KYUNGPOOK Math. J. 54(2014), 531-543
http://dx.doi.org/10.5666/KMJ.2014.54.4.531

Some Properties Subclasses of Analytic Functions

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Abstract. The object of the present paper is to discuss some interesting properties of analytic functions $f(z)$ associated with the subclasses $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, $\mathcal{G}(\theta, \alpha)$ and $\mathcal{Q}(\theta, \alpha)$. Also, radius problems of $\frac{1}{\delta} f(\delta z)$ for $f(z)$ in the class $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, $\mathcal{G}(\theta, \alpha)$ and $\mathcal{Q}(\theta, \alpha)$ are considered.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a function $f(z)$ in the class $\mathcal{A}$, Sãlãgean [1] defined the differential operator D^k, by

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^2 f(z) &= D(D^1 f(z) = z(zf'(z))', \end{aligned}$$

and

$$D^k f(z) = D(D^{k-1} f(z)), \quad (k \in \mathbb{N}).$$

Thus

$$D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \quad (k \in \mathbb{N} \cup \{0\}). \tag{2}$$

Received October 13, 2011; accepted March 27, 2013.
2010 Mathematics Subject Classification: 30C45.
Key words and phrases: Analytic, univalent functions, Cauchy-Schwarz inequality, radius problems.

Let $\mathcal{G}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\frac{\mathrm{D^3f(z)}}{\mathrm{z}}\right\} > \alpha \tag{3}$$

for some $\alpha(0 \le \alpha < 1)$ and for all $z \in \mathcal{U}$. Also, Let $\mathcal{Q}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\frac{\mathrm{D^2f(z)}}{\mathrm{D^1f(z)}}\right\} > \alpha$$

for some $\alpha(0 \le \alpha < 1)$ and for all $z \in \mathcal{U}$.

For analytic functions $f(z)$, Uyanik and Owa [2], obtained some interesting properties for analytic functions in the subclass $\mathcal{A}(\beta_1, \beta_2, \beta_3; \lambda)$ defined by

$$\left|\beta_1 z\left(\frac{f(z)}{z}\right)' + \beta_2 z^2\left(\frac{f(z)}{z}\right)'' + \beta_3 z^3\left(\frac{f(z)}{z}\right)'''\right| \le \lambda$$

$$(\beta_1, \beta_2, \beta_3 \in \mathbb{C}; \lambda > 0;\ z \in \mathcal{U}),$$

associated with close-to-convex functions and starlike functions of order α.

In this paper, we define the following subclass of analytic functions.

Definition 1.1. A function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ if it satisfies

$$\left|\beta_1 z\left(\frac{D^2f(z)}{z}\right)' + \beta_2 z^2\left(\frac{D^2f(z)}{z}\right)'' + \beta_3 z^3\left(\frac{D^2f(z)}{z}\right)'''\right| \le \lambda \qquad (z \in \mathcal{U}), \tag{4}$$

for some complex numbers $\beta_1, \beta_2, \beta_3$, and for some real $\lambda > 0$.

Example 1.2. Let us consider the function $f_\gamma(z)$, $\gamma \in \mathbb{R}$, given by

$$f_\gamma(z) = z(1+z)^\gamma,$$

then, we have

$$D^2 f_\gamma(z) = z + \sum_{n=2}^{\infty} n^2 \binom{\gamma}{n-1} z^n \tag{5}$$

where

$$\binom{\gamma}{n-1} = \frac{\gamma(\gamma-1)(\gamma-2)...(\gamma-n+2)}{(n-1)!}.$$

From (5), it follows that

$$\begin{aligned}&\left|\beta_1 z\left(\frac{D^2 f_\gamma(z)}{z}\right)' + \beta_2 z^2\left(\frac{D^2 f_\gamma(z)}{z}\right)'' + \beta_3 z^3\left(\frac{D^2 f_\gamma(z)}{z}\right)'''\right| \\ =& \left|\sum_{n=2}^{\infty} n^2(n-1)\binom{\gamma}{n-1}(\beta_1 + (n-2)\beta_2 + (n-2)(n-3)\beta_3)z^{n-1}\right|.\end{aligned}$$

