcal{M}(S)$ is denoted by $\sharp(A)$.

For a linear operator T on $\mathcal{M}(S)$, we say that T *preserves all line matrices* if $T(L)$ is a line matrix for all line matrix L.

Lemma 2.4. Assume that $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is an invertible linear operator. Then T preserves all line matrices if and only if T is a (P, Q)-operator.

Proof. By Lemma 2.3, T permutes $\mathcal{E}_{m,n}$. Suppose that T preserves all line matrices and let $\mathcal{R} = \{R_1, \ldots, R_m\}$ and $\mathcal{C} = \{C_1, \ldots, C_n\}$. Now we will claim that either

(1) T maps $\mathcal{R}$ onto $\mathcal{R}$ and maps $\mathcal{C}$ onto $\mathcal{C}$, or

(2) T maps $\mathcal{R}$ onto $\mathcal{C}$ and maps $\mathcal{C}$ onto $\mathcal{R}$.

If $m \neq n$, (1) is satisfied since T preserves all line matrices. Thus we assume that $m = n$. Suppose that the claim is not true. Then there are two row matrices R_i and R_j such that $T(R_i) \in \mathcal{R}$ and $T(R_j) \in \mathcal{C}$. But then $\sharp(R_i + R_j) = 2n$, while $\sharp(T(R_i + R_j)) = 2n - 1$, a contradiction to the fact that T is invertible. Hence the claim is true.

If (1) holds, there are permutations α and β of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively, such that $T(R_i) = R_{\alpha(i)}$ for all i and $T(C_j) = C_{\beta(j)}$ for all j. Let P and Q be the permutation matrices corresponding to α and β, respectively. Then we have that

$$T(E_{i,j}) = E_{\alpha(i),\beta(j)} = PE_{i,j}Q$$

for all cells $E_{i,j}$. By the action of T on $\mathcal{E}_{m,n}$, we have that $T(X) = PXQ$ for all X. Hence T is a (P, Q)-operator. If (2) holds, then $m = n$ and a parallel argument shows that there are permutation matrices P and Q of order n such that $T(X) = PX^tQ$ for all X. Thus T is a (P, Q)-operator.

The converse is obvious. □

For Boolean matrices A and B in $\mathcal{M}(\mathbb{B})$ with $B \sqsubseteq A$, we define $A \setminus B$ to be the matrix C such that $c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise.} \end{cases}$

Theorem 2.5. *Suppose that $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is an invertible linear operator and $1 \le k \le m$. Then T preserves zero-term rank k if and only if T is a (P, Q)-operator.*

Proof. By Lemma 2.3, T permutes $\mathcal{E}_{m,n}$. Assume that T preserves zero-term rank k, where $1 \le k \le m$. Since T permutes $\mathcal{E}_{m,n}$, we have that $T(J) = J$ and hence $J \sqsubseteq T(J)$. Thus by Theorem , T is a (P, Q)-operator for the case of $k = 1$. Now let $k \ge 2$. Suppose that T does not preserve a line matrix. Then there are two cells E and F that are not dominated by the same line matrix such that $T(E)$ and

$T(F)$ are dominated by the same line matrix. Without loss of generality, we may assume that $T(E_{1,1}+E_{2,2}) = E_{1,1}+E_{1,2}$. Let $A = E_{1,1}+E_{2,2}+\cdots E_{k,k}$ so that $z(J \setminus A) = k$. Since at least two cells in $T(A)$ are dominated by the same line matrix, we have that $z(T(J \setminus A)) = z(J \setminus T(A)) \leq k-1$. This is a contradiction to the fact that T preserves zero-term rank k. Therefore T must preserve all line matrices. Thus T is a (P, Q)-operator by Lemma 2.4.

The converse is obvious. □

3. Zero-Term Rank Preservers of Boolean Matrices

In this section, we give necessary and sufficient conditions for a linear operator $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ to preserve zero-term rank.

