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DYNAMICAL COCYCLES ASSOCIATED WITH CERTAIN NON-MEDIAL LEFT-DISTRIBUTIVE QUASIGROUPS

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Abstract. The purpose of this paper is to explicitly present Andruskiewitsch-Grañ̃a's dynamical cocycles associated with the two known non-medial left-distributive quasigroups of order 15 which are extensions of the dihedral quandle of order 3 by those cocycles.

1. Introduction

In knot theory, quandles were considered by G. Wraith and J. Conway in 1959 as a generalization of a group with the binary operation given by conjugation, and further developed by D. Joice [7] in 1980 for invariants of knots. In particular, connected finite quandles receive attentions for generalization of the classical Fox's n-colorings of knots [12].

A family of connected finite quandles were already investigated in the other area of mathematics with terms such as distributive (both left and right) or left-distributive quasigroups which include all connected finite Alexander quandles, a major class of finite quandles in knot theory.

Since it is known that for order < 81 any distributive quasigroup is necessarily medial (c.f. [10] or [8]), one may ask if there are non-medial left-distributive quasigroups of order < 81. We take this opportunity to keep tracking of the status of the above question. Around 2008, Baik-Sim-Song circulated an unpublished manuscript [2] reporting existence of two non-medial left-distributive quasigroups of order 15. One of them was referred to as Galkin and denoted by G_{15}; the other Stanovsky and S_{15}. Indeed Galkin [5] showed G_{15} in his survey article of quasigroups. On the other hand Stanovsky [11] constructed S_{15} by using a computer. Later on Clark and et al.[3] extended Galikin's construction; in particular they pointed out that there are only two Galkin type quasigroups $G[Z_5, [0])$ and $G[Z_5, [1])$ of order 15. Finally Vendramin [14] and Vlachy [15]

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independently proved that there are no more non-medial left-distributive quasigroups of order 15 except for $G[Z_5, [0])$ and $G[Z_5, [1])$. In his survey article [4] Elhamdadi reviewed on the progress mentioned in the above. The purpose of this paper is to explicitly present Andruskiewitsch-Graña's dynamical cocycles [1] associated with $G_{15} = G(Z_5, [0])$ and $S_{15} = G(Z_5, [1])$, which are extensions of the dihedral quandle of order 3 (referred to as the Tait quandle for short) by those cocycles.

2. Preliminaries

In this paper we adopt the left-handed definition of a quandle in order to be in concordance with treatment of previous results used in the work on which our research is based.

We recall some terminologies for binary systems.

A groupoid $(X, *)$, a non-empty set X with a binary operation $*$, is said to be:

(1) *idempotent* if for each $x \in X$ $x * x = x$,

(2) *left-invertible* if for each $x \in X$ the function $L_x : X \to X$ defined by $L_x(y) = x * y (y \in X)$ is bijective,

(3) *right-invertible* if for each $y \in X$ the function $R_y : X \to X$ defined by $R_y(x) = x * y (x \in X)$ is bijective,

(4) *left-distributive* if for each $x, y, z \in X$ $x * (y * z) = (x * y) * (x * z)$,

(5) *right-distributive* if for each $x, y, z \in X$ $(x * y) * z = (x * z) * (y * z)$,

(6) *distributive* if both left- and right-distributive,

(7) *medial* if for each $x, y, z, w \in X$ $(x * y) * (z * w) = (x * z) * (y * w)$.

An idempotent, left-invertible and left-distributive groupoid is called a *quandle*, and a left- as well as right- invertible groupoid is called a *quasigroup*.

For a quandle $(Q, *)$, we call each bijection L_x, $(x \in Q)$ a *left-translation* which is an automorphisms of Q due to left-distributivity. The group generated by $\{L_x | x \in Q\}$ is called the inner automorphism group of Q and denoted by $Inn_*(Q)$. A quandle $(Q, *)$ is said to be *connected* if $Inn_*(Q)$ acts transitively on Q, i.e., for each $x, y \in Q$ there exists $\phi \in Inn_*(Q)$ such that $\phi(x) = y$.

In the sequel we assume that all groupoids we deal with are finite. From Lemma 2.1 to Theorem 2.4 we recall some well-known facts in quasigroup theory.