Therefore, if $\gamma = 1$, then

$$\left|\beta_1 z\left(\frac{D^2 f_1(z)}{z}\right)' + \beta_2 z^2\left(\frac{D^2 f_1(z)}{z}\right)'' + \beta_3 z^3\left(\frac{D^2 f_1(z)}{z}\right)'''\right| = |4\beta_1 z| \leq 4\,|\beta_1|\,.$$

This implies that $f_1(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 4\,|\beta_1|$. If $\gamma = 2$, then

$$\begin{aligned}&\left|\beta_1 z\left(\frac{D^2 f_2(z)}{z}\right)' + \beta_2 z^2\left(\frac{D^2 f_2(z)}{z}\right)'' + \beta_3 z^3\left(\frac{D^2 f_2(z)}{z}\right)'''\right| \\ =& \left|8\beta_1 z + 18(\beta_1 + \beta_2)z^2\right| \leq 26\,|\beta_1| + 18\,|\beta_2|\,.\end{aligned}$$

Therefore, $f_2(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 26\,|\beta_1| + 18\,|\beta_2|$. Further, if $\gamma = 3$, then we have

$$\begin{aligned}&\left|\beta_1 z\left(\frac{D^2 f_3(z)}{z}\right)' + \beta_2 z^2\left(\frac{D^2 f_3(z)}{z}\right)'' + \beta_3 z^3\left(\frac{D^2 f_3(z)}{z}\right)'''\right| \\ =& \left|12\beta_1 z + 54(\beta_1 + \beta_2)z^2 + 48(\beta_1 + 2\beta_2 + 2\beta_3)z^3\right| \\ \leq& \ 114\,|\beta_1| + 150\,|\beta_2| + 96\,|\beta_3|\,.\end{aligned}$$

Thus, $f_3(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ for $\lambda \geq 114\,|\beta_1| + 150\,|\beta_2| + 96\,|\beta_3|$.

Now, let $\mathcal{A}_\theta$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ with

$$a_n = |a_n|\, e^{i((n-1)\theta+\pi)} \qquad (n = 2, 3, \ldots).$$

Also, we introduce the subclasses $\mathcal{G}(\theta, \alpha)$ and $\mathcal{Q}(\theta, \alpha)$ of $\mathcal{A}_\theta$ as follows:

$$\mathcal{G}(\theta, \alpha) = \mathcal{A}_\theta \cap \mathcal{G}(\alpha) \quad \text{and } \mathcal{Q}(\theta, \alpha) = \mathcal{A}_\theta \cap \mathcal{Q}(\alpha).$$

2. Properties of the Class $\mathcal{D}(\beta_1,\beta_2,\beta_3;\lambda)$

We first prove

Theorem 2.1. *If $f(z)\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|a_n|\le\lambda \tag{6}$$

for some complex numbers β_1,β_2,β_3 and for some real $\lambda>0$, then $f(z)\in\mathcal{D}(\beta_1,\beta_2,\beta_3;\lambda)$.

Proof. We observe that

$$\begin{aligned}
&\left|\beta_1 z\left(\frac{D^2f(z)}{z}\right)'+\beta_2 z^2\left(\frac{D^2f(z)}{z}\right)''+\beta_3 z^3\left(\frac{D^2f(z)}{z}\right)'''\right|\\
=&\left|\sum_{n=2}^{\infty} n^2(n-1)(\beta_1+(n-2)\beta_2+(n-2)(n-3)\beta_3)a_n z^{n-1}\right|\\
\le&\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|a_n||z|^{n-1}\\
<&\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|a_n|.
\end{aligned}$$

Therefore, if $f(z)$ satisfies the inequality (6), then $f(z)\in\mathcal{D}(\beta_1,\beta_2,\beta_3;\lambda)$. □

Next, we prove

Theorem 2.2. *If $f(z)\in\mathcal{D}(\beta_1,\beta_2,\beta_3;\lambda)$ with $\arg\beta_1=\arg\beta_2=\arg\beta_3=\phi$ and $a_n=|a_n|e^{i((n-1)\theta-\phi)}$ $(n=2,3,...)$, then we have*