An operator $T : \mathcal{M}(S) \to \mathcal{M}(S)$ is *singular* if $T(X) = O$ for some nonzero $X \in \mathcal{M}(S)$; otherwise T is *nonsingular*. Evidently every (P, Q, B)-operator on $\mathcal{M}(S)$ is nonsingular. In particular, if $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is a singular linear operator, then we can easily check that $T(E) = O$ for some cell E.

Example 3.1. For $0 \leq k \leq m$, let $A = J \setminus (E_{1,1}+E_{2,2}+\cdots+E_{k,k})$. Define an operator $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ by $T(O) = O$ and $T(X) = A$ for all nonzero $X \in \mathcal{M}(\mathbb{B})$. Clearly, T is linear, nonsingular and preserves zero-term rank k since $z(A) = k$. But T does not preserve zero-term rank. □

The above Example implies that the condition on T that it preserves a zero-term rank k is not sufficient for T to be a zero-term rank preserver. So we want to find some conditions for T to be a zero-term rank preserver.

For a linear operator $T : \mathcal{M}(S) \to \mathcal{M}(S)$ and $0 \leq l \leq m$, we say that T *strongly preserves zero-term rank* l if $z(T(X)) = l$ if and only if $z(X) = l$ for all X.

Lemma 3.2. Let $0 \leq k \leq m-1$ and $0 \leq l \leq m$. Assume that $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is a linear operator. If

(i) T preserves zero-term ranks k and $k+1$, or

(ii) T strongly preserves zero-term rank l,

then T is nonsingular.

Proof. If T is singular, then $T(E) = O$ for some cell E. Without loss of generality, we assume that $T(E_{1,1}) = O$ so that $T(J) = T(J \setminus E_{1,1})$. Notice that $z(J) = 0$ and $z(J \setminus E_{1,1}) = 1$. Hence we have a contradiction for the case of $k = 0$ and $l \in \{0, 1\}$. Thus we assume that $k \geq 1$ and $l \geq 2$. Let $A = \sum_{i=1}^{k+1} E_{i,i}$. Then $z(J \setminus A) = k+1$ and $z(J \setminus A + E_{1,1}) = k$. But $T(J \setminus A) = T(J \setminus A + E_{1,1})$ contradicts the condition (i). Let $B = \sum_{i=1}^{l} E_{i,i}$. Then $z(J \setminus B) = l$ and $z(J \setminus B + E_{1,1}) = l-1$. But $T(J \setminus B) = T(J \setminus B + E_{1,1})$ contradicts the condition (ii).
Hence T is nonsingular. □

For $0 \leq k \leq m$, we let $\mathbb{D}_k = [d_{i,j}]$ be a Boolean matrix in $\mathcal{M}(\mathbb{B})$ such that $d_{i,j} = 0$ if and only if $i + j \leq k + 1$. Then we have that $z(\mathbb{D}_k) = k$. Notice that if $1 \leq k \leq m - 1$ then we have that $z(\mathbb{D}_{k+1}) = k + 1$, while $z(\mathbb{D}_{k+1} + E_{2,k}) = k$ since all zero entries of $\mathbb{D}_{k+1} + E_{2,k}$ are contained in the first row and the first $k - 1$ columns. Similarly $z(\mathbb{D}_{k+1} + E_{k,2}) = k$. For example, consider the 5×6 matrix

$$\mathbb{D}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $z(\mathbb{D}_4) = 4$ and $z(\mathbb{D}_4 + E_{2,3}) = z(\mathbb{D}_4 + E_{3,2}) = 3$.

Lemma 3.3. Let $m \geq 3$ and $1 \leq k \leq m - 1$. If $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is a linear operator that preserves zero-term ranks k and $k+1$, then $\sharp(T(E_{i,j})) = 1$ for all cells $E_{i,j}$.

Proof. By Lemma 3.2(i), T is nonsingular and hence $\sharp(T(E_{i,j})) \geq 1$ for all cells $E_{i,j}$. Suppose that $\sharp(T(E)) \geq 2$ for some cell E. By permuting we may assume that $T(E) \sqsupseteq E + F$ for some cell $F \neq E$.

