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A NEW GENERALIZED RESOLVENT AND APPLICATION IN BANACH MAPPINGS

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Abstract. In this paper, we introduce a new generalized resolvent in a Banach space and discuss its some properties. Using these properties, we obtain an iterative scheme for finding a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping. Furthermore, strong convergence of the scheme to a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping is proved.

1. Introduction

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E into 2^{E^*}. defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*} = I^*$ and $J^{-1}J = I_E = I$ (see [3]). We note that in a Hilbert space H, J is the identity mapping. Let E be a smooth, reflexive, and strictly convex Banach space. We define the function $V_2 : E \times E \to R$ by

$$V_2(y, x) = \|x\|^2 - 2\langle Jy, x \rangle + \|y\|^2, \tag{1.1}$$

for $\forall x \in E, y \in E$. Let C be a nonempty closed convex subset of E. For an arbitrary point x of E, consider the set $\{z \in C : V_2(z, x) = \min_{y \in C} V_2(y, x)\}$.

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It is known that this set is always a singleton(see [7])Let Π_C be a mapping of E onto C satisfying

$$V_2(\Pi_C x, x) = \min_{y\in C} V_2(y, x). \tag{1.2}$$

Such a mapping Π_C is called the generalized projection.

Applying the definitions of V_2 and J , a functional $V : E^* \times E \to R$ is defined by the formula:

$$V(x^*, y) = V_2(J^{-1}x^*, y), \quad \forall x^* \in E^*, y \in E.$$

In the following, we shall make use of the following lemmas.

Lemma 1.1. *([1]) Let E be a real smooth Banach space, $A : E \to 2^{E^*}$ be a maximal monotone mapping, then $A^{-1}0$ is a closed and convex subset of E and the graph of A, $G(A)$, is demiclosed in the following sense: $\forall x_n \in D(A)$ with $x_n \to x$ in E, and $\forall y_n \in Ax_n$ with $y_n \to y$ in E imply that $x \in D(A)$ and $y \in Ax$.*

Lemma 1.2. *([7]) Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then $y \in C$,*

$$V_2(y, \Pi_C x) + V_2(\Pi_C x, x) \leq V_2(y, x).$$

Lemma 1.3. *([7]) Let C be a convex subset of a real smooth Banach space E. Let $x \in E$ and $x_0 \in C$. Then $V_2(x_0, x) = \inf\{V_2(z, x) : z \in C\}$ if and only if*

$$\langle z - x_0, Jx_0 - Jx\rangle \geq 0.$$

Lemma 1.4. *([4]) Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $V_2(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Let E^* be a smooth Banach space and let D^* be a nonempty closed convex subset of E^*. A mapping $R^* : D^* \to D^*$ is called generalized nonexpansive if $F(R^*) \neq \emptyset$ and

$$V(R^*x^*, J^{-1}y^*) \leq V(x^*, J^{-1}y^*), \quad \forall x^* \in D^*, y^* \in F(R^*),$$

where $F(R^*)$ is the set of fixed points of R^*.

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point of p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called weak relatively nonexpansive if $\widetilde{F}(T) = F(T)$ and $V_2(p, Tx) \leq V_2(p, x)$ for all $x \in C$ and $p \in F(T)$.(see[8])

In this paper, motivated by Alber [7], Iiduka and Takahashi [6] and Habtu [2], we first introduce the generalized resolvent and discuss its properties. Secondly, we give an iterative scheme for finding a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping.

Finally we show its convergence.

2. Second section

Let E^* be a reflexive and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. For each $\lambda > 0$ and $x \in E$, consider the set

$$J_\lambda^* x^* := \{z^* \in E^* : x^* \in z^* + \lambda B J^{-1}(z^*)\}.$$

If $z_1^* + \lambda w_1^* = x^*, z_2^* + \lambda w_2^* = x^*, w_1^* \in BJ^{-1}(z_1^*), w_2^* \in BJ^{-1}(z_2^*)$, then we have from the monotonicity of B that

$$\langle w_1^* - w_2^*, J^{-1}(z_1^*) - J^{-1}(z_2^*)\rangle \geq 0$$

and hence

$$\left\langle \frac{x^* - z_1^*}{\lambda} - \frac{x^* - z_2^*}{\lambda}, J^{-1}(z_1^*) - J^{-1}(z_2^*)\right\rangle \geq 0.$$