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|a_n|\le\lambda.$$

Proof. For $f(z)\in\mathcal{D}(\beta_1,\beta_2,\beta_3;\lambda)$, we see that

$$\begin{aligned}
&\left|\beta_1 z\left(\frac{D^2f(z)}{z}\right)'+\beta_2 z^2\left(\frac{D^2f(z)}{z}\right)''+\beta_3 z^3\left(\frac{D^2f(z)}{z}\right)'''\right|\\
=&\left|\sum_{n=2}^{\infty} n^2(n-1)(\beta_1+(n-2)\beta_2+(n-2)(n-3)\beta_3)a_n z^{n-1}\right|\\
=&\left|\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1|+(n-2)|\beta_2|+(n-2)(n-3)|\beta_3|)|a_n|e^{i(n-1)\theta}z^{n-1}\right|\\
\le&\ \lambda.
\end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z| e^{-i\theta}$. Then we have

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n||z|^{n-1} \leq \lambda.$$

Letting $|z| \to 1^-$, we obtain

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \leq \lambda.$$

□

Corollary 2.3. If $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \phi$ and $a_n = |a_n| e^{i((n-1)\theta-\phi)}$ $(n = 2, 3, ...)$, then we have

$$|a_n| \leq \frac{\lambda}{n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)} \qquad (n = 2, 3, ...).$$

Example 2.4. Let us consider the function $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \phi$ and

$$a_n = \frac{\lambda e^{i((n-1)\theta-\phi)}}{n^3(n-1)^2(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)} \qquad (n = 2, 3...).$$

Then we see that

$$\begin{aligned} & \sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n| \\ = \quad & \lambda \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lambda \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \lambda. \end{aligned}$$

Corollary 2.5. If $f(z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ with $\arg\beta_1 = \arg\beta_2 = \arg\beta_3 = \phi$ and $a_n = |a_n| e^{i((n-1)\theta-\phi)}$ $(n = 2, 3, ...)$, then we have

$$|z| - \sum_{n=2}^{j} |a_n||z|^n - A_j |z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n||z|^n + A_j |z|^{j+1}$$

with

$$A_j = \frac{\left(\lambda - \sum_{n=2}^{j} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|a_n|\right)}{j(j+1)^2(|\beta_1| + (j-1)|\beta_2| + (j-1)(j-2)|\beta_3|)}$$

and

$$1-\sum_{n=2}^{j}|a_n|\,|z|^{n-1}-B_j\,|z|^j\le |f'(z)|\le 1+\sum_{n=2}^{j}|a_n|\,|z|^{n-1}+B_j\,|z|^j$$

with

$$B_j=\frac{\left(\lambda-\sum\limits_{n=2}^{j}n^2(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\right)}{j(j+1)(|\beta_1|+(j-1)\,|\beta_2|+(j-1)(j-2)\,|\beta_3|)}.$$

Proof. In view of Theorem 2.1, we know that

$$\begin{aligned}&\sum_{n=j+1}^{\infty}n^2(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\\ \le\;&\lambda-\sum_{n=2}^{j}n^2(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\,.\end{aligned}$$

Further, we note that

$$\begin{aligned}&j(j+1)^2(|\beta_1|+(j-1)\,|\beta_2|+(j-1)(j-2)\,|\beta_3|)\sum_{n=j+1}^{\infty}|a_n|\\ \le\;&\sum_{n=j+1}^{\infty}n^2(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\,,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\sum_{n=j+1}^{\infty}|a_n|\;&\le\;\frac{\left(\lambda-\sum\limits_{n=2}^{j}n^2(n-1)(|\beta_1|+(n-2)\,|\beta_2|+(n-2)(n-3)\,|\beta_3|)\,|a_n|\right)}{j(j+1)^2(|\beta_1|+(j-1)\,|\beta_2|+(j-1)(j-2)\,|\beta_3|)}\\ &=\;A_j.\end{aligned}$$

Thus, we have

$$|f(z)|\le |z|+\sum_{n=2}^{j}|a_n|\,|z|^n+\sum_{n=j+1}^{\infty}|a_n|\,|z|^n\le |z|+\sum_{n=2}^{j}|a_n|\,|z|^n+A_j\,|z|^{j+1}$$

and

$$|f(z)|\ge |z|-\sum_{n=2}^{j}|a_n|\,|z|^n-\sum_{n=j+1}^{\infty}|a_n|\,|z|^n\ge |z|-\sum_{n=2}^{j}|a_n|\,|z|^n-A_j\,|z|^{j+1}\,.$$

