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Uniqueness and Value-Sharing of Meromorphic Functions

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Abstract. In this paper, we prove two uniqueness theorem on meromorphic functions sharing one value which generalize a recent result of R. S. Dyavanal [2], and on the other hand, we relax the nature of sharing value from CM to IM.

1. Introduction

In this section, let f be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory:

$$T(r,f),\ \ m(r,f),\ \ N(r,f),\ \ \overline{N}(r,f),...$$

(See Hayman [3], Yang [5] and Yi and Yang [6]). We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to +\infty$, possibly outside of a set with finite measure. For any constant $'a'$, we define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{(f-a)}\right)}{T(r,f)}.$$

Let $'a'$ be a finite complex number and k a positive integer. We denote by $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ the counting function for the zeros of $f(z)-a$ with the multiplicity $\leq k$, and by $\overline{N}_{k)}\left(r, \frac{1}{(f-a)}\right)$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the counting function for the zeros of $f(z)-a$ with multiplicity atleast k, and $\overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the corresponding one for which the multiplicity is not counted. Set

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$$N_k\left(r,\frac{1}{(f-a)}\right)=\overline{N}\left(r,\frac{1}{(f-a)}\right)+\overline{N}_{(2}\left(r,\frac{1}{(f-a)}\right)+....+\overline{N}_{(k}\left(r,\frac{1}{(f-a)}\right).$$

We define

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}_k\left(r,\frac{1}{(f-a)}\right)}{T(r,f)}.$$

Let $g(z)$ be a meromorphic function. If $f(z)-a$ and $g(z)-a$, assume the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share the value $'a'$ CM, where $'a'$ is a complex number. Similarly, we say that $f(z)$ and $g(z)$ share a IM, provided that $f(z)-a$ and $g(z)-a$ have same multiplicities.

Recently, R. S. Dyavanal [2] proved the following theorems.

Theorem A.([2]) *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $n\geq 2$ be an integer satisfying $(n+1)s\geq 12$. If f^nf' and g^ng' share the value 1 CM, then either $f=dg$, for some $(n+1)$-th root of unity d or $g(z)=c_1e^{cz}$ and $f(z)=c_2e^{-cz}$ where c_1, c_2 and c are constants satisfying $(c_1c_2)^{n+1}c^2=-1$.*

Theorem B.([2]) *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let n be an integer satisfying $(n-2)s\geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then*

$$g=\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})},\quad f=\frac{(n+2)(1-h^{n+1})h}{(n+1)(1-h^{n+2})}$$

where h is a non-constant meromorphic function.

Theorem C.([2]) *Let f and g be two transcendental entire functions, whose zeros are of multiplicities atleast s, where s is a positive integer. Let n be an integer satisfying $(n-2)s\geq 7$. If f^nf' and g^ng' share the value 1 CM, then either $f=dg$, for some $(n+1)$-th root of unity d or $g(z)=c_1e^{cz}$ and $f(z)=c_2e^{-cz}$, where c_1, c_2 and c are constants satisfying $(c_1c_2)^{n+1}c^2=-1$.*

Theorem D.([2]) *Let f and g be two transcendental entire functions, whose zeros are of multiplicities atleast s, where s is a positive integer. Let n be an integer satisfying $(n-2)s\geq 5$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f\equiv g$.*

From the above results we can ask whether there exists a corresponding unicity theorem for $[f^nP(f)]^{(k)}$ where $P(f)$ is a polynomial. In this paper, we give a positive answer to above question by proving the following Theorems.

Theorem 1.1. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let*

$P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let n, k, m be three positive integers with $s(n+m) > 4k+12$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 CM, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, ...n+m-i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

Corollary 1. Let f and g be two non-constant entire functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let n, k, m be three positive integers with $s(n+m) > 2k+6$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 CM, then the conclusions of Theorem 1.1 hold.

Theorem 1.2. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let n, k, m be three positive integers with $s(n+m) > 9k+16$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 IM, then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, ...n+m-i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.*

Corollary 2. Let f and g be two non-constant entire functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and $a_i (i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let n, k, m be three positive integers with $s(n+m) > 5k+9$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 IM, then the conclusions of Theorem 1.2 hold.

Remark 1.1. In Theorem 1.1 giving specific values for s in Theorem 1.1, we get the following interesting cases:

(i) If $s = 1$, then $n > 4k + 12 - m$.

(ii) If $s = 2$, then $n > 2k + 6 - m$.

(iii) If $s = 3$, then $n > \frac{4}{3}k + 4 - m$.

We conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n.