So, we obtain

$$\langle x^* - z_1^* - (x^* - z_2^*), J^{-1}(z_1^*) - J^{-1}(z_2^*)\rangle \geq 0.$$

and hence

$$\langle z_2^* - z_1^*, J^{-1}(z_1^*) - J^{-1}(z_2^*)\rangle \geq 0.$$

This implies $z_1^* = z_2^*$. Then $J_\lambda^* x^*$ consists of one point. We also denote the domain and the range of $J_\lambda^* x^*$ by $D(J_\lambda^*) = R(I^* + \lambda BJ^{-1})$ and $R(J_\lambda^*) = D(BJ^{-1})$, respectively, where I^* is the identity on E^*. Such a $J_\lambda^* : E^* \to E^*$ is called the generalized resolvent of B and is denoted by

$$J_\lambda^* = (I^* + \lambda BJ^{-1})^{-1}. \tag{2.1}$$

We get some properties of J_λ^* and $(BJ^{-1})^{-1}0$.

Proposition 2.1. *Let E^* be a reflexive and strictly convex Banach space with a Fréchet differentiable norm and let $B \subset E \times E^*$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) $D(J_\lambda^*) = E^*$ for each $\lambda > 0$.

(2) $(BJ^{-1})^{-1}0 = F(J_\lambda^*)$ for each $\lambda > 0$, where $F(J_\lambda^*)$ is the set of fixed points of J_λ^*.

(3) $(BJ^{-1})^{-1}0$ is closed.

(4) $J_\lambda^* : E^* \to E^*$ is generalized nonexpansive for each $\lambda > 0$.

Proof. (1) From the maximality of B, we have

$$R(J + \lambda B) = E^*, \quad \forall \lambda > 0.$$

Hence, for each $x^* \in E^*$, there exists $x \in E$. such that $x^* \in Jx + \lambda Bx$. Since E is reflexive and strictly convex, then J is bijective. Therefore, there exists $z^* \in E^*$ such that $x = J^{-1}(z^*)$. Therefore, we have

$$x^* \in JJ^{-1}(z^*) + \lambda BJ^{-1}(z^*) = z^* + \lambda BJ^{-1}(z^*) \subset R(I^* + \lambda BJ^{-1}) = D(J_\lambda^*).$$

This implies $E^* \subset D(J_\lambda^*)$. $D(J_\lambda^*) \subset E^*$ is clear. So, we have $D(J_\lambda^*) = E^*$.

(2) Let $\lambda > 0$. Then we have

$$x^* \in F(J_\lambda) \Leftrightarrow J_\lambda^* x^* = x^* \Leftrightarrow x^* \in x^* + \lambda BJ^{-1}(x^*)$$

$$\Leftrightarrow 0 \in \lambda BJ^{-1}(x^*) \Leftrightarrow 0 \in BJ^{-1}(x^*) \Leftrightarrow x^* \in (BJ^{-1})^{-1}0.$$

(3) Let $\{x_n^*\} \subset (BJ^{-1})^{-1}0$ with $x_n^* \to x^*$. From $x_n^* \in (BJ^{-1})^{-1}0$, we have $J^{-1}(x_n^*) \in B^{-1}0$. Since J^{-1} is norm to norm continuous, and $B^{-1}0$ is closed, we have that $J^{-1}(x_n^*) \to J^{-1}(x^*) \in B^{-1}0$. This implies $x^* \in (BJ^{-1})^{-1}0$. That is, $(BJ^{-1})^{-1}0$ is closed.

