East Asian Mathematical Journal
Vol. 30 (2014), No. 1, pp. 063–067
http://dx.doi.org/10.7858/eamj.2014.006

MINIMAL BASICALLY DISCONNECTED COVERS OF P'-SPACES

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Abstract. Observing that for any P'-space X, ΛvX is a P'-space if vX is a weakly Lindelöf space, $(\Lambda vX \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$ for a countably locally weakly Lindelöf space Y.

1. Introduction

All spaces in this paper are assumed to be Tychonoff and $(\beta X, \beta_X)$ $\big((vX, v_X)$, *resp.*$\big)$ denotes the Stone-Čech compactification (Hewitt realcompactification, resp.) of X.

Iliadis constructed the absolute of a Hausdorff space X, which is the minimal extremally disconnected cover (EX, π_X) of X and they turn out to be the perfect onto projective covers ([5]). To generalize extremally disconnected spaces, basically disconnected spaces, quasi-F spaces and cloz-spaces have been introduced and their minimal covers have been studied by various aurthors. In these ramifications, minimal covers of compact spaces can be nisely characterized.

In particular, Vermeer ([7]) showed that every Tychonoff space X has the minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and that for any compact space X, ΛX, is given by the Stone space $S(\sigma Z(X)^{\#})$ of a σ-complete Boolean subalgebra $\sigma Z(X)^{\#}$ of $R(X)$.

In [1], Comfort, Hindman, and Negrepontis showed that X is a P-space and Y is a countably locally weakly Lindelöf space, then $X \times Y$ is a basically disconnected space.

The purpose of this paper is to construct the minimal basically disconnected covers of P'-spaces.

In [4], it showed that if X is a weakly P-space and Y is a countably locally weakly Lindelöf space, then $(\Lambda X \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$.

Received October 23, 2013; Accepted November 22, 2013.
2000 *Mathematics Subject Classification.* Primary 54G05, 54C10, 54D20, 54G10.
Key words and phrases. basically disconnected space, covering map, weakly Lindelöf space, P'-spaces.
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In this paper, we will show that for any P'-space ([6]) X such that vX is a weakly Lindelöf space, ΛvX is a P'-space and that for any countably locally weakly Lindelöf space Y, $(\Lambda vX \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$.

For the terminology, we refer to [2] and [5].

2. Basically disconnected covers of P'-spaces

Let X be a space. The collection $R(X)$ of all regular closed sets in X, when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows :
For any $A \in R(X)$ and any $\mathcal{F} \subseteq R(X)$,
$\bigvee \mathcal{F} = cl_X\big(\cup \{F \mid F \in \mathcal{F}\}\big)$,
$\bigwedge \mathcal{F} = cl_X\big(int_X\big(\cap \{F \mid F \in \mathcal{F}\}\big)\big)$, and
$A' = cl_X(X - A)$.
A sublattice of $R(X)$ is a subset of $R(X)$ that contains $\emptyset$, X and is closed under finite joins and finite meets ([5]).

Let $Z(X)$ be the set of all zero-sets in X and $Z(X)^\# = \{cl_X\big(int_X(A)\big) \mid A \in Z(X)\}$. Then $Z(X)^\#$ is a sublattice of $R(X)$.

Recall that a map $f : Y \to X$ is called a *covering map* if it is a continuous, onto, perpect, and irreducible map.

Lemma 2.1. ([3], [5]) (1) *Let $f : Y \to X$ be a covering map. Then the map $\psi : R(Y) \to R(X)$, defined by $\psi(A) = f(A)$, is a Boolean isomorphism and the inverse map ψ^{-1} of ψ is given by $\psi^{-1}(B) = cl_Y\big(f^{-1}\big(int_X(B)\big)\big) = cl_Y\big(int_Y\big(f^{-1}(B)\big)\big)$.*

(2) *Let X be a dense subspace of a space K. Then the map $\phi : R(K) \to R(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ϕ^{-1} of ϕ is given by $\phi^{-1}(B) = cl_K(B)$.*

A lattice L is called *σ- complete* if every countable subset of L has join and meet. For any subset M of a Boolean algebra L, there is the smallest σ-complete Boolean algebra σM of L containing M.

Let X be a space. Then $Z(X)^\#$ is a sublattice of $R(X)$. Note that for any zero-set A in X, there is a zero-set B in βX such that $A = B \cap X$. Hence, by Lemma 2.1, $Z(X)^\#$, $Z(vX)^\#$ and $Z(\beta X)^\#$ are Boolean isomorphic. Moreover $\sigma Z(X)^\#$, $\sigma Z(vX)^\#$ and $\sigma Z(\beta X)^\#$ are Boolean isomorphic.

Definition 1. A space X is called *basically disconnected* if for any zero-set Z in X, $int_X(Z)$ is closed in X, equivalently, $Z(X)^\# = B(X)$, where $B(X)$ is the set of all clopen sets in X.

A space X is a basically disconnected space if and only if βX is a basically disconnected space.

