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The $*$-Nagata Ring of almost Prüfer $*$-multiplication Domains

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Abstract. Let D be an integral domain with quotient field K, $\overline{D}$ denote the integral closure of D in K and $*$ be a star-operation on D. In this paper, we study the $*$-Nagata ring of AP$*$MDs. More precisely, we show that D is an AP$*$MD and $D[X] \subseteq \overline{D}[X]$ is a root extension if and only if the $*$-Nagata ring $D[X]_{N_*}$ is an AB-domain, if and only if $D[X]_{N_*}$ is an AP-domain. We also prove that D is a P$*$MD if and only if D is an integrally closed AP$*$MD, if and only if D is a root closed AP$*$MD.

1. Introduction

For the sake of clarity, we first review some definitions and notation. Let D be an integral domain with quotient field K and $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. A *star-operation* on D is a mapping $I \mapsto I_*$ from $\mathbf{F}(D)$ into itself which satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathbf{F}(D)$:

(1) $(aD)_* = aD$ and $(aI)_* = aI_*$;

(2) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$; and

(3) $(I_*)_* = I_*$.

An $I \in \mathbf{F}(D)$ is said to be a *$*$-ideal* if $I = I_*$. A $*$-ideal of D is called a *maximal $*$-ideal* of D if it is maximal among proper integral $*$-ideals of D. Given any star-operation $*$ on D, we can construct a new star-operation $*_f$ as follows: For all $I \in \mathbf{F}(D)$, the *$*_f$-operation* is defined by $I_{*_f} = \bigcup\{J_* \mid J$ is a nonzero finitely generated fractional subideal of $I\}$. A star-operation $*$ on D is said to be of *finite character* (or *finite type*) if $I_* = I_{*_f}$ for each $I \in \mathbf{F}(D)$. It is easy to see that the $*_f$-operation is of finite character. Let $*'$ be a finite character star-operation on D. It is well known that if D is not a field, then each proper integral $*'$-ideal of D is

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contained in a maximal $*'$-ideal of D, and hence a maximal $*'$-ideal of D always exists. An $I \in \mathbf{F}(D)$ is said to be *$*_f$-invertible* if $(II^{-1})_{*_f} = D$, or equivalently, $II^{-1} \nsubseteq M$ for any maximal $*_f$-ideal M of D. If $*_1$ and $*_2$ are star-operations on D, then we mean by $*_1 \leq *_2$ that $I_{*_1} \subseteq I_{*_2}$ for all $I \in \mathbf{F}(D)$. Clearly, if $*_1$ and $*_2$ are star-operations of finite character with $*_1 \leq *_2$, then a $*_1$-invertible ideal is $*_2$-invertible.

The simplest example of a star-operation is the d-operation. Other well-known examples are the v- and t-operations. The *d-operation* is just the identity map on $\mathbf{F}(D)$, *i.e.*, $I_d = I$ for all $I \in \mathbf{F}(D)$. The *v-operation* is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} := \{a \in K \mid aI \subseteq D\}$, and the *$t$-operation* is defined by $I_t = \bigcup\{J_v \mid J$ is a nonzero finitely generated fractional subideal of $I\}$, *i.e.*, $t = v_f$. Clearly, if an $I \in \mathbf{F}(D)$ is finitely generated, then $I_v = I_t$. It is also well known that $d \leq * \leq v$ for all star-operations $*$. For more on star-operations, the readers can refer to [8, Section 32].

Let $\mathrm{T}(D)$ be the abelian group of t-invertible fractional t-ideals of an integral domain D under the t-multiplication $I * J = (IJ)_t$, and let $\mathrm{Prin}(D)$ be the subgroup of $\mathrm{T}(D)$ of principal fractional ideals of D. Then the *t-class group* of D is the quotient group $\mathrm{Cl}_t(D) := \mathrm{T}(D)/\mathrm{Prin}(D)$. Let $\mathrm{Inv}(D)$ be the group of invertible fractional ideals of D. Clearly, $\mathrm{Inv}(D)$ is a subgroup of $\mathrm{T}(D)$ containing $\mathrm{Prin}(D)$. The *Picard group* is the group $\mathrm{Pic}(D) := \mathrm{Inv}(D)/\mathrm{Prin}(D)$, and $\mathrm{Pic}(D)$ is obviously a subgroup of $\mathrm{Cl}_t(D)$.