Lemma 2.1. *If $(X, *)$ is a left-distributive quasigroup then, $(X, *)$ is idempotent.*

Remark. A left-distributive quasigroup is referred to as a *Latin quandle* in some literatures due to the fact that a multiplication table of a quasigroup of order n constitutes a *Latin square*, namely an $n \times n$ array (matrix in which each of the n^2 cells contain a number from $I = \{1, 2, \cdots, n\}$ so that each number of I occurs just once in each row and once in each column.

Lemma 2.2. *If $(X, *)$ is a medial idempotent quasigroup, then $(X, *)$ is a distributive quasigroup.*

However the converse is not true in general as stated in the introduction.

Lemma 2.3. *Let $A = (A, +)$ be an abelian group with two commuting automorphisms f, g of A and let c be a fixed element of A. Then for the binary operation $*$ defined by $x * y = f(x) + g(y) + c$, $(A, *)$ is a medial quasigroup.*

Conversely, we have the Toyoda representation theorem [13].

Theorem 2.4. *Let $(Q, *)$ be a medial quasigroup, then there exists a abelian group $(Q, +)$, two commuting automorphisms f, g of $(Q, +)$ and a fixed element of Q such that $x * y = f(x) + g(y) + c$.*

Lemma 2.5. *Let A be an abelian group and g be an automorphism of A. Then for an Alexander quandle $(A, *)$ defined by*

$$a * b = (I - g)(a) + g(b), (a, b \in A),$$

where I denotes the identity automorphism of A, the following statements are all equivalent.
(i) *$(A, *)$ is a quasigroup.*
(ii) *$(A, *)$ is connected.*
(iii) *$f = I - g$ is an automorphism of A.*

Proof. Implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (i)$ are elementary.
$(ii) \Rightarrow (iii)$: Let 0 be the neutral element of $(A, +)$. For an arbitrary element b of A consider an automorphism $\phi \in Inn_*(A)$ such that $\phi(0) = b$. Then there exist $a_1, a_2, \cdots, a_n \in A$ such that $\phi = L_{a_1} L_{a_2} \cdots L_{a_n}$. Hence we have

$$b = \phi(0) = a_1 * (a_2 * \cdots * (a_n * 0) \cdots) \in 0 + f(A).$$

Thus f is an epimorphism and hence an automorphism of A since A is finite. □

Corollary 2.6. *Any connected Alexander quandle is a medial idempotent quasigroup and vice versa.*

Proof. It is easy to verify mediality and idempotency of an Alexander quandle. Hence one direction is followed from Lemma 2.5. From the Toyoda representation theorem and idempotency of $(Q, *)$ we have

$$0 = 0 * 0 = f(0) + g(0) + c;\ c = 0.$$

Then applying idempotency of $(Q, *)$ once more, we see that

$$a = a * a = f(a) + g(a)\ \forall a \in Q;\ f = I - g.$$

Thus we have the other direction. □

The last part of this section is a brief introduction of Andruskiewitsch-Grañ*a*' extension theory of quandles by dynamical cocycles. For more details see [1].

For a quandle $(X, \cdot)$ and a non-empty set S, a function $\alpha : X \times X \to S^{S\times S}$ is said to be a dynamical cocycle if the following conditions hold for $\alpha_{xy} = \alpha(x, y)$:

(i) $\alpha_{xx}(a, a) = a$ for all $x \in X$ and $a \in S$

(ii) $\alpha_{xy}(a, -) : S \to S$ is bijection for all $x, y \in X$ and for all $a \in S$

(iii) $\alpha_{x(y\cdot z)}(a, \alpha_{yz}(b, c)) = \alpha_{(x\cdot y)(x\cdot z)}(\alpha_{xy}(a, b), \alpha_{xz}(a, c))$ for all $x, y, z \in X$ and for all $a, b, c \in S$

Given a quandle $(X, \cdot)$ with a dynamical cocycle α, by defining a binary operation $*$ on a set $X \times S$ so that

$$(x, a) * (y, b) = (x \cdot y, \alpha_{xy}(a, b)) \ \forall (x, a), (y, b) \in X \times S$$

we have a quandle which is called an extension of X by a dynamical 2-cocycle α and denoted by $X \times_\alpha S$.