Next, we observe that

$$\begin{aligned} & j(j+1)(|\beta_1| + (j-1)\,|\beta_2| + (j-1)(j-2)\,|\beta_3|) \sum_{n=j+1}^{\infty} n\,|a_n| \\ \leq & \sum_{n=j+1}^{\infty} n^2(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n| \\ \leq & \ \lambda - \sum_{n=2}^{j} n^2(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|, \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{n=j+1}^{\infty} n\,|a_n| &\leq \frac{\left(\lambda - \sum_{n=2}^{j} n^2(n-1)(|\beta_1| + (n-2)\,|\beta_2| + (n-2)(n-3)\,|\beta_3|)\,|a_n|\right)}{j(j+1)(|\beta_1| + (j-1)\,|\beta_2| + (j-1)(j-2)\,|\beta_3|)} \\ &= B_j. \end{aligned}$$

Therefore, we obtain that

$$|f'(z)| \leq 1 + \sum_{n=2}^{j} n\,|a_n|\,|z|^{n-1} + \sum_{n=j+1}^{\infty} n\,|a_n|\,|z|^{n-1} \leq 1 + \sum_{n=2}^{j} |a_n|\,|z|^{n-1} + B_j\,|z|^j$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{j} n\,|a_n|\,|z|^{n-1} - \sum_{n=j+1}^{\infty} n\,|a_n|\,|z|^{n-1} \geq 1 - \sum_{n=2}^{j} |a_n|\,|z|^{n-1} - B_j\,|z|^j.$$

□

3. Radius Problem for the Class $\mathcal{G}(\theta, \alpha)$

To obtain the radius problem for the class $\mathcal{G}(\theta, \alpha)$, we need the following lemma.

Lemma 3.1. *If* $f(z) \in \mathcal{G}(\theta, \alpha)$, *then*

$$\sum_{n=2}^{\infty} n^3\,|a_n| \leq 1 - \alpha. \tag{7}$$

Proof. Let $f(z) \in \mathcal{G}(\theta, \alpha)$. Then, we have

$$\begin{aligned} \mathrm{Re}\left\{\frac{\mathrm{D^3f(z)}}{\mathrm{z}}\right\} &= \mathrm{Re}\left\{1 + \sum_{\mathrm{n=2}}^{\infty} \mathrm{n^3 a_n z^{n-1}}\right\} = \mathrm{Re}\left\{1 + \sum_{\mathrm{n=2}}^{\infty} \mathrm{n^3}\,|\mathrm{a_n}|\,\mathrm{e^{i((n-1)\theta+\pi)} z^{n-1}}\right\} \\ &= \mathrm{Re}\left\{1 - \sum_{\mathrm{n=2}}^{\infty} \mathrm{n^3}\,|\mathrm{a_n}|\,\mathrm{e^{i((n-1)\theta} z^{n-1}}\right\} > \alpha \end{aligned}$$

for all $z \in \mathcal{U}$. Let us consider a point $z \in \mathcal{U}$ such that $z = |z| e^{-i\theta}$. Then we have

$$1 - \sum_{n=2}^{\infty} n^3 |a_n| |z|^{n-1} > \alpha$$

Letting $|z| \to 1^-$, we obtain the inequality (7). $\square$

Corollary 3.2. If $f(z) \in \mathcal{G}(\theta, \alpha)$, then

$$|a_n| \leq \frac{1-\alpha}{n^3} \qquad (n = 2, 3, ...).$$

Remark 3.3. By Lemma 3.1, we observe that if $f(z) \in \mathcal{G}(\theta, \alpha)$, then

$$\sum_{n=2}^{\infty} n^2(n-1) |a_n| \leq \sum_{n=2}^{\infty} n^3 |a_n| \leq 1 - \alpha.$$