(4) Let $x^* \in E^*$, $y^* \in E^*, z^* \in E^*$ and $\lambda > 0$. By definition (1.1) and calculated that

$$\begin{aligned} V(x^*, J^{-1}z^*) + V(z^*, J^{-1}y^*) &= \|x^*\|^2 + \|z^*\|^2 - 2\langle x^*, J^{-1}z^*\rangle \\ &\quad + \|y^*\|^2 + \|z^*\|^2 - 2\langle z^*, J^{-1}y^*\rangle \\ &= V(x^*, J^{-1}y^*) + 2\langle z^* - x^*, J^{-1}z^* - J^{-1}y^*\rangle, \end{aligned}$$

we have that

$$V(x^*, J^{-1}y^*) = V(x^*, J^{-1}z^*) + V(z^*, J^{-1}y^*) + 2\langle x^* - z^*, J^{-1}z^* - J^{-1}y^*\rangle.$$

Let $x^* \in E^*, y^* \in F(J_\lambda)$ and $\lambda > 0$. From above formula , we have

$$V(x^*, J^{-1}y^*) = V(x^*, J^{-1}J_\lambda^* x^*) + V(J_\lambda^* x^*, J^{-1}y^*) + 2\langle x^* - J_\lambda^* x^*, J^{-1}J_\lambda x^* - J^{-1}y^*\rangle.$$

Since $\frac{x^* - J_\lambda^* x^*}{\lambda} \in BJ^{-1}(J_\lambda^* x^*)$ and $0 \in BJ^{-1}(y^*)$, we have

$$\langle x^* - J_\lambda^* x^*, J^{-1}J_\lambda^* x^* - J^{-1}y^*\rangle \geq 0.$$

Therefore we get

$$V(x^*, J^{-1}y^*) \geq V(x^*, J^{-1}J_\lambda^* x^*) + V(J_\lambda^* x^*, J^{-1}y^*) \geq V(J_\lambda^* x^*, J^{-1}y^*).$$

That is, J_λ^* is generalized nonexpansive on E^*. □

Theorem 2.2. ([5]) *Let E be a Banach space and let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. If E^* is strictly convex and has a Fréchet differentiable norm, then, for each $x \in E$, $\lim_{\lambda\to\infty}(J + \lambda A)^{-1}J(x)$ exists and belongs to $A^{-1}0$.*

Using Theorem 2.2, we get the following result.

Theorem 2.3. *Let E^* be a uniformly convex Banach space with a Fréchet differentiable norm and let $B \subset E \times E^*$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) For each $x^* \in E^*$, $\lim_{\lambda\to\infty} J_\lambda^* x^*$ exists and belongs to $(BJ^{-1})^{-1}0$.

(2) If $R^*x^* := \lim_{\lambda\to\infty} J_\lambda^* x^*$ for each $x^* \in E^*$, then R^* is a sunny generalized nonexpansive retraction of E^* onto $(BJ^{-1})^{-1}0$.

Proof. (1) Defining a mapping Q_λ from E to E by

$$Q_\lambda x := (I + \lambda J^{-1}B))^{-1}x, \quad \forall x \in E, \lambda > 0,$$

we have, for $\forall x^* \in E^*, \lambda > 0,\ J_\lambda^* x^* = JQ_\lambda J^{-1}(x^*)$. In fact, define

$$x_\lambda^* := JQ_\lambda J^{-1}(x^*) = \left[J(I + \lambda J^{-1}B)J^{-1}\right]^{-1}(x^*).$$

Then, we have

$$x^* \in J(I + \lambda J^{-1}B)J^{-1}(x_\lambda^*) = (I^* + \lambda BJ^{-1})x_\lambda^*$$

and hence $x_\lambda^* = J_\lambda^* x^*$. From Theorem 2.1, we get

$$\lim_{\lambda\to\infty} Q_\lambda J^{-1}(x^*) = u \in B^{-1}0.$$

If E^* is uniformly convex, then E has a Fréchet differentiable norm. So, then J is norm to norm continuous. Since $B^{-1}0$ is closed, we have

$$\lim_{\lambda\to\infty} J_\lambda^* x^* = \lim_{\lambda\to\infty} JQ_\lambda J^{-1}(x^*) = Ju \in JB^{-1}0 = (BJ^{-1})^{-1}0.$$