For an extension $E = X \times_\alpha S$, $X = (X, \cdot)$ is called the *base* (of E). And we have the natural projection $\pi : E \to X$ defined by $\pi(x, a) = x$ ($x \in X,\ a \in S$), which is a quandle homomorphism. Then for each $x \in X$ a subquandle $E_x = \pi^{-1}(x)$ of E is called a *fiber* (of E).

The following lemma may be utilized to see if a given quandle $(E, *)$ is extended from a quandle $(X, \cdot)$ by a dynamical cocycle α.

Lemma 2.7. [1] *Let $(E, *)$ be a quandle satisfying following conditions:*

(i) *E is a disjoint union $E = \cup_{x\in X} E_x$ for a non-empty set X,*

(ii) *a binary operation $\cdot$ is defined on X so that $E_x * E_y = E_{x\cdot y}$ for all $x, y \in X$,*

(iii) *$|card(E_x)| = |card(E_y)|$ for all $x, y \in X$.*

Then $(X, \cdot)$ is a quandle. Furthermore, let S be a set such that $card(S) = card(E_x)$ and take a bijection $g_x : E_x \to S$ for each $x \in X$. Then a function $\alpha : X \times X \to S^{S\times S}$ defined by

$$\alpha_{xy}(a, b) = g_{x\cdot y}(g_x^{-1}(a) * g_y^{-1}(y))$$

is a dynamical cocycle and $E \cong X \times_\alpha S$

3. Main results

3.1. A dynamical cocycle associated with the Galkin quandle G_{15}

For examples of non-medial left-distributive quasigroups, in his survey article [5] Galkin considered a binary operation on a set $G_{3p} = \mathbf{Z}_p \times \mathbf{Z}_3$ defined by

$$(a, x) \circ (b, y) = (\mu(-x + y)a - b + \tau(-x + y), -x - y)$$

where a function $\mu : \mathbf{Z}_3 \to \mathbf{Z}_5$ is defined so that $\mu(x) = 2$ for $x = 0$, $\mu(x) = -1$ for $x \neq 0$, and a function $\tau : \mathbf{Z}_3 \to \mathbf{Z}_p$ is defined so that $\tau(0) = 0$. In particular he showed that for odd prime p, the above quandles are indeed non-medial left-distributive quasigroups, We call them Galkin quandles. One notices that the function τ is rather ambiguously defined. Unfortunately, we could not access to his unpublished Russian paper [6] possibly containing more accurate information regarding τ. Indeed Clark and et al. [3] more rigorously denoted Galkin quandles by $G(Z_p, c_1, c_2)$ where $c_i = \tau(i)$ for $i = 1, 2$. Then they showed that $G(Z_p, c_1, c_2)$ is isomorphic to $G[Z_p, 0, c_2 - c_1)$ which is denoted by $G(Z_p, c_2 - c_1)$ for short. Furthermore they proved that up to isomorphism there are two Galkin quandles of order 15; $G(Z_5, [0])$ and $G(Z_5, [1])$ where $[c]$ denotes the modulo class of c with respect to p. In [2] we took $c_1 = c_2 = 0$ for a Galkin quandle G_{15} which corresponds to $G(Z_5, [0])$.

To see that G_{15} is indeed an extension of the Tait quandle by a dynamical cocycle. We work with a multiplication table of G_{15}. Although our strategy for a proof of the following theorem is based on Lemma 2.7 we do not assume left-distributivity of G_{15} for a self-contained proof. We derive a function $\alpha : X \times X \to S^{S\times S}$ from the multiplication table of G_{15} and show that α is indeed a dynamical cocyle associated with an extension of the Tait quandle.