Applying Theorem 2.1 and Lemma 3.1, we derive

Theorem 3.4. *If* $f(z) \in \mathcal{G}(\theta, \alpha)$,*and* $\delta \in \mathbb{C}$ $(0 < |\delta| < 1)$. *Then the function* $\frac{1}{\delta} f(\delta z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$ *for* $(0 < |\delta| \leq |\delta_0(\lambda)|$, *where* $|\delta_0(\lambda)|$ *is the smallest positive root of the equation*

$$\begin{aligned} & |\beta_1| \frac{|\delta| \sqrt{2(|\delta|^2 + 2)(1-\alpha)}}{\left(1 - |\delta|^2\right)^2} \\ & + |\beta_2| \frac{|\delta|^2 \sqrt{6(3|\delta|^4 + 14|\delta|^2 + 3)(1 - \alpha - 4|a_2|^2)}}{(1-|\delta|^2)^3} \qquad (8) \\ & + |\beta_3| \frac{|\delta|^3 \sqrt{48(6|\delta|^6 + 52|\delta|^4 + 43|\delta|^2 + 4)(1 - \alpha - 4|a_2|^2 - 18|a_3|^2)}}{\left(1-|\delta|^2\right)^4} \\ = \ & \lambda \end{aligned}$$

in $0 < |\delta| < 1$.

Proof. For $f(z) \in \mathcal{G}(\theta, \alpha)$, we see that

$$\frac{1}{\delta} f(\delta z) = z + \sum_{n=2}^{\infty} \delta^{n-1} a_n z^n$$

and

$$\sum_{n=2}^{\infty} n^2(n-1) |a_n|^2 \leq 1 - \alpha.$$

Thus, to show that $\frac{1}{\delta}f(\delta z) \in \mathcal{D}(\beta_1, \beta_2, \beta_3; \lambda)$, from Theorem 2.1, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^{n-1}|a_n| \leq \lambda.$$

Applying Cauchy-Schwarz inequality, we note that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^{n-1}|a_n| \\
\leq\ & \frac{|\beta_1|}{|\delta|}\left(\sum_{n=2}^{\infty} n^2(n-1)|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty} n^2(n-1)|a_n|^2\right)^{\frac{1}{2}} \\
&+\frac{|\beta_2|}{|\delta|}\left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=3}^{\infty} n^2(n-1)|a_n|^2\right)^{\frac{1}{2}} \\
&+\frac{|\beta_3|}{|\delta|}\left(\sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2|\delta|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=4}^{\infty} n^2(n-1)|a_n|^2\right)^{\frac{1}{2}} \\
\leq\ & \frac{|\beta_1|}{|\delta|}\left(\sum_{n=2}^{\infty} n^2(n-1)|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha} \\
&+\frac{|\beta_2|}{|\delta|}\left(\sum_{n=3}^{\infty} n^2(n-1)(n-2)^2|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha-4|a_2|^2} \\
&+\frac{|\beta_3|}{|\delta|}\left(\sum_{n=4}^{\infty} n^2(n-1)(n-2)^2(n-3)^2|\delta|^{2n}\right)^{\frac{1}{2}}\sqrt{1-\alpha-4|a_2|^2-18|a_3|^2}. \qquad (9)
\end{aligned}$$

Making use of

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad (|x|<1),$$

we have

$$\sum_{n=2}^{\infty} n^2(n-1)x^n = \frac{2x^2(x+2)}{(1-x)^4}. \qquad (10)$$

Since

$$\sum_{n=3}^{\infty}(n-2)x^{n-1} = x^2\left(\sum_{n=3}^{\infty}(n-2)x^{n-3}\right) = x^2\left(\sum_{n=3}^{\infty}x^{n-2}\right)' = \frac{x^2}{(1-x)^2},$$

we see that

$$\sum_{n=3}^{\infty}(n-1)(n-2)^2x^n = x^3\left(\frac{x^2}{(1-x)^2}\right)'' = \frac{2x^3+4x^4}{(1-x)^4}.$$

and thus, we obtain

$$\sum_{n=3}^{\infty}n(n-1)(n-2)^2x^n = \frac{6x^3+18x^4}{(1-x)^5}$$

which yields

$$\sum_{n=3}^{\infty}n^2(n-1)(n-2)^2x^n = \frac{6x^3(3x^2+14x+3)}{(1-x)^6}. \tag{11}$$