2. Some Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1.([3]) Let f be a non-constant meromorphic function, let k be a positive integer, and let c be a non-zero finite complex number. Then

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-c}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.2.([1]) Let f and g be two meromorphic functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and

$$\begin{aligned} \Delta &= [(k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g)] \\ &> k+7 \end{aligned}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Lemma 2.3.([4]) Let f and g be two meromorphic functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and

$$\begin{aligned} \Delta =&[(2k+3)\Theta(\infty,f) + (2k+4)\Theta(\infty,g) + \Theta(0,f) \\ &+ \Theta(0,g) + 2\delta_{k+1}(0,f) + 3\delta_{k+1}(0,g)] \\ >&4k+13 \end{aligned} \tag{1.1}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Lemma 2.4. Let f and g be two non-constant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 1)$ be a integers. Then

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \neq 1.$$

Proof. Let

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv 1. \tag{1.2}$$

Let z_0 be a zero of f of order p_0. From (2.1) we get z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0. Again by (2.1), we obtain $np_0 - k = nq_0 + mq_0 + k$, i.e., $n(p_0 - q_0) = mq_0 + 2k$. which implies that $q_0 \geq \frac{n-2k}{m}$ and so we have $p_0 \geq \frac{n+m-2k}{m}$.

Let z_1 be a zero of $f - 1$ of order p_1, then z_1 is a zero of $[f^nP(f)]^{(k)}$ of order $p_1 - k$. Therefore from (2.1) we obtain $p_1 - k = nq_1 + mq_1 + k$ i.e., $p_1 \geq (n+m)s + 2k$.

Let z_2 be a zero of f' of order p_2 that is not a zero of $fP(f)$, then from (2.1) z_2 is a pole of g of order q_2. Again by (2.1) we get $p_2 - (k-1) = nq_2 + mq_2 + k$ i.e., $p_2 \geq (n+m)s + 2k - 1$.

In the same manner as above, we have similar results for the zeros of $[g^nP(g)]^{(k)}$.

On other hand, suppose that z_3 is a pole of f. From (2.1), we get that z_3 is the zero of $[g^nP(g)]^{(k)}$.

Thus

(1.3)

$$\begin{aligned}\overline{N}(r,f) &\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\le \frac{1}{p_0}N\left(r,\frac{1}{g}\right)+\frac{1}{p_1}N\left(r,\frac{1}{g-1}\right)+\frac{1}{p_2}N\left(r,\frac{1}{g'}\right)\\ &\le \left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,g)+S(r,g).\end{aligned}$$

By second fundamental theorem and equation (2.2), we have

$$\begin{aligned}T(r,f) &\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)\\ &\le \frac{m}{n+m-2k}N\left(r,\frac{1}{f}\right)+\frac{1}{(n+m)s+2k}N\left(r,\frac{1}{f-1}\right)\\ &+ \left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,g)\\ &+S(r,g)+S(r,f).\end{aligned}$$

$$\begin{aligned}T(r,f) \le{}& \left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}\right]T(r,f)\\ &+\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,g)\\ &+S(r,g)+S(r,f).\end{aligned}\tag{1.4}$$

Similarly, we have

$$\begin{aligned}T(r,g) \le{}& \left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}\right]T(r,g)\\ &+\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,f)\\ &+S(r,g)+S(r,f).\end{aligned}\tag{1.5}$$

Adding (2.3) and (2.4) we get

$$\begin{aligned}&T(r,f)+T(r,g)\\ &\le \left[\frac{2m}{n+m-2k}+\frac{2}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]\{T(r,f)+T(r,g)\}\\ &+S(r,g)+S(r,f).\end{aligned}$$

which is a contradiction. Thus Lemma proved. □

3. Proofs of the Theorems

Proof of Theorem 1.1.

Let $F = f^nP(f)$ and $G = g^nP(g)$ then $[F]^{(k)}$ and $[G]^{(k)}$ share 1CM. We have

$$\Delta = [(k+2)\Theta(\infty, F) + 2\Theta(\infty, G) + \Theta(0, F) + \Theta(0, G) + \delta_{k+1}(0, F) + \delta_{k+1}(0, G)]$$

Consider

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{f^nP(f)}\right) \leq \frac{2}{s(n+m)} N\left(r, \frac{1}{F}\right) \leq \frac{2}{s(n+m)}[T(r, F) + O(1)].$$

$$\Theta(0, F) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r, F)} \geq 1 - \frac{2}{s(n+m)}. \tag{1.6}$$

Similarly,

$$\Theta(0, G) \geq 1 - \frac{2}{s(n+m)}. \tag{1.7}$$

$$\Theta(\infty, F) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, F)}{T(r, F)} \geq 1 - \frac{1}{s(n+m)}. \tag{1.8}$$