Theorem 3.1. *Let $G_{15} = (\mathbf{Z}_{15}, *)$ be a quasigroup defined by a binary operation in TABLE 1. Then G_{15} is an extension of the Tait quandle by a dynamical cocycle.*

*	0	3	6	9	12	1	4	7	10	13	2	5	8	11	14
0	0	12	9	6	3	2	14	11	8	5	1	13	10	7	4
3	6	3	0	12	9	14	11	8	5	2	13	10	7	4	1
6	12	9	6	3	0	11	8	5	2	14	10	7	4	1	13
9	3	0	12	9	6	8	5	2	14	11	7	4	1	13	10
12	9	6	3	0	12	5	2	14	11	8	4	1	13	10	7
1	2	14	11	8	5	1	13	10	7	4	0	12	9	6	3
4	14	11	8	5	2	7	4	1	13	10	12	9	6	3	0
7	11	8	5	2	14	13	10	7	4	1	9	6	3	0	12
10	8	5	2	14	11	4	1	13	10	7	6	3	0	12	9
13	5	2	14	11	8	10	7	4	1	13	3	0	12	9	6
2	1	13	10	7	4	0	12	9	6	3	2	14	11	8	5
5	13	10	7	4	1	12	9	6	3	0	8	5	2	14	11
8	10	7	4	1	13	9	6	3	0	12	14	11	8	5	2
11	7	4	1	13	10	6	3	0	12	9	5	2	14	11	8
14	4	1	13	10	7	3	0	12	9	6	11	8	5	2	14

TABLE 1 : the Galkin quandle G_{15}

Proof. Let $E = G_{15}$ and $E_x = \{x, 3 + x, 6 + x, 9 + x, 12 + x\}$ for each $x \in \mathbf{Z}_3$. And take X in Lemma 2.7 as the Tait quandle $R_3 = (\mathbf{Z}_3, \cdot)$ which has a multiplication table:

	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

TABLE 2: the Tait quandle R_3

Further let $S = \mathbf{Z}_5$ and take bijections $g_x : E_x \to S$ in Lemma 2.7 so that $g_x(x) = 0, g_x(3+x) = 1, g_x(6+x) = 2, g_x(9+x) = 3$ and $g_x(12+x) = 4$. Then one can easily see that the condition (ii) of Lemma 2.7 holds for G_{15}. Moreover the function $\alpha : X \times X \to S^{S\times S}$ defined by $\alpha_{xy}(a,b) = g_{x\cdot y}(g_x^{-1}(a) * g_y^{-1}(y))$ can be explicitly determined as shown in TABLE 3.

	$g_0^{-1}(E_0)$	$g_1^{-1}(E_1)$	$g_2^{-1}(E_2)$
$g_0^{-1}(E_0)$	$\alpha_{00}(a,b) = 2a - b$	$\alpha_{01}(a,b) = -a - b$	$\alpha_{02}(a,b) = -a - b$
$g_1^{-1}(E_1)$	$\alpha_{10}(a,b) = -a - b$	$\alpha_{11}(a,b) = 2a - b$	$\alpha_{12}(a,b) = -a - b$
$g_2^{-1}(E_2)$	$\alpha_{20}(a,b) = -a - b$	$\alpha_{21}(a,b) = -a - b$	$\alpha_{22}(a,b) = 2a - b$

TABLE 3

Note for instance that for each $x \in R_3$, $(\mathbf{Z}_5, \alpha_{xx})$ is R_5, the dihedral quandle of order 5.

Now we show that α is indeed a dynamical cocycle. It is enough to check that the identity

$$\alpha_{x(y\cdot z)}(a, \alpha_{yz}(b,c)) = \alpha_{(x\cdot y)(x\cdot z)}(\alpha_{xy}(a,b), \alpha_{xz}(a,c))$$

holds for every $x, y, z \in R_3$ and $a, b, c \in \mathbf{Z}_5$. Put the left and right hand side of above identity (LHS) and (RHS) respectively.