Suppose that X is a basically disconnected space. Then for any sequence (B_n) in $B(X)$, $\bigwedge\{B_n \mid n \in N\} = cl_X\big(int_X\big(\cap \{B_n \mid n \in N\}\big)\big) \in Z(X)^\#$

and $\bigvee\{B_n \mid n \in N\} = cl_X\big(int_X\big(\cup\{B_n \mid n \in N\}\big)\big) \in Z(X)^{\#}$. Hence X is a basically disconnected space if and only if $Z(X)^{\#}$ is a σ- complete Boolean algebra.

Lemma 2.2. *Let $f : X \to Y$ be a covering map and (A_n) a decreasing sequence of closed sets in X. Then $f\big(\cap\{A_n \mid n \in N\}\big) = \cap\{f(A_n) \mid n \in N\}$*

Proof. Clearly, we have $f\big(\cap\{A_n \mid n \in N\}\big) \subseteq \cap\{f(A_n) \mid n \in N\}$.

Let $x \in \cap\{f(A_n) \mid n \in N\}$. Since (A_n) is a decreasing sequence of closed sets in X, $\{A_n \cap f^{-1}(x) \mid n \in N\}$ has a family of closed sets in $f^{-1}(x)$ with the finite intersection property. Since $f^{-1}(x)$ is compact, $\cap\{A_n \cap f^{-1}(x) \mid n \in N\} \neq \emptyset$ and so $\cap\{A_n \mid n \in N\} \cap f^{-1}(x) \neq \emptyset$. Note that

$$\begin{aligned}\emptyset &\neq f\big(\cap\{A_n \mid n \in N\}\big) \cap \{x\} \\ &= f\big(\cap\{A_n \mid n \in N\} \cap f^{-1}(x)\big).\end{aligned}$$

Hence $x \in f\big(\cap\{A_n \mid n \in N\}\big)$ and so $f\big(\cap\{A_n \mid n \in N\}\big) \supseteq \cap\{f(A_n) \mid n \in N\}$. Thus we have the result.

□

A space X is called *a P-space* if every zero-set in X is open in X. The concept of P'-spaces is a generalization of the concept of P-spaces ([6]).

Definition 2. A space X is called *a P'-space* if every zero-set in X is a regular closed sets in X, equivalently, for any non-empty zero set Z in X, $int_X(Z) = \emptyset$.

A space X is called *a weakly Lindelöf space* if every open cover $\mathcal{U}$ of X has a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\cup\{V \mid V \in \mathcal{V}\}$ is dence in X.

We recall that a covering map $f : X \to Y$ is called *$z^{\#}$-irreducible* if $f\big(Z(X)^{\#}\big) = Z(Y)^{\#}$ and that if Y is a weakly Lindelöf space, then $f : X \to Y$ is a $z^{\#}$-irreducible map.

Definition 3. Let X be a space. Then a pair (Y, f) is called

(1) *a cover of* X if $f : X \to Y$ is a covering map,

(2) *a basically disconnected cover of* X if (Y, f) is a cover of X and Y is a basically disconnected space, and

(3) *a minimal basically disconnected cover of* X if (Y, f) is a basically disconnected cover of X and for any basically disconnected cover (Z, g) of X, there is a covering map $h : Z \to Y$ such that $f \circ h = g$.

Vermeer([7]) showed that every space X has a minimal basically disconnected cover $(\Lambda X, \Lambda_X)$ and that if X is a compact space, then ΛX is the Stone-space $S(\sigma Z(X)^{\#})$ of $\sigma Z(X)^{\#}$ and $\Lambda_X(\alpha) = \cap\{A \mid A \in \alpha\}$ $(\alpha \in \Lambda X)$.

Let X be a space. Since $\sigma Z(X)^{\#}$ and $\sigma Z(\beta X)^{\#}$ are Boolean isomorphic, $S(\sigma Z(X)^{\#})$ and $S(\sigma Z(\beta X)^{\#})$ are homeomorphic.

Let X, Y be spaces and $f : Y \to X$ a map. For any $U \subseteq X$, let $f_U : f^{-1}(U) \to U$ denote the restriction and co-restriction of f with respect to $f^{-1}(U)$ and U, respectively.

For any space X, let $\big(\Lambda\beta X, \Lambda_\beta\big)$ denote the minimal basically disconnected cover of βX.

Lemma 2.3. ([3], [5]) *Let X be a space. Then we have the following :*

(1) *if ${\Lambda_\beta}^{-1}(X)$ is al basically disconnected space, then $({\Lambda_\beta}^{-1}(X), \Lambda_{\beta_X})$ is the minimal basically disconnected cover of X, and*

(2) *if $\Lambda_X : \Lambda X \to X$ is $z^\#$-irreducible, then ${\Lambda_\beta}^{-1}(X) = \Lambda X$, $\Lambda_X = \Lambda_{\beta_X}$ and $\beta\Lambda X = \Lambda\beta X$.*

Theorem 2.4. *Let X be a P'-space such that vX is a weaky Lindelöf space. Then ΛvX is a P'-space.*