If E and F are in the same row, we may assume by permuting that $E = E_{2,k+1}$ and $F = E_{2,k}$. If E and F are in the same column, we may assume by permuting that $E = E_{k+1,2}$ and $F = E_{k,2}$. If E and F are in different rows and different columns, we may assume by permuting that $E = E_{3,k+1}$ and $F = E_{2,k}$. Then we have that $E \sqsubseteq \mathbb{D}_{k+1}$ and $F \not\sqsubseteq \mathbb{D}_{k+1}$. It follows that $z(\mathbb{D}_{k+1} + E) = k + 1$ and $z(\mathbb{D}_{k+1} + F) = k$.

Let $L = T^d$ where d is chosen so that $L : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is an idempotent operator ($L^2 = L$). Then we can easily check that L preserves zero-term ranks k and $k + 1$, and $L(E) \sqsupseteq E + F$. Since $L(E) = F + X$ for some matrix $X \in \mathcal{M}(\mathbb{B})$, we have that

$$L(E) + F = (F + X) + F = F + X = L(E).$$

Since L is idempotent, we have that $L(E) = L^2(E) = L(L(E)) = L(L(E) + F) = L^2(E) + L(F) = L(E) + L(F) = L(E + F)$. It follows that $L(\mathbb{D}_{k+1} + E) = L(\mathbb{D}_{k+1} + E + F)$, equivalently, $L(\mathbb{D}_{k+1}) = L(\mathbb{D}_{k+1} + F)$ since $E \sqsubseteq \mathbb{D}_{k+1}$. This is a contradiction to the fact that L preserves zero-term ranks k and $k + 1$ since $z(\mathbb{D}_{k+1}) = k + 1$ and $z(\mathbb{D}_{k+1} + F) = k$. Hence we have that $\sharp(T(E_{i,j})) = 1$ for all cells $E_{i,j}$. □

Theorem 3.4. *Let $m \geq 3$ and $0 \leq k \leq m - 1$. Suppose that $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is a linear operator. Then T preserves zero-term rank if and only if T preserves zero-term ranks k and $k + 1$.*

Proof. By Theorem 1.1 and Lemma 2.2, the result is obvious for the case of $k = 0$. Now consider the case $k \geq 1$. Assume that T preserves zero-term ranks k and $k+1$. Then $\sharp(T(E_{i,j})) = 1$ for all cells $E_{i,j}$ by Lemma 3.3. Now, suppose that T is not invertible. Then $T(E) = T(F)$ for some distinct cells E and F by Lemma 2.3. If E and F are in the same row, we may assume by permuting that $E = E_{2,k+1}$ and $F = E_{2,k}$. If E and F are in the same column, we may assume by permuting that $E = E_{k+1,2}$ and $F = E_{k,2}$. If E and F are in different rows and different columns, we may assume by permuting that $E = E_{3,k+1}$ and $F = E_{2,k}$. Then we have that $E \sqsubseteq \mathbb{D}_{k+1}$ and $F \not\sqsubseteq \mathbb{D}_{k+1}$. It follows that $z(\mathbb{D}_{k+1}+E) = k+1$ and $z(\mathbb{D}_{k+1}+F) = k$. But then $T(\mathbb{D}_{k+1}+E) = T(\mathbb{D}_{k+1}+F)$, a contradiction to the fact that T preserves zero-term ranks k and $k+1$. Hence T must be invertible. By Theorem 2.5, T is a (P, Q)-operator and hence T preserves zero-term rank by Theorem 1.1.

The converse is obvious. □

Lemma 3.5. If $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is a linear operator that strongly preserves zero-term rank 1, then T preserves zero-term rank.