Let D be an integral domain with quotient field K, and $\overline{D}$ be the integral closure of D in K. In [1, Definition 4.1], Anderson and Zafrullah first introduced the notions of almost Prüfer domains and almost Bézout domains. They defined D to be an *almost Prüfer domain* (AP-domain) (respectively, *almost Bézout domain* (AB-domain)) if for any $0 \neq a, b \in D$, there exists a positive integer $n = n(a, b)$ such that (a^n, b^n) is invertible (respectively, principal). It was shown that D is an AP-domain with torsion (t-)class group if and only if D is an AB-domain [1, Lemma 4.4]; and D is an AP-domain (respectively, AB-domain) if and only if $\overline{D}$ is a Prüfer domain (respectively, Prüfer domain with torsion Picard group) and $D \subseteq \overline{D}$ is a root extension [1, Corollary 4.8]. In [1, Definition 5.1], the authors also defined D to be an *almost valuation domain* (AV-domain) if for any $0 \neq a, b \in D$, there exists an integer $n = n(a, b) \geq 1$ such that $a^n \mid b^n$ or $b^n \mid a^n$. Later, Li gave the notion of almost Prüfer v-multiplication domains which is the t-operation analogue of AP-domains. She defined D to be an *almost Prüfer v-multiplication domain* (APvMD) if for any $0 \neq a, b \in D$, there exists a positive integer $n = n(a, b)$ such that (a^n, b^n) is t-invertible. It was shown in [1, Theorem 5.8] (respectively, [14, Theorem 2.3]) that D is an AP-domain (respectively, APvMD) if and only if D_M is an AV-domain for all maximal ideals (respectively, maximal t-ideals) M of D. Following [13, Definition 2.1], D is an *almost Prüfer $*$-multiplication domain* (AP$*$MD) if for each $0 \neq a, b \in D$, there exists an integer $n = n(a, b) \geq 1$ such that (a^n, b^n) is $*_f$-invertible, where $*$ is a star-operation on D. It was shown in [13, Theorem 2.4] that D is an AP$*$MD if and only if D_M is an AV-domain for all maximal $*_f$-ideals M of D. Also, it is clear that if $*_1$ and $*_2$ are star-operations with $*_1 \leq *_2$, then an AP$*_1$MD is an AP$*_2$MD; so for any star-operation $*$, an AP-domain is an AP$*$MD, and an AP$*$MD is an APvMD.

In this paper, we study the $*$-Nagata ring of AP$*$MDs, where $*$ is a star-operation. More precisely, we show that D is an AP$*$MD and $D[X] \subseteq \overline{D}[X]$ is a root extension if and only if the $*$-Nagata ring $D[X]_{N_*}$ is an AB-domain, if and only if $D[X]_{N_*}$ is an AP-domain. We also prove that D is a P$*$MD if and only if D is an integrally closed AP$*$MD, if and only if D is a root closed AP$*$MD. (Preliminaries related to P$*$MDs will be reviewed before Lemma 5.) As a corollary, we recover a well-known fact that D is a P$*$MD if and only if $D[X]_{N_*}$ is a Bézout domain, if and only if $D[X]_{N_*}$ is a Prüfer domain.