(2) Defining a mapping R^* from E^* to E^* by

$$R^*x^* := \lim_{\lambda\to\infty} J_\lambda^* x^* \quad \forall x^* \in E^*.$$

Let $u^* \in (BJ^{-1})^{-1}0 = F(J_\lambda^* x^*)$. Then $R^*u^* = \lim_{\lambda\to\infty} J_\lambda^* u^* = \lim_{\lambda\to\infty} u^* = u^*$. Therefore R^* is a retraction of E^* onto $(BJ^{-1})^{-1}0$. Since $x^* \in J_\lambda^* x^* + \lambda BJ^{-1}(J_\lambda^* x^*)$, we have

$$\left\langle \frac{x^* - J_\lambda^* x^*}{\lambda}, J^{-1}(J_\lambda^* x^*) - J^{-1}(z^*) \right\rangle \geq 0, \quad \forall z^* \in (BJ^{-1})^{-1}0,$$

and hence

$$\langle x^* - J_\lambda^* x^*, J^{-1}(J_\lambda^* x^*) - J^{-1}(z^*)\rangle \geq 0.$$

Letting $\lambda \to 0$, we get

$$\langle x^* - R^*x^*, J^{-1}(R^*x^*) - J^{-1}(z^*)\rangle \geq 0, \quad \forall z^* \in (BJ^{-1})^{-1}0$$

From Proposition 2.1, R^* is sunny and generalized nonexpansive. This implies that R^* is a sunny generalized nonexpansive retraction of E^* onto $(BJ^{-1})^{-1}0$.
□ □

Now we construct an iterative scheme which converges strongly to a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping.

Theorem 2.4. *Let E^* be a uniformly convex Banach space and uniformly smooth Banach space. let $A \subset E \times E^*$ be a maximal monotone operator. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a relatively weak* *nonexpansive mapping with* $A^{-1}0 \cap F(T) \neq \emptyset$. *Assume that* $0 \leq \alpha_n < a < 1$ *is a sequence of real numbers. Then the sequence* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 \in C, \quad \lambda_n \to +\infty, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J^*_{\lambda_n}Jx_n), \quad J^*_{\lambda_n} = (I^* + \lambda_n AJ^{-1})^{-1}, \\ z_n = Ty_n, \\ H_0 = \{v \in C : V_2(v, z_0) \leq V_2(v, y_0) \leq V_2(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : V_2(v, z_n) \leq V_2(v, y_n) \leq V_2(v, x_n)\}, \\ W_0 = C, \\ W_n = \{v \in H_{n-1} \cap W_{n-1} : \langle v - x_n, Jx_0 - Jx_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x_0), \quad n \geq 1, \end{cases} \tag{3.1}$$

converges strongly to $\Pi_{A^{-1}0 \cap F(T)}(x_0)$, *where* $\Pi_{A^{-1}0 \cap F(T)}$ *is the generalized projection from* E *onto* $A^{-1}0 \cap F(T)$.

Proof. We first show that H_n and W_n are closed and convex for each $n \geq 0$. From the definition of H_n and W_n , it is obvious that H_n is closed and W_n is closed and convex for each $n \geq 0$. We show that H_n is convex. Since

$$H_n = \{v \in H_{n-1} \cap W_{n-1} : V_2(v, z_n) \leq V_2(v, y_n)\} \cap \{v \in H_{n-1} \cap W_{n-1} : V_2(v, y_n) \leq V_2(v, x_n)\},$$

and that $V_2(v, y_n) \leq V_2(v, x_n)$ is equivalent to

$$2\langle v, Jx_n - Jy_n \rangle + \|y_n\|^2 + \|x_n\|^2 \leq 0,$$

$V_2(v, z_n) \leq V_2(v, y_n)$ is equivalent to

$$2\langle v, Jy_n - Jz_n \rangle + \|z_n\|^2 + \|x_n\|^2 \leq 0,$$

it follows that H_n is convex.

Next, we show that $F =: A^{-1}0 \cap F(T) \subset H_n \cap W_n$ for each $n \geq 0$. Let $p \in F$, then relatively weak nonexpansiveness of T and generalized nonexpansiveness of J^*_λ give that