Furthermore, we have

$$\begin{aligned}\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n &= x^4\left(\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^{n-4}\right)\\ &= x^4\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1}\right)''',\end{aligned}$$

but

$$\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-1} = x^3\left(\sum_{n=4}^{\infty}(n-2)(n-3)x^{n-4}\right) = \frac{2x^3}{(1-x)^3}$$

thus, we have

$$\sum_{n=4}^{\infty}(n-1)(n-2)^2(n-3)^2x^n = \frac{12x^4+72x^5+36x^6}{(1-x)^6},$$

which yields

$$\sum_{n=4}^{\infty}n(n-1)(n-2)^2(n-3)^2x^n = \frac{48x^4+384x^5+288x^6}{(1-x)^7}.$$

This gives us that

$$\sum_{n=4}^{\infty}n^2(n-1)(n-2)^2(n-3)^2x^n = \frac{48x^4(6x^3+52x^2+43x+4)}{(1-x)^8}. \tag{12}$$

Therefore, from (9)- (12) with $|\delta|^2 = x$, we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} n^2(n-1)(|\beta_1| + (n-2)|\beta_2| + (n-2)(n-3)|\beta_3|)|\delta|^{n-1}|a_n| \\
\leq\ & |\beta_1| \frac{|\delta|\sqrt{2(|\delta|^2+2)(1-\alpha)}}{\left(1-|\delta|^2\right)^2} \\
&+|\beta_2| \frac{|\delta|^2\sqrt{6(3|\delta|^4+14|\delta|^2+3)(1-\alpha-4|a_2|^2)}}{(1-|\delta|^2)^3} \\
&+|\beta_3| \frac{|\delta|^3\sqrt{48(6|\delta|^6+52|\delta|^4+43|\delta|^2+4)(1-\alpha-4|a_2|^2-18|a_3|^2)}}{\left(1-|\delta|^2\right)^4}
\end{aligned}$$

Now, let us consider the complex number δ $(0 < |\delta| < 1)$ such that

$$\begin{aligned}
& |\beta_1| \frac{|\delta|\sqrt{2(|\delta|^2+2)(1-\alpha)}}{\left(1-|\delta|^2\right)^2} \\
&+|\beta_2| \frac{|\delta|^2\sqrt{6(3|\delta|^4+14|\delta|^2+3)(1-\alpha-4|a_2|^2)}}{(1-|\delta|^2)^3} \\
&+|\beta_3| \frac{|\delta|^3\sqrt{48(6|\delta|^6+52|\delta|^4+43|\delta|^2+4)(1-\alpha-4|a_2|^2-18|a_3|^2)}}{\left(1-|\delta|^2\right)^4} \\
=\ & \lambda
\end{aligned}$$

If we define the function $h(|\delta|)$ by

$$\begin{aligned}
h(|\delta|) =\ & |\beta_1||\delta|(1-|\delta|^2)^2\sqrt{2(|\delta|^2+2)(1-\alpha)} \\
&+|\beta_2||\delta|^2\left(1-|\delta|^2\right)\sqrt{6(3|\delta|^4+14|\delta|^2+3)(1-\alpha-4|a_2|^2)} \\
&+|\beta_3||\delta|^3\sqrt{48(6|\delta|^6+52|\delta|^4+43|\delta|^2+4)(1-\alpha-4|a_2|^2-18|a_3|^2)} \\
&-\lambda\left(1-|\delta|^2\right)^4,
\end{aligned}$$

then we have $h(0) = -\lambda < 0$ and $h(1) = 4\sqrt{3}|\beta_3|\sqrt{105(1-\alpha-4|a_2|^2-18|a_3|^2)} > 0$. This means that there exists some δ_0 such that $h(|\delta_0|) = 0$ $(0 < |\delta_0| < 1)$. This completes the proof of the theorem. □

4. Radius Problem for the Class $\mathcal{Q}(\theta,\alpha)$

For the class $\mathcal{Q}(\theta,\alpha)$, we prove the following lemma.