Proof. Suppose that $J \not\sqsubseteq T(J)$. Since T strongly preserves zero-term rank 1, we have that $z(T(J)) \geq 2$. Let $E_{i,j}$ be an arbitrary cell. Then $T(J \setminus E_{i,j}) \sqsubseteq T(J)$ and hence by Lemma 2.1(ii), we have that $z(T(J \setminus E_{i,j})) \geq z(T(J)) \geq 2$, a contradiction since $z(J \setminus E_{i,j}) = 1$. Thus we have that $J \sqsubseteq T(J)$. By Theorem , T preserves zero-term rank. □

Theorem 3.6. *Let $m \geq 3$ and $1 \leq l \leq m$. Suppose that $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is a linear operator. Then T preserves zero-term rank if and only if T strongly preserves zero-term rank l.*

Proof. For $1 \leq l \leq m$, assume that T strongly preserves zero-term rank l. If $l = 1$, then T preserves zero-term rank by Lemma 3.5. So we assume that $l \geq 2$. Then T is nonsingular by Lemma 3.2(ii) and hence $\sharp(T(E_{i,j})) \geq 1$ for all cells $E_{i,j}$.

First, suppose that $\sharp(T(E)) \geq 2$ for some cell E. By permuting we may assume that $T(E) \sqsupseteq E + F$ for some cell $F \neq E$. If E and F are in the same row, we may assume by permuting that $E = E_{2,l}$ and $F = E_{2,l-1}$. If E and F are in the same column, we may assume by permuting that $E = E_{l,2}$ and $F = E_{l-1,2}$. If E and F are in different rows and different columns, we may assume by permuting that $E = E_{3,l}$ and $F = E_{2,l-1}$. Then we have that $E \sqsubseteq \mathbb{D}_l$ and $F \not\sqsubseteq \mathbb{D}_l$. It follows that $z(\mathbb{D}_l + E) = l$ and $z(\mathbb{D}_l + F) = l - 1$.

Let $L = T^d$ where d is chosen so that $L : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ is an idempotent operator. Then we can easily check that L strongly preserves zero-term ranks l and $L(E) \sqsupseteq E + F$. By the similar argument in the proof of Lemma 3.3, we have that $L(\mathbb{D}_l) = L(\mathbb{D}_l + F)$, a contradiction to the fact that T strongly preserves zero-term rank l. Hence we have established that $\sharp(T(E_{i,j})) = 1$ for all cells $E_{i,j}$.

Next, suppose that T is not invertible. Then $T(E) = T(F)$ for some distinct cells E and F by Lemma 2.3. By the similar argument in the proof of Theorem , we have that $T(\mathbb{D}_l + E) = T(\mathbb{D}_l + F)$ with $z(\mathbb{D}_l + E) = l$ and $z(\mathbb{D}_l + F) = l - 1$, a contradiction to the fact that T strongly preserves zero-term rank l. Thus T must be invertible. By Theorem 2.5, T is a (P, Q)-operator and hence T preserves zero-term rank by Theorem 1.1.

The converse is obvious. □

If we combine three Theorems 1.1, 3.4 and 3.6, we obtain that:

Theorem 3.7. *Let $m \geq 3$. For a linear operator $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$, the following are equivalent*:

(i) *T preserves zero-term rank*;

(ii) *T preserves zero-term ranks k and $k+1$, where $0 \leq k \leq m-1$*;

(iii) *T strongly preserves zero-term rank l, where $1 \leq l \leq m$*;

(iv) *T is a (P, Q)-operator.*

We remarks that the condition $m \geq 3$ is essential in the Theorem 3.7. For $m = 2$, the conditions (ii) or (iii) in the Theorem 3.7 may not imply the condition (i). Consider the cases of $k = m - 1 = 1$ and $l = m = 2$, respectively. Let $\mathcal{M}(\mathbb{B})$ be the set of all 2×2 Boolean matrices. Define $T : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ by

$$T(E_{1,1}) = T(E_{2,2}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad T(E_{1,2}) = T(E_{2,1}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

equivalently, $T\left(\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}\right) = \begin{bmatrix} x_{1,1} + x_{2,2} & x_{1,2} + x_{2,1} \\ x_{1,2} + x_{2,1} & 0 \end{bmatrix}$ for all $X \in \mathcal{M}(\mathbb{B})$. Then we can easily check that T is a nonsingular linear operator such that

(1) T preserves zero-term ranks 1 and 2, and

(2) T strongly preserves zero-term rank 2.