CASE 1: $x \neq y$ and $y \neq z$

$$\begin{aligned} (LHS) &= \alpha_{xx}(a, -b-c) \\ &= 2a + b + c \\ (RHS) &= \alpha_{xy}(-a-b, -a-c) \\ &= 2a + b + c \end{aligned}$$

CASE 2: $x \neq y$ and $y = z$

$$\begin{aligned} (LHS) &= \alpha_{xy}(a, \alpha_{yy}(b,c)) \\ &= \alpha_{xy}(a, 2b - c) \\ &= -a - 2b + c \\ (RHS) &= \alpha_{(x\cdot y)(x\cdot y)}(\alpha_{xy}(a,b), \alpha_{xy}(a,c)) \\ &= \alpha_{(x\cdot y)(x\cdot y)}(-a-b, -a-c) \\ &= -a - 2b + c \end{aligned}$$

CASE 3: $x = y$ and $y \neq z$

$$\begin{aligned}(LHS) &= \alpha_{x(x\cdot z)}(a, -b-c)\\ &= -a+b+c\\ (RHS) &= \alpha_{x(x\cdot z)}(\alpha_{xx}(a,b), \alpha_{xz}(a,c))\\ &= \alpha_{x(x\cdot z)}(2a-b, -a-c)\\ &= -a+b+c\end{aligned}$$

CASE 4: $x = y = z$

$$\begin{aligned}(LHS) &= \alpha_{xx}(a, \alpha_{xx}(b,c))\\ &= \alpha_{xx}(a, 2b-c)\\ &= 2a-2b+c\\ (RHS) &= \alpha_{xx}(\alpha_{xx}(a,b), \alpha_{xx}(a,c))\\ &= \alpha_{xx}(2a-b, 2a-c)\\ &= 2a-2b+c\end{aligned}$$

□

3.2. A dynamical cocycle associated with the Stanovsky quandle S_{15}

We say that quasigroups $(Q, *)$ and$(R, \circ)$ are *isotopic*, if there are bijections $\alpha, \beta, \gamma : Q \to R$ such that $\alpha(x * y) = \beta(x) \circ \gamma(y)$ for all $x, y \in Q$. Then the Toyoda representation theorem says that a medial quasigroup $(Q, *)$ is isotopic to an abelian group $(Q, +)$.

Likewise, it is known that any distributive quasigroup $(Q, \cdot)$ is isotopic to a commutative Moufang loop $(Q, \circ)$ in such a way that

$$a \circ b = L_e(a) \cdot R_e(b)$$

where e is a fixed element of Q, and L_e, R_e are the left and right translation of e respectively. As for a left-distributive quasigroup it is expected that it is isotopic to so called a Bol loop. But Stanovsky [11] came up with a counter-example to this idea by using a computer as shown in TABLE 4.

His example attracts our attentions because of its non-mediality. Through quandle isomorphisms, we can transform TABLE 4 into TABLE 5 which may be thought of as a kind of a normalized multiplication table for an extension of the Tait quandle with fiber isomorphic to R_5 as in the case of the Galkin quandle G_{15}.

*	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	2	4	1	6	3	8	5	10	7	9	12	11	14	13
1	3	1	0	11	5	4	12	14	9	8	13	7	2	6	10
2	1	12	2	6	0	13	3	10	11	14	7	4	5	9	8
3	5	0	7	3	11	12	10	2	14	13	6	1	9	8	4
4	2	8	11	14	4	9	0	13	1	5	12	3	6	10	7
5	7	9	12	0	8	5	13	11	4	1	14	10	3	2	6
6	4	13	10	7	12	14	6	3	0	11	2	8	1	5	9
7	9	11	6	10	14	0	2	7	13	12	3	5	8	4	1
8	6	5	14	13	9	1	11	12	8	4	0	2	10	7	3
9	10	4	13	12	1	8	14	0	5	9	11	6	7	3	2
10	8	14	3	2	13	11	7	6	12	0	10	9	4	1	5
11	14	3	8	4	2	7	9	1	6	10	5	11	13	12	0
12	13	6	1	5	10	2	4	9	7	3	8	14	12	0	11
13	11	10	5	9	7	6	1	8	3	2	4	0	14	13	12
14	12	7	9	8	3	10	5	4	2	6	1	13	0	11	14

TABLE 4

Theorem 3.2. *Let $S_{15} = (\mathbf{Z}_{15}, *)$ be a quasigroup defined by a binary operation in TABLE 5. Then S_{15} is an extension of the Tait quandle by a dynamical cocycle.*