Lemma 4.1. *If $f(z)\in\mathcal{Q}(\theta,\alpha)$, then*

$$\sum_{n=2}^{\infty} n(n-\alpha)\left|a_n\right| \leq 1-\alpha. \tag{13}$$

Proof. Let $f(z)\in\mathcal{Q}(\theta,\alpha)$. Then, we have

$$\begin{aligned}
\operatorname{Re}\left\{\frac{\mathrm{D}^2\mathrm{f(z)}}{\mathrm{D}^1\mathrm{f(z)}}\right\} &= \operatorname{Re}\left\{\frac{1+\sum\limits_{n=2}^{\infty}\mathrm{n^2a_nz^{n-1}}}{1+\sum\limits_{n=2}^{\infty}\mathrm{na_nz^{n-1}}}\right\}\\
&= \operatorname{Re}\left\{\frac{1-\sum\limits_{n=2}^{\infty}\mathrm{n^2}\left|\mathrm{a_n}\right|\mathrm{e}^{\mathrm{i(n-1)}\theta}\mathrm{z^{n-1}}}{1-\sum\limits_{n=2}^{\infty}\mathrm{n}\left|\mathrm{a_n}\right|\mathrm{e}^{\mathrm{i(n-1)}\theta}\mathrm{z^{n-1}}}\right\} > \alpha
\end{aligned}$$

for all $z\in\mathcal{U}$. Let us consider a point $z\in\mathcal{U}$ such that $z=|z|\,e^{-i\theta}$. Then we have

$$\frac{1-\sum\limits_{n=2}^{\infty} n^2\left|a_n\right|\left|z\right|^{n-1}}{1-\sum\limits_{n=2}^{\infty} n\left|a_n\right|\left|z\right|^{n-1}} > \alpha$$

Letting $|z|\to 1^-$, we obtain the inequality (13). □

Corollary 4.2. If $f(z)\in\mathcal{Q}(\theta,\alpha)$, then

$$\left|a_n\right| \leq \frac{1-\alpha}{n(n-\alpha)} \qquad (n=2,3,...).$$

Remark 4.3. If $f(z)\in\mathcal{Q}(\theta,\alpha)$, then

$$\sum_{n=2}^{\infty} n^2(n-1)\left|a_n\right| \leq \sum_{n=2}^{\infty} n(n-\alpha)\left|a_n\right| \leq 1-\alpha.$$

Applying Theorem 2.1, Lemma 4.1 and using the same technique as in the proof of Theorem 3.4, we derive

Theorem 4.4. *If $f(z)\in\mathcal{Q}(\theta,\alpha)$, and $\delta\in\mathbb{C}$ $(0<|\delta|<1)$. Then the function $\frac{1}{\delta}f(\delta z)\in\mathcal{D}(\beta_1,\beta_2,\beta_3;\lambda)$ for $(0<|\delta|\leq|\delta_0(\lambda)|$, where $|\delta_0(\lambda)|$ is the smallest positive* *root of the equation*

$$
\begin{aligned}
(14)\qquad & |\beta_1| \frac{|\delta| \sqrt{2(|\delta|^2+2)(1-\alpha)}}{\left(1-|\delta|^2\right)^2} \\
& + |\beta_2| \frac{|\delta|^2 \sqrt{6(3\,|\delta|^4 + 14\,|\delta|^2 + 3)(1-\alpha-4\,|a_2|^2)}}{(1-|\delta|^2)^3} \\
& + |\beta_3| \frac{|\delta|^3 \sqrt{48(6\,|\delta|^6 + 52\,|\delta|^4 + 43\,|\delta|^2 + 4)(1-\alpha-4\,|a_2|^2 - 18\,|a_3|^2)}}{\left(1-|\delta|^2\right)^4} \\
= \quad & \lambda
\end{aligned}
$$

in $0 < |\delta| < 1$.

References

[1] G. S. Sãlãgean, *Subclasses of univalent functions*, Lecture Notes in Math. Springer-Verlag, **1013**(1983), 362-372.

[2] Neslihan Uyanik and Shigeyoshi Owa, *New extensions for classes of analytic functions associated with close-to-convex and starlike of order* α, Mathematical and Computer Modelling, **54**(2011), 359–366.