But it follows from $J \not\sqsubseteq T(J)$ that T does not preserve zero-term rank 0. Hence T does not preserve zero-term rank.

4. Zero-Term Rank Preservers of Matrices over Antinegative Semirings

Throughout this section, S denotes any commutative and antinegative semiring. In this section we provide characterizations of linear operators $T : \mathcal{M}(S) \to \mathcal{M}(S)$ that preserve zero-term rank.

The *pattern*, $\overline{A}$, of a matrix A in $\mathcal{M}(S)$ is the Boolean matrix in $\mathcal{M}(\mathbb{B})$ whose (i, j)th entry is 0 if and only if $a_{i,j} = 0$. Notice that $\overline{A} \sqsubseteq \overline{B}$ if and only if $A \sqsubseteq B$ for all A and B in $\mathcal{M}(S)$. It follows that $z(A) = z(\overline{A})$ for all $A \in \mathcal{M}(S)$. Thus, the zero-term rank of $A \in \mathcal{M}(S)$ depends only on its pattern $\overline{A}$.

For a linear operator $T : \mathcal{M}(S) \to \mathcal{M}(S)$, define $\overline{T} : \mathcal{M}(\mathbb{B}) \to \mathcal{M}(\mathbb{B})$ by $\overline{T}(E_{i,j}) = \overline{T(E_{i,j})}$ for all cells $E_{i,j}$. Then $\overline{T}$ is a linear operator on $\mathcal{M}(\mathbb{B})$.

Lemma 4.1. Let $0 \leq k \leq m$. Suppose that $T : \mathcal{M}(S) \to \mathcal{M}(S)$ is a linear operator. Then T preserves zero-term rank k on $\mathcal{M}(S)$ if and only if $\overline{T}$ preserves zero-term rank k on $\mathcal{M}(\mathbb{B})$.

Proof. Since $z(A) = z(\overline{A})$ and $\overline{T}(\overline{A}) = \overline{T(A)}$ for all $A \in \mathcal{M}(S)$, we have $z(T(A)) = z(\overline{T}(\overline{A}))$, and hence the result follows. □

Theorem 4.2. *Let $m \geq 3$. For a linear operator $T : \mathcal{M}(S) \to \mathcal{M}(S)$, the following are equivalent*:

(i) *T preserves zero-term rank*;

(ii) *T preserves zero-term ranks k and $k+1$, where $0 \leq k \leq m-1$*;

(iii) *T strongly preserves zero-term rank l, where $1 \leq l \leq m$*;

(iv) *T is a (P, Q, B)-operator.*

Proof. The result follows from Lemma 4.1 and Theorems 1.1 and 3.7. □

As a concluding remark, we suggest to prove the following conjecture:

Conjecture. Let $T : \mathcal{M}(S) \to \mathcal{M}(S)$ be a linear operator. Then T preserves zero-term rank if and only if T preserves any two zero-term ranks h and k with $1 \leq h < k \leq m \leq n$.

References

[1] L. B. Beasley and N. J. Pullman, *Boolean rank preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl., **59**(1984), 55-77.

[2] L. B. Beasley and N. J. Pullman, *Semiring rank versus column rank*, Linear Algebra Appl., **101**(1988), 33-48.

[3] L. B. Beasley and N. J. Pullman, *Linear operators that preserve term rank-1*, Proc. Roy. Irish Acad., **91**(1990), 71-78.

[4] L. B. Beasley, S. Z. Song and S. G. Lee, *Linear operators that preserve zero-term rank of Boolean matrices*, J. Korean Math. Soc., **36(6)**(1999), 1181-1190.

[5] L. B. Beasley, S. Z. Song and S. G. Lee, *Zero-term rank preservers*, Linear and Multilinear Algebra, **48**(2001), 313-318.

[6] K. T. Kang, S. Z. Song and L. B. Beasley, *Linear preservers of term ranks of matrices over semirings*, Linear Algebra Appl., **436**(2012), 1850-1862.