*	0	3	6	9	12	1	4	7	10	13	2	5	8	11	14
0	0	12	9	6	3	2	11	5	14	8	10	1	7	13	4
3	6	3	0	12	9	5	14	8	2	11	4	10	1	7	13
6	12	9	6	3	0	8	2	11	5	4	13	4	10	1	7
9	3	0	12	9	6	11	5	4	8	2	7	13	4	10	1
12	9	6	3	0	12	14	8	2	11	5	1	7	13	4	10
1	5	8	11	14	2	1	13	10	7	4	0	3	6	9	12
4	14	2	5	8	11	7	4	1	13	10	6	9	12	0	3
7	8	11	14	2	5	13	10	7	4	1	12	0	3	6	9
10	2	5	8	11	14	4	1	13	10	7	3	6	9	12	0
13	11	14	2	5	8	10	7	4	1	13	9	12	0	3	6
2	1	10	4	13	7	12	3	9	0	6	2	14	11	8	5
5	7	1	10	4	13	0	6	12	3	9	8	5	2	14	11
8	13	7	1	10	4	3	9	0	6	12	14	11	8	5	2
11	4	13	7	1	10	6	12	3	9	0	5	2	14	11	8
14	10	4	13	7	1	9	0	6	12	3	11	8	5	2	14

TABLE 5: the Stanovsky quandle S_{15}

Proof. With the notations in the proof of Theorem 3.1, from TABLE 5 we can come up with a function $\alpha : X \times X \to S^{S\times S}$ as shown in TABLE 6.

	$g_0^{-1}(E_0)$	$g_1^{-1}(E_1)$	$g_2^{-1}(E_2)$
$g_0^{-1}(E_0)$	$\alpha_{00}(a,b)=2a-b$	$\alpha_{01}(a,b)=a+3b$	$\alpha_{02}(a,b)=3a+2b+3$
$g_1^{-1}(E_1)$	$\alpha_{10}(a,b)=3a+b+1$	$\alpha_{11}(a,b)=2a-b$	$\alpha_{12}(a,b)=2a+b$
$g_2^{-1}(E_2)$	$\alpha_{20}(a,b)=2a+3b$	$\alpha_{21}(a,b)=a+2b+4$	$\alpha_{22}(a,b)=2a-b$

TABLE 6

The following rutin calculations show that α is indeed a dynamical cocycle.

CASE I) $x \neq y,\ y \neq z$

(1) $x=0,\ y=1,\ z=2\ :\ \alpha_{0(1\cdot 2)}(a,\alpha_{12}(b,c)) = \alpha_{(0\cdot 1)(0\cdot 2)}(\alpha_{01}(a,b),\alpha_{02}(a,c))$

$$\begin{aligned}(LHS) &= \alpha_{00}(a,2b+c)\\ &= 2a+3b+4c\ (\texttt{in}\ \mathbf{Z}_5)\\ (RHS) &= \alpha_{21}(a+3b,3a+2c+3)\\ &= 2a+3b+4c\ (\texttt{in}\ \mathbf{Z}_5)\end{aligned}$$

(2) $x=0,\ y=2,\ z=1\ :\ \alpha_{0(2\cdot 1)}(a,\alpha_{21}(b,c)) = \alpha_{(0\cdot 2)(0\cdot 1)}(\alpha_{02}(a,b),\alpha_{01}(a,c))$

$$\begin{aligned}(LHS) &= \alpha_{00}(a,b+2c+4)\\ &= 2a+4b+3c+1\ (\texttt{in}\ \mathbf{Z}_5)\\ (RHS) &= \alpha_{12}(3a+2b+3,a+3c)\\ &= 2a+4b+3c+1\ (\texttt{in}\ \mathbf{Z}_5)\end{aligned}$$

(3) $x=1,\ y=0,\ z=2\ :\ \alpha_{1(0\cdot 2)}(a,\alpha_{02}(b,c)) = \alpha_{(1\cdot 0)(1\cdot 2)}(\alpha_{10}(a,b),\alpha_{12}(a,c))$

$$\begin{aligned}(LHS) &= \alpha_{11}(a,3b+2c+3)\\ &= 2a+2b+3c+2\ (\texttt{in}\ \mathbf{Z}_5)\\ (RHS) &= \alpha_{20}(3a+b+1,2a+c)\\ &= 2a+2b+3c+2\ (\texttt{in}\ \mathbf{Z}_5)\end{aligned}$$