Similarly,

$$\Theta(\infty, G) \geq 1 - \frac{1}{s(n+m)}. \tag{1.9}$$

Consider

$$\begin{aligned} N_{k+1}\left(r, \frac{1}{F}\right) &= N_{k+1}\left(r, \frac{1}{f^nP(f)}\right) = (k+1)\overline{N}\left(r, \frac{1}{f^nP(f)}\right) \\ &\leq \frac{(k+1)}{s(n+m)}[T(r, F) + O(1)]. \end{aligned}$$

Next, we have

$$\delta_{k+1}(0,F) = 1 - \limsup_{r\to\infty} \frac{N_{k+1}\left(r, \frac{1}{F}\right)}{T(r,F)} \geq 1 - \frac{(k+1)}{s(n+m)}. \tag{1.10}$$

Similarly,

$$\delta_{k+1}(0,G) \geq 1 - \frac{(k+1)}{s(n+m)}. \tag{1.11}$$

From (2.5) to (2.10), we get

$$\Delta \geq (k+4)\left(1 - \frac{2}{s(n+m)}\right) + 2\left(1 - \frac{1}{s(n+m)}\right) + 2\left(1 - \frac{(k+1)}{s(n+m)}\right).$$

Since $s(n+m) > 4k+12$, we get $\Delta > k+7$.
Therefore, by Lemma 2.2, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.
If $F^{(k)}G^{(k)} \equiv 1$, that is

$$\begin{aligned}&[f^n(a_m f^m + a_{m-1}f^{m-1} + \cdots + a_1 f + a_0)]^{(k)}\\ &\cdot [g^n(a_m g^m + a_{m-1}g^{m-1} + \cdots + a_1 g + a_0)]^{(k)} \equiv 1\end{aligned} \tag{1.12}$$

then by Lemma 2.4 we can get a contradiction.
Hence, we deduce that $F \equiv G$, that is
(1.13)
$$f^n(a_m f^m + a_{m-1}f^{m-1} + ... + a_1 f + a_0) = g^n(a_m g^m + a_{m-1}g^{m-1} + ... + a_1 g + a_0).$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (2.12) we obtain

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1}g^{n+m-1}(h^{n+m-1} - 1) + ... + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = (n+m, ..., n+m-i, ...n)$, $a_{m-1} \neq 0$ for some $i = 0, 1, ...m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, ..., n+m-i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ...m$.
If h is not a constant , then we know (2.12) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.
This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2.

Let $F = f^n P(f)$ and $G = g^n P(g)$ then $[F]^{(k)}$ and $[G]^{(k)}$ share 1IM. We have

$$\begin{aligned}\Delta \;=\; &[(2k+3)\Theta(\infty,F) + (2k+4)\Theta(\infty,G) + \Theta(0,F) + \Theta(0,G)\\ &+2\delta_{k+1}(0,F) + 3\delta_{k+1}(0,G)]\end{aligned}$$

Consider

$$\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{f^nP(f)}\right)\leq\frac{2}{s(n+m)}N\left(r,\frac{1}{F}\right)\leq\frac{2}{s(n+m)}[T(r,F)+O(1)].$$

$$\Theta(0,F)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F}\right)}{T(r,F)}\geq 1-\frac{2}{s(n+m)}. \tag{1.14}$$

Similarly,

$$\Theta(0,G)\geq 1-\frac{2}{s(n+m)}. \tag{1.15}$$

$$\Theta(\infty,F)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,F\right)}{T(r,F)}\geq 1-\frac{1}{s(n+m)}. \tag{1.16}$$

Similarly,

$$\Theta(\infty,G)\geq 1-\frac{1}{s(n+m)}. \tag{1.17}$$

Consider

$$\begin{aligned}N_{k+1}\left(r,\frac{1}{F}\right) &= N_{k+1}\left(r,\frac{1}{f^nP(f)}\right)=(k+1)\overline{N}\left(r,\frac{1}{f^nP(f)}\right)\\ &\leq \frac{(k+1)}{s(n+m)}[T(r,F)+O(1)].\end{aligned}$$

Next, we have

$$\delta_{k+1}(0,F)=1-\limsup_{r\to\infty}\frac{N_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)}\geq 1-\frac{(k+1)}{s(n+m)}. \tag{1.18}$$

Similarly,

$$\delta_{k+1}(0,G)\geq 1-\frac{(k+1)}{s(n+m)}. \tag{1.19}$$

From (2.13) to (2.18), we get

$$\Delta\geq 2\left(1-\frac{2}{s(n+m)}\right)+(4k+7)\left(1-\frac{1}{s(n+m)}\right)+5\left(1-\frac{(k+1)}{s(n+m)}\right).$$

Since $s(n+m)>9k+16$, we get $\Delta>4k+13$.

Now proceeding as in Theorem 1.1 we can prove the Theorem 1.2. This completes the proof of Theorem 1.2. □

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