2. Main Results

Throughout this section, D always denotes an integral domain with quotient field K, $\overline{D}$ is the integral closure of D in K and $D[X]$ means the polynomial ring over D. For a polynomial $g \in D[X]$, $c(g)$ stands for the *content ideal* of D, *i.e.*, the ideal of D generated by the coefficients of g. Let $*$ be a star-operation on D and set $N_* := \{g \in D[X] \mid c(g)_* = D\}$. If we need to make the integral domain D explicit, then we use $N_*(D)$ instead of N_*. Clearly, $N_* = N_{*_f}$. Also, note that $N_* = D[X] \setminus \bigcup MD[X]$, where M runs over all maximal $*_f$-ideals of D [11, Proposition 2.1(1)]; so N_* is a saturated multiplicative subset of $D[X]$. We call the quotient ring $D[X]_{N_*}$ the $*$-*Nagata ring* of D. Recently, the authors in [3] studied the t-Nagata ring of APvMDs. In fact, they showed that D is an APvMD and $D[X] \subseteq \overline{D}[X]$ is a root extension if and only if $D[X]_{N_v}$ is an AP-domain, if and only if $D[X]_{N_v}$ is an AB-domain [3, Theorem 2.5]. (Recall that an extension $R \subseteq T$ of integral domains is a *root extension* if for each $z \in T$, $z^n \in R$ for some integer $n = n(z) \geq 1$.)

In order to study the $*$-Nagata ring of AP$*$MDs, we need the following lemma.

Lemma 1. The following assertions hold.

(1) If D is an AV-domain and F is a subfield of K, then $D \cap F$ is an AV-domain.

(2) Let $*$ be a star-operation on D. Then D is an AP$*$MD if and only if D_M is an AV-domain for all maximal $*_f$-ideals M of D.

Proof. (1) Let $0 \neq x \in F$. Then $x = \frac{b}{a}$ for some $0 \neq a, b \in D$. Since D is an AV-domain, we can find a suitable integer $n = n(a, b) \geq 1$ such that $a^n \mid b^n$ or $b^n \mid a^n$; so $x^n \in D$ or $x^{-n} \in D$. Hence $x^n \in D \cap F$ or $x^{-n} \in D \cap F$. Thus $D \cap F$ is an AV-domain.

(2) This appears in [13, Theorem 2.4]. □

Recall that D is *root closed* if for $a \in K$, $a^n \in D$ for some positive integer n implies that $a \in D$.

Lemma 2. Let S be a (not necessarily saturated) multiplicative subset of D. Then the following assertions hold.

(1) If $D \subseteq \overline{D}$ is a root extension, then $D_S \subseteq \overline{D}_S$ is a root extension.

(2) If D is root closed, then D_S is root closed.

Proof. (1) Let $\frac{e}{s} \in \overline{D}_S$, where $e \in \overline{D}$ and $s \in S$. Since $D \subseteq \overline{D}$ is a root extension, $e^n \in D$ for some integer $n = n(e) \geq 1$; so $(\frac{e}{s})^n \in D_S$. Thus $D_S \subseteq \overline{D}_S$ is a root extension.

(2) Let $a \in K$ such that $a^n \in D_S$ for some integer $n \geq 1$. Then $sa^n \in D$ for some $s \in S$; so $(sa)^n \in D$. Since D is root closed, $sa \in D$, and hence $a \in D_S$. Thus D_S is root closed. $\square$

Now, we give the main result in this article.

Theorem 3. *Let $*$ be a star-operation on D and let $N_* := \{g \in D[X] \mid c(g)_* = D\}$. Then the following statements are equivalent.*

(1) *D is an AP$*$MD and $D[X] \subseteq \overline{D}[X]$ is a root extension.*

(2) *$D[X]_{N_*}$ is an AB-domain.*

(3) *$D[X]_{N_*}$ is an AP-domain.*

Proof. (1) $\Rightarrow$ (2) Assume that D is an AP$*$MD, and let Q be a maximal ideal of $D[X]_{N_*}$. Then $Q = MD[X]_{N_*}$ for some maximal $*_f$-ideal M of D [11, Proposition 2.1(2)]. Note that D_M is an AV-domain by Lemma 1(2); so D_M is an APvMD and MD_M is a maximal t-ideal of D_M [1, Proof of Theorem 5.6]. Also, note that $\overline{D_M[X]} = \overline{D}_{D\setminus M}[X]$ (cf. [7, Theorem 12.10(2)]); so by Lemma 2(1), $D_M[X] \subseteq \overline{D_M[X]}$ is a root extension, because $D[X] \subseteq \overline{D}[X]$ is a root extension. Therefore $D_M[X]$ is an APvMD [14, Theorem 3.13]. Since $MD_M[X]$ is a maximal t-ideal of $D_M[X]$ [5, Lemma 2.1(4)], $D_M[X]_{MD_M[X]}$ is an AV-domain by Lemma 1(2). Note that $(D[X]_{N_*})_Q = (D[X]_{N_*})_{MD[X]_{N_*}} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$ [2, Lemma 2]; so $(D[X]_{N_*})_Q$ is an AV-domain. Hence $D[X]_{N_*}$ is an AP-domain by Lemma 1(2). Note that $\mathrm{Pic}(D[X]_{N_*}) = 0$ [11, Theorem 2.14]. Thus $D[X]_{N_*}$ is an AB-domain [1, Lemma 4.4].