(4) $x=1,\ y=2,\ z=0\ :\ \alpha_{1(2\cdot 0)}(a,\alpha_{20}(b,c)) = \alpha_{(1\cdot 2)(1\cdot 0)}(\alpha_{12}(a,b),\alpha_{10}(a,c))$

$$\begin{aligned}(LHS) &= \alpha_{11}(a,2b+3c)\\ &= 2a+3b+2c\ (\texttt{in}\ \mathbf{Z}_5)\\ (RHS) &= \alpha_{02}(2a+b,3a+c+1)\\ &= 2a+3b+2c\ (\texttt{in}\ \mathbf{Z}_5)\end{aligned}$$

(5) $x=2,\ y=0,\ z=1\ :\ \alpha_{2(0\cdot 1)}(a,\alpha_{01}(b,c)) = \alpha_{(2\cdot 0)(2\cdot 1)}(\alpha_{20}(a,b),\alpha_{21}(a,c))$

$$\begin{aligned}(LHS) &= \alpha_{22}(a,b-2c)\\ &= 2a+4b+2c\ (\texttt{in}\ \mathbf{Z}_5)\\ (RHS) &= \alpha_{10}(2a+3b,a+2c+4)\\ &= 2a+4b+2c\ (\texttt{in}\ \mathbf{Z}_5)\end{aligned}$$

(6) $x = 2,\ y = 1,\ z = 0\ :\ \alpha_{2(1\cdot 0)}(a, \alpha_{10}(b, c)) = \alpha_{(2\cdot 1)(2\cdot 0)}(\alpha_{21}(a, b), \alpha_{20}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{22}(a, 3b + c + 1)\\ &= 2a + 2b + 4c + 4 \text{ (in } \mathbf{Z}_5)\\ (RHS) &= \alpha_{01}(a + 2b + 4, 2a + 3c)\\ &= 2a + 2b + 4c + 4 (\text{in } \mathbf{Z}_5)\end{aligned}$$

CASE II) $x = y,\ y \neq z$

(1) $x = y = 0,\ z = 1\ :\ \alpha_{0(0\cdot 1)}(a, \alpha_{01}(b, c)) = \alpha_{(0\cdot 0)(0\cdot 1)}(\alpha_{00}(a, b), \alpha_{01}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{02}(a, b + 3c)\\ &= 3a + 2b + c + 3 \text{ (in } \mathbf{Z}_5)\\ (RHS) &= \alpha_{02}(2a - b, a + 3c)\\ &= 3a + 2b + c + 3 \text{ (in } \mathbf{Z}_5)\end{aligned}$$

(2) $x = y = 0,\ z = 2\ :\ \alpha_{0(0\cdot 2)}(a, \alpha_{02}(b, c)) = \alpha_{(0\cdot 0)(0\cdot 2)}(\alpha_{00}(a, b), \alpha_{02}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{01}(a, 3b + 2c + 3)\\ &= a + 4b + c + 4 \text{ (in } \mathbf{Z}_5)\\ (RHS) &= \alpha_{01}(2a - b, 3a + 2c + 3)\\ &= a + 4b + c + 4 \text{ (in } \mathbf{Z}_5)\end{aligned}$$

(3) $x = y = 1,\ z = 0\ :\ \alpha_{1(1\cdot 0)}(a, \alpha_{10}(b, c)) = \alpha_{(1\cdot 1)(1\cdot 0)}(\alpha_{11}(a, b), \alpha_{10}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{12}(a, 3b + c + 1)\\ &= 2a + 3b + c + 1 \text{ (in } \mathbf{Z}_5)\\ (RHS) &= \alpha_{12}(2a - b, 3a + c + 1)\\ &= 2a + 3b + c + 1 \text{ (in } \mathbf{Z}_5)\end{aligned}$$