$$\begin{aligned} V_2(p, z_0) &= V_2(p, Ty_0) \leq V_2(p, y_0) \\ &= V_2(p, J^{-1}(\alpha_0 Jx_0 + (1-\alpha_0)J^*_{\lambda_0}Jx_0)) \\ &= \|p\|^2 + \|\alpha_0 Jx_0 + (1-\alpha_0)J^*_{\lambda_0}Jx_0\|^2 - 2\langle p, \alpha_0 Jx_0 + (1-\alpha_0)J^*_{\lambda_0}Jx_0 \rangle \\ &\leq \|p\|^2 - 2\alpha_0\langle p, Jx_0 \rangle - 2(1-\alpha_0)\langle p, J^*_{\lambda_0}Jx_0 \rangle + \alpha_0\|Jx_0\|^2 + (1-\alpha_0)\|J^*_{\lambda_0}Jx_0\|^2 \\ &= \alpha_0(\|p\|^2 - 2\alpha_0\langle p, Jx_0 \rangle + \|x_0\|^2) + (1-\alpha_0)(\|p\|^2 - 2\langle p, J^*_{\lambda_0}Jx_0 \rangle + \|J^*_{\lambda_0}Jx_0\|^2) \\ &= \alpha_0 V_2(p, x_0) + (1-\alpha_0)V_2(p, J^{-1}J^*_{\lambda_0}Jx_0) \\ &= \alpha_0 V_2(p, x_0) + (1-\alpha_0)V(p, J^*_{\lambda_0}Jx_0) \\ &\leq \alpha_0 V_2(p, x_0) + (1-\alpha_0)V(p, Jx_0) \\ &\leq \alpha_0 V_2(p, x_0) + (1-\alpha_0)V_2(p, x_0) = V_2(p, x_0). \end{aligned} \tag{3.2}$$

Thus, we give that $p \in H_0$. On the other hand it is clear that $p \in C$. Thus $F \subset H_0 \cap W_0$ and therefore, $x_1 = \Pi_{H_0 \cap W_0}$ is well defined. Suppose that $F \subset H_{n-1} \cap W_{n-1}$ and $\{x_n\}$ is well defined. Then the methods in (3.2) imply that $V_2(p, z_n) \leq V_2(p, y_n) \leq V_2(p, x_n)$ and that $p \in H_n$. Moreover, it follows from Lemma 1.3 that

$$\langle p - x_n, Jx_n - Jx_0 \rangle \geq 0,$$

which implies that $p \in W_n$. Hence $F \subset H_n \cap W_n$ and $x_{n+1} = \Pi_{H_n \cap W_n}$ is well defined. Then by induction, $F \subset H_n \cap W_n$ and the sequence generated by (3.1) is well defined for each $n \geq 0$.

Now we show that $\{x_n\}$ is a bounded sequence and converges to a point of F. Let $p \in F$. Since $x_{n+1} = \Pi_{H_n \cap W_n}(x_0)$ and $H_n \cap W_n \subset H_{n-1} \cap W_{n-1}$ for all $n \geq 1$, we have

$$V_2(x_n, x_0) \leq V_2(x_{n+1}, x_0)$$

for all $n \geq 0$. Therefore, $\{V_2(x_n, x_0)\}$ is nondecreasing. In addition, it follows from definition of W_n and Lemma 1.3 that $x_n = \Pi_{W_n}(x_0)$. Therefore, by Lemma 1.2 we have

$$V_2(x_n, x_0) = V_2(\Pi_{W_n}(x_0), x_0) \leq V_2(p, x_0) - V_2(p, x_n) \leq V_2(p, x_0),$$

for each $p \in F(T) \subset W_n$ for all $n \geq 0$. Therefore, $\{V_2(x_n, x_0)\}$ is bounded. This together with (3.2) implies that the limit of $\{V_2(x_n, x_0)\}$ exists. Put $\lim_{n\to\infty} V_2(x_n, x_0) = d$. From Lemma 1.2, we have, for any positive integer m, that

$$\begin{aligned} V_2(x_{n+m}, x_n) = V_2(x_{n+m}, \Pi_{W_n}(x_0)) &\leq V_2(x_{n+m}, x_0) - V_2(\Pi_{W_n}(x_0), x_0) \\ &= V_2(x_{n+m}, x_0) - V_2(x_n, x_0), \end{aligned} \tag{3.3}$$

for all $n \geq 0$. The existence of $\lim_{n\to\infty} V_2(x_n, x_0)$ implies that $\lim_{n\to\infty} V_2(x_{m+n}, x_n) = 0$. Thus, Lemma 1.4 implies that