(2) $\Rightarrow$ (3) This implication is obvious.

(3) $\Rightarrow$ (1) Let M be a maximal $*_f$-ideal of D. Then $MD[X]_{N_*}$ is a maximal ideal of $D[X]_{N_*}$ [11, Proposition 2.1(2)]. Since $D[X]_{N_*}$ is an AP-domain, $(D[X]_{N_*})_{MD[X]_{N_*}}$ is an AV-domain by Lemma 1(2). Note that $(D[X]_{N_*})_{MD[X]_{N_*}} = D[X]_{MD[X]} = D_M[X]_{N_d(D_M)}$ [2, Lemma 2]; so $D_M[X]_{N_d(D_M)}$ is an AV-domain. Since $D_M = D_M[X]_{N_d(D_M)} \cap K$ [11, Proposition 2.8(1)], D_M is an AV-domain by Lemma 1(1). Thus by Lemma 1(2), D is an AP$*$MD.

Let $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then $N_* \subseteq N_v$; so $D[X]_{N_v} = (D[X]_{N_*})_{N_v}$. Since $D[X]_{N_*}$ is an AP-domain, $D[X]_{N_*}$ is an APvMD; so $D[X]_{N_v}$ is also an APvMD [3, Lemma 2.4]. Thus $D[X] \subseteq \overline{D}[X]$ is a root extension [3, Theorem 2.5]. $\square$

By applying $* = d$ to Theorem 3, we obtain

Corollary 4. The following assertions are equivalent.

(1) D is an AP-domain and $D[X] \subseteq \overline{D}[X]$ is a root extension.

(2) $D[X]_{N_d}$ is an AB-domain.

(3) $D[X]_{N_d}$ is an AP-domain.

Let $*$ be a star-operation on D. Recall that D is a *Prüfer $*$-multiplication domain* (P∗MD) if every nonzero finitely generated ideal of D is $*_f$-invertible, or equivalently, D_M is a valuation domain for all maximal $*_f$-ideals M of D [10, Theorem 1.1]. When $* = d$ or t, it was shown in [1, Theorem 4.7] (respectively, [14, Theorem 2.4]) that D is a Prüfer domain (respectively, PvMD) if and only if D is an integrally closed AP-domain (respectively, APvMD), if and only if D is a root closed AP-domain (respectively, APvMD). We next extend these results to P∗MDs for any star-operation $*$.

Lemma 5. Let $*$ be a star-operation on D. Then the following assertions are equivalent.

(1) D is a P∗MD.

(2) D is an integrally closed AP∗MD.

(3) D is a root closed AP∗MD.

Proof. (1) ⇒ (2) Clearly, a P∗MD is an AP∗MD. Thus this implication follows directly from a well-known fact that a P∗MD is integrally closed [10, Theorem 1.1].

(2) ⇒ (3) It suffices to note that an integrally closed domain is always root closed.

(3) ⇒ (1) Assume that D is a root closed AP∗MD, and let M be a maximal $*_f$-ideal of D. Then D_M is an AV-domain by Lemma 1(2). Let a and b be nonzero elements of D_M. Then there exists a positive integer $n = n(a, b)$ such that $a^n \mid b^n$ or $b^n \mid a^n$. Hence $(\frac{b}{a})^n \in D_M$ or $(\frac{a}{b})^n \in D_M$. Note that D_M is root closed by Lemma 2(2); so $\frac{b}{a} \in D_M$ or $\frac{a}{b} \in D_M$, which indicates that D_M is a valuation domain. Thus D is a P∗MD [10, Theorem 1.1]. □

Recall that D is a *Bézout domain* if every finitely generated ideal of D is principal. It is well known that D is a Bézout domain if and only if D is a Prüfer domain with trivial Picard group.