Proof. Take any zero-set Z in ΛvX such that $\emptyset \neq Z$ and $int_{\Lambda vX}(Z) = \emptyset$. Then there is a continuous function $f : \Lambda vX \to R$ such that $Z = f^{-1}(0)$. For any $n \in N$, let $Z_n = cl_{\Lambda vX}\big(int_{\Lambda vX}(f^{-1}([0, \frac{1}{n}]))\big)$. Then for any $n \in N$, $Z_{n+1} \subseteq int_{\Lambda vX}(Z_n)$ and (Z_n) is a decreasing sequence in $Z(\Lambda vX)^\#$ such that $Z = \cap\{Z_n \mid n \in N\}$. Since Λ_{vX} is a covering map, by Lamma 2.2, $\Lambda_{vX}(Z) = \cap\{\Lambda_{vX}(Z_n) \mid n \in N\}$. Since vX is a weakly Lindelöf space, $\Lambda_{vX} : \Lambda vX \to vX$ is $z^\#$-irreducible and so for any $n \in N$, $\Lambda_{vX}(Z_n) \in Z(vX)^\#$.

Let $n \in N$. Then there exists a zero-set A_n in $Z(vX)^\#$ such that $\Lambda_{vX}(Z_n) = cl_{vX}\big(int_{vX}(A_n)\big)$. Since vX is a P'-space, $cl_{vX}\big(int_{vX}(A_n)\big) = A_n$ and so $\Lambda_{vX}(Z_n) \in Z(vX)$. Hence $\Lambda_{vX}(Z) = \cap\{\Lambda_{vX}(Z_n) \mid n \in N\} \in Z(vX)$. Since $\Lambda_{vX} : \Lambda vX \to vX$ is $z^\#$-irreducible, by Lamma 2.3, ${\Lambda_{vX}}^{-1}(vX) = \Lambda vX$ and $\Lambda_{vX} = \Lambda_{\beta_{vX}}$. Note that

$$\begin{aligned}\Lambda_{vX}(Z \cap \Lambda vX) &= \Lambda_{vX}\big(Z \cap {\Lambda_\beta}^{-1}(vX)\big)\\ &= \Lambda_\beta\big(Z \cap {\Lambda_\beta}^{-1}(vX)\big)\\ &= \Lambda_\beta(Z) \cap vX.\end{aligned}$$

Since $int_{\Lambda vX}(Z) = \emptyset$, $\emptyset = \Lambda_{vX}(Z \cap vX) = \Lambda_{vX}(Z) \cap X = \emptyset$. Since vX is a P'-space and $\Lambda_{vX}(Z) \in Z(vX)$, $\Lambda_{vX}(Z) = \emptyset$ and hence $Z = \emptyset$. This is a contradiction and so $int_{\Lambda vX}(Z) \neq \emptyset$. Therefore vX is a P'-space.

□

It is well-know that a basically disconnected P'-space is a P-space. Using this, we have the following.

Corollary 2.5. *Let X be a P'-space such that X or vX has a dense weakly Lindelöf subspace. Then ΛvX is a P'-space.*

Proof. Suppose that D is a dense weakly Lindelöf subspace of vX. Let $\mathcal{U}$ be an open cover of vX. Then $\mathcal{U}_D = \{U \cap D \mid U \in \mathcal{U}\}$ is an open cover of D. Since D is a weakly Lindelöf space, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\{V \cap D \mid V \in \mathcal{V}\}$ is dense in D. Since D is dense in vX and $\cup\{V \mid V \in \mathcal{V}\}$ is open in vX, $\cup\{V \mid V \in \mathcal{V}\}$ is dense in vX. Hence vX is a weakly Lindelöf space. By Theorem 2.4, ΛvX is a P'-space. □

Suppose that D is a dense weakly Lindelöf space of X. Then similarly we can show that X is a dense weakly Lindelöf subspace of vX.

Theorem 2.6. ([4]) *Let X, Y be spaces such that ${\Lambda_\beta}^{-1}(X) = \Lambda X$ and ${\Lambda_\beta}^{-1}(Y) = \Lambda Y$. If $\Lambda X \times \Lambda Y$ is a basically disconnected space, then $(\Lambda X \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$, where $(\Lambda_X \times \Lambda_Y)(x, y) = \big(\Lambda_X(x) \times \Lambda_Y(y)\big)$.*

A space X is called *a countably locally weakly Lindelöf space* if for any countable collection $\{\mathcal{U}_n \mid n \in N\}$ of open covers of X and for any $x \in X$, there is a neighborhood G of x in X and for any $n \in N$, there is a subfamily $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $G \subseteq cl_X(\cup \mathcal{V}_n)$.

In [1], it was shown that if X is a P-space and Y is a countably locally weakly Lindelöf space, then $X \times Y$ is a basically disconnected space. By Corollary 2.5 and Theorem2.6, we have the following corollary :

Corollary 2.7. *Let X be a P'-space such that X or vX has a weakly Lindelöf dense subspace and Y a countably locally weakly Lindelöf space. Then $(\Lambda vX \times \Lambda Y, \Lambda_X \times \Lambda_Y)$ is the minimal basically disconnected cover of $X \times Y$.*

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