$$x_{m+n} - x_n \to 0 \quad \text{as} \quad n \to \infty \tag{3.4}$$

and hence $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $q \in E$ such that $x_n \to q$ as $n \to \infty$. Since $x_{n+1} \in H_n$, we have $V_2(x_{n+1}, z_n) \leq V_2(x_{n+1}, y_n) \leq V_2(x_{n+1}, x_n)$. Thus by Lemma 1.4 and (3.4) we get that

$$x_{n+1} - z_n \to 0, \quad x_{n+1} - y_n \to 0 \quad \text{as} \quad n \to \infty \tag{3.5}$$

and hence $\|x_n - y_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Furthermore, since J is uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = \lim_{n\to\infty} \|Jx_n - Jy_n\| = 0, \tag{3.6}$$

which implies that

$$\|Jx_{n+1} - JTy_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Since J^{-1} is also uniformly norm-norm-continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - Ty_n\| = \lim_{n\to\infty} \|J^{-1}Jx_{n+1} - J^{-1}JTy_n\| = 0. \tag{3.8}$$

Therefore, from (3.5), (3.8) and $\|y_n - Ty_n\| \leq \|x_{n+1} - Ty_n\| + \|x_n - y_n\|$, we obtain that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$. This together with the fact that $\{x_n\}$ (and hence $\{y_n\}$) converges strongly to $q \in E$ and the definition of relatively weak nonexpansive mapping implies that $q \in F(T)$. Furthermore, from (3.1) and (3.6), we have that $(1-\alpha_n)\|J^*_{\lambda_n} Jx_n - Jx_n\| = \|Jx_n - Jy_n\| \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} J^*_{\lambda_n} Jx_n = \lim_{n\to\infty} Jx_n = Jq \in JA^{-1}0 = (AJ^{-1})^{-1}0$, we obtain that $q \in A^{-1}0$.Finally, we show that $q = \Pi_{A^{-1}0\cap F(T)}(x_0)$ as $n \to \infty$. From Lemma 1.2, we have

$$V_2(q, \Pi_{A^{-1}0\cap F(T)}(x_0)) + V_2(\Pi_{A^{-1}0\cap F(T)}(x_0), x_0) \leq V_2(q, x_0). \tag{3.9}$$

On the other hand, since $x_{n+1} = \Pi_{H_n\cap W_n}(x_0)$ and $F \subset H_n \cap W_n$ for all $n \geq 0$ we have by Lemma 1.2 that

$$V_2(\Pi_{A^{-1}0\cap F(T)}(x_0), x_{n+1}) + V_2(x_{n+1}, x_0) \leq V_2(\Pi_{A^{-1}0\cap F(T)}(x_0), x_0). \tag{3.10}$$

Moreover, by the definition of $V_2(x, y)$ we get that

$$\lim_{n\to\infty} V_2(x_{n+1}, x_0) = V_2(q, x_0). \tag{3.11}$$

Combining (3.9),(3.11) we obtain that $V_2(q, x_0) = V_2(\Pi_{A^{-1}0\cap F(T)}(x_0), x_0)$. Therefore, it follows from the uniqueness of $\Pi_{A^{-1}0\cap F(T)}(x_0)$ that $q = \Pi_{A^{-1}0\cap F(T)}(x_0)$. □

Remark 1. If in Theorem 3.1 we have that $T = I$, the identity map on E then we get the following:

Corollary 2.5. *Let E^* be a uniformly convex Banach space and uniformly smooth Banach space. let $A \subset E \times E^*$ be a maximal monotone operator. Let C be a nonempty closed convex subset of E with $A^{-1}0 \neq \emptyset$. Assume that $0 \leq \alpha_n < a < 1$ is a sequence of real numbers. Then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C, \quad \lambda_n \to +\infty, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J^*_{\lambda_n} Jx_n), \quad J^*_{\lambda_n} = (I^* + \lambda_n AJ^{-1})^{-1}, \\ H_0 = \{v \in C : V_2(v, z_0) \leq V_2(v, y_0) \leq V_2(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : V_2(v, z_n) \leq V_2(v, y_n) \leq V_2(v, x_n)\}, \\ W_0 = C, \\ W_n = \{v \in H_{n-1} \cap W_{n-1} : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n\cap W_n}(x_0), \quad n \geq 1, \end{cases}$$

converges strongly to $\Pi_{A^{-1}0}$, where $\Pi_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.

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