Corollary 6. ([6, Theorem 3.1]) Let $*$ be a star-operation on D. Then the following statements are equivalent.

(1) D is a P∗MD.

(2) $D[X]_{N_*}$ is a Bézout domain.

(3) $D[X]_{N_*}$ is a Prüfer domain.

Proof. Note that by suitable combinations of [12, Theorems 51 and 52], [7, Corollary 12.11(2)] and [11, Proposition 2.8(1)]), it is easy to see that D is integrally closed if and only if $D[X]_{N_*}$ is integrally closed.

(1) $\Rightarrow$ (2) If D is a P$*$MD, then by Lemma 5, D is an integrally closed AP$*$MD; so $D[X]_{N_*}$ is an integrally closed AP-domain by Theorem 3. Hence $D[X]_{N_*}$ is a Prüfer domain [1, Theorem 4.7] (or Lemma 5). Note that $\mathrm{Pic}(D[X]_{N_*}) = 0$ [11, Theorem 2.14]. Thus $D[X]_{N_*}$ is a Bézout domain.

(2) $\Rightarrow$ (3) This implication is clear.

(3) $\Rightarrow$ (1) Assume that $D[X]_{N_*}$ is a Prüfer domain. Then $D[X]_{N_*}$ is an integrally closed AP-domain [1, Theorem 4.7] (or Lemma 5). Hence D is an integrally closed AP$*$MD by Theorem 3. Thus the result follows from Lemma 5. $\square$

A particular case of Corollary 6 is when $* = d$ or t.

Corollary 7. ([2, Theorem 4] (respectively, [11, Theorem 3.7])) The following assertions are equivalent.

(1) D is a Prüfer domain (respectively, PvMD).

(2) $D[X]_{N_d}$ (respectively, $D[X]_{N_v}$) is a Bézout domain.

(3) $D[X]_{N_d}$ (respectively, $D[X]_{N_v}$) is a Prüfer domain.

Let $*$ be a star-operation on D. Note that the $*$-Nagata ring $D[X]_{N_*}$ is a quotient ring of the polynomial ring $D[X]$. We end this article by mentioning a remark for the polynomial extensions of AP$*$MDs.

Remark 8. (1) Let $*$ be a star-operation on $D[X]$. Then the mapping $\overline{*} : \mathbf{F}(D) \to \mathbf{F}(D)$ defined by $I_{\overline{*}} = (ID[X])_* \cap D$ for all $I \in \mathbf{F}(D)$ is a star-operation on D [15, Proposition 2.1]. It is well known that if $*$ denotes the d-operation (respectively, t-operation, v-operation) on $D[X]$, then $\overline{*}$ is the d-operation (respectively, t-operation, v-operation) on D [15, Remark 2.2] (or [9, Proposition 4.3]).

(2) If D is an APvMD and $D[X] \subseteq \overline{D}[X]$ is a root extension, then $D[X]$ is also an APvMD [14, Theorem 3.13]. (Note that the condition "$D[X] \subseteq \overline{D}[X]$ is a root extension" is essential [14, Remark 3.12(3)].)

(3) Let $*$ and $\overline{*}$ be star-operations as in (1). By (2), it might be natural to ask whether AP$\overline{*}$MD properties of the base ring can be ascended to AP$*$MD properties of the polynomial extension (under some assumptions if needed), *i.e.*, if D is an AP$\overline{*}$MD with some additional conditions, then $D[X]$ is an AP$*$MD. However the answer is not generally affirmative. For example, the polynomial ring over an AP-domain is not generally an AP-domain. In fact, $D[X]$ is an AP-domain if and only if D is a field (cf. [4, Theorem 2.15]).

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