(4) $x = y = 1,\ z = 2\ :\ \alpha_{1(1\cdot 2)}(a, \alpha_{12}(b, c)) = \alpha_{(1\cdot 1)(1\cdot 2)}(\alpha_{11}(a, b), \alpha_{12}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{10}(a, 2b + c)\\ &= 3a + 2b + c + 1 \text{ (in } \mathbf{Z}_5)\\ (RHS) &= \alpha_{10}(2a - b, 2a + c)\\ &= 3a + 2b + c + 1 \text{ (in } \mathbf{Z}_5)\end{aligned}$$

(5) $x = y = 2,\ z = 0\ :\ \alpha_{2(2\cdot 0)}(a, \alpha_{20}(b, c)) = \alpha_{(2\cdot 2)(2\cdot 0)}(\alpha_{22}(a, b), \alpha_{20}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{21}(a, 2b + 3c)\\ &= a + 4b + c + 4 \text{ (in } \mathbf{Z}_5)\\ (RHS) &= \alpha_{21}(2a - b, 2a + 3c)\\ &= a + 4b + c + 4 \text{ (in } \mathbf{Z}_5)\end{aligned}$$

(6) $x = y = 2,\ z = 1\ :\ \alpha_{2(2\cdot 1)}(a, \alpha_{21}(b, c)) = \alpha_{(2\cdot 2)(2\cdot 1)}(\alpha_{22}(a, b), \alpha_{21}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{20}(a, b + 2c + 4)\\ &= 2a + 3b + c + 2 \ (\texttt{in}\ \mathbf{Z}_5)\\ (RHS) &= \alpha_{20}(2a - b, a + 2c + 4)\\ &= 2a + 3b + c + 2 \ (\texttt{in}\ \mathbf{Z}_5)\end{aligned}$$

CASE III) $x = y = z\ :\ \alpha_{x(x\cdot x)}(a, \alpha_{xx}(b, c)) = \alpha_{(x\cdot x)(x\cdot x)}(\alpha_{xx}(a, b), \alpha_{xx}(a, c))$

$$\begin{aligned}(LHS) &= \alpha_{xx}(a, 2b - c)\\ &= 2a + 3b + c \ (\texttt{in}\ \mathbf{Z}_5)\\ (RHS) &= \alpha_{xx}(2a - b, 2a - c)\\ &= 2a + 3b + c \ (\texttt{in}\ \mathbf{Z}_5)\end{aligned}$$

□

Using a computer programming GAP, Vendramin [14] classified connected quandles of orders ≤ 32, The j-th quandle of order i is denoted by $C[i, j]$.

Theorem 3.3. *The quandle G_{15} is isomorphic to $C[15, 6] = G(Z_5, [0])$, and the quandle S_{15} is isomorphic to $C[15, 5] = G(Z_5, [1])$.*

Proof. There are 7 connected quandles of order 15 consisting of 3 Alexander quandles: $C[15, 1] = Z_{15}[t]/(t + 1)$, $C[15, 3] = Z_{15}[t]/(t + 7)$, $C[15, 4] = Z_{15}[t]/(t+13)$, two non-medial quasigroups $C[15, 6] = G(Z_5, [0])$ and $C[15, 5] = G(Z_5, [1])$ and two non-medial, non-quasigroups $C[15, 2]$, $C[15, 7]$. Furthermore $C[15, 6] = G(Z_5, [0])$ is involutive but $C[15, 5] = G(Z_5, [1])$ is not. Thus the claims are followed from the following observations (a) and (b):

(a) Non-medialty of G_{15} and S_{15}: Take $x = 0, y = 3, z = 1$ and $w = 2$. Then from TABLE 1 of G_{15}, we have

$$(x * y) * (z * w) = (0 * 3) * (1 * 2) = 9 \neq 3 = (0 * 1) * (3 * 2) = (x * z) * (y * w)$$

Likewise from TABLE 5 of S_{15}, we have

$$(x * y) * (z * w) = (0 * 3) * (1 * 2) = 9 \neq 3 = (0 * 1) * (3 * 2) = (x * z) * (y * w).$$

(b) Isomorphic invariance of types of cycles in the disjoint cyclic decomposition of a left-translation [9]: Indeed each left-translation of G_{15} is an involution consisting of 7-disjoint transpositions whereas that of S_{15} consists of a single 10-cycle and two transpositions which are mutually disjoint. □

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