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EXISTENCE OF PROPER CONTACT CR PRODUCT AND MIXED FOLIATE CONTACT CR SUBMANIFOLDS OF $E^{2m+1}(-3)$

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Abstract. The first purpose of this paper is to study contact CR submanifolds of Sasakian manifolds and investigate some properties concernig with ϕ-holomorphic bisectional curvature. The second purpose is to show an existence theorem of mixed foliate proper contact CR submanifolds in the standard Sasakian space form $E^{2m+1}(-3)$ with constant ϕ-sectional curvature -3.

1. Introduction

A submanifold M^{n+1} of a Sasakian manifold $\bar{M}^{2m+1}$ with structure tensors $(\phi,\ \xi,\ \eta,\ g)$ is called a *contact CR submanifold* if there exists two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M such that

$(a)\ TM = \mathcal{D}\oplus\mathcal{D}^{\perp}\oplus\mathrm{Span}\{\xi\}$ and $(b)\ \phi\mathcal{D}_x = \mathcal{D}_x,\ \phi\mathcal{D}_x^{\perp} \subset T_xM^{\perp}$ for each $x \in M$,

where $\mathcal{D}, \mathcal{D}^{\perp}$ and $\mathrm{Span}\{\xi\}$ are mutually orthogonal to each other. A contact CR submanifold is said to be *proper* if neither $\dim\mathcal{D} = 0$ nor $\dim\mathcal{D}^{\perp} = 0$. A contact CR submanifold is said to be *mixed foliate* if

$(a)\ \mathcal{D} \oplus \mathrm{Span}\{\xi\}$ is integrable and $(b)\ h(X, Y) = 0,\ X \in \mathcal{D},\ Y \in \mathcal{D}^{\perp}$,

where h is the second fundamental form of M. A contact CR submanifold M is called a *contact CR product* if

$(a)\ \mathcal{D}\oplus\mathrm{Span}\{\xi\}$ is integrable and $(b)\ M$ is locally a Riemannain product $M^{\top}\times M^{\perp}$,

where $M^{\top}$and $M^{\perp}$ are leafs of $\mathcal{D} \oplus \mathrm{Span}\{\xi\}$ and $\mathcal{D}^{\perp}$, respectively.

In 1986, Bejancu[1] proved that there is no proper contact CR product in Sasakian space form $\bar{M}(c)$ with constant ϕ-sectional curvature $c < -3$.

The first purpose of this paper is to study contact CR submanifolds of Sasakian manifolds and to investigate some properties concernig with ϕ-holomorphic bisectional curvature $\bar{H}_B$ and prove Theorem A which yields Bejancu's result[1].

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The second purpose is to show Theorem B as an existence theorem of mixed foliate proper contact CR submanifolds in the standard Sasakian space form $E^{2m+1}(-3)$ with constant ϕ-sectional curvature $c = -3$.

Theorem A. Let $\bar{M}$ be a Sasakian manifold with $\bar{H}_B < -2$. Then every contact CR product in $\bar{M}$ is either an invariant submanifold or an anti-invariant submanifold. In other words, there exists no proper contact CR product in any Sasakian manifold with $\bar{H}_B < -2$.

Theorem B. Let M be a mixed foliate proper contact CR submanifold of the standard Sasakian space form $E^{2m+1}(-3)$. If

$$h(X, Y) \in \phi\mathcal{D}^{\perp},\ X, Y \in \mathcal{D}^{\perp},$$

then for a point $x \in M$ there exists a unique complete totally geodesic invariant submanifold M' of $E^{2m+1}(-3)$ such that $x \in M'$ and $T_xM' = T_xM \oplus \phi\mathcal{D}_x^{\perp}$.

2. Submanifolds of Sasakian manifold

Let $\bar{M}$ be a $(2m+1)$-dimensional Sasakian manifold with structure tensors (ϕ, ξ, η, g). Then, by definition(cf. [2], [3], [7], [8], [9]), the structure tensors satisfy

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any vector fields X, Y tangent to $\bar{M}$. Moreover, denoting by $\bar{\nabla}$ the operator of covariant differentiation with respect to the metric g on $\bar{M}$, $\bar{M}$ also satisfy

$$\bar{\nabla}_X \xi = \phi X, \quad (\bar{\nabla}_X \phi)Y = \bar{R}(X, \xi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.2}$$

where $\bar{R}$ denotes the Riemannian curvature tensor of $\bar{M}$.

Let M be an $(n+1)$-dimensional submanifold isometrically immersed in $\bar{M}$ tangent to the structure vector field ξ. We denote by the same g the Riemannian metric tensor field induced on M from that of $\bar{M}$. The operator of covariant differentiation with respect to the induced connection on M will be denoted by ∇. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{2.3}$$

for any vector fields X, Y tangent to M and any vector field V normal to M, where $\nabla^{\perp}$ denotes the operator of covariant differentiation with respect to the connection induced in the normal bundle $TM^{\perp}$ of M. h and A_V appeared in (2.3) are called the *second fundamental form* of M and the *shape operator* in the direction of V, respectively and they are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.4}$$

If the second fundamental form h vanishes identically, then M is said to be *totally geodesic*. The covariant derivative $\nabla_X h$ of h is defined to be

$$(\nabla_X h)(Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{2.5}$$

and the covariant derivative $\nabla_X A$ of A is defined to be

$$(\nabla_X A)_V Y = \nabla_X(A_V Y) - A_{\nabla^{\perp}_X V} Y - A_V \nabla_X Y.$$

Let R and $R^{\perp}$ be the Riemannian curvature tensor field of M and the curvature tensor field of the normal bundle $TM^{\perp}$ of M, respectively. Then we have equations of Gauss, Codazzi and Ricci respectively

$$g(\bar{R}(X,Y)Z,W) = g(R(X,Y)Z,W) - g(h(X,W),h(Y,Z)) + g(h(Y,W),h(X,Z)), \tag{2.6}$$

$$(\bar{R}(X,Y)Z)^{\perp} = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.7}$$

$$g(\bar{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V) + g([A_V, A_U]X,Y) \tag{2.8}$$

for any tangent vector fields X, Y, Z, W and any normal vector fields U, V to M, where $(\bar{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\bar{R}(X,Y)Z$(cf. [1], [4], [8], [9]).

For any vector field X tangent to M, we put

$$\phi X = PX + FX, \tag{2.9}$$

where PX is the tangential part and FX the normal part of ϕX. Then P is an endomorphism on the tangent bundle TM and F is a normal bundle valued 1-form on TM. Similarly, for any vector field V normal to M, we put

$$\phi V = tV + fV, \tag{2.10}$$

where tV is the tangential part and fV the normal part of ϕV. Then f is an endomorphism of the normal bundle $TM^{\perp}$, and t is a tangent bundle valued 1-form on $TM^{\perp}$. For any vector fields X, Y tangent to M, $g(\phi X, Y) = g(PX, Y)$ because of (2.9) and consequently $g(PX, Y)$ is skew-symmetric. Similarly, for any vector fields U, V normal to M, (2.10) yields $g(\phi V, U) = g(fV, U)$ and hence $g(fV, U)$ is also skew-symmetric. From (2.9) and (2.10) we also have the relation between F and t such that

$$g(FX, V) = -g(X, tV) \tag{2.11}$$

for any tangent vector field X and any normal vector field V to M. Since the structure vector field ξ is assumed to be tangent to M, it follows immediately from (2.1) and (2.9) that

$$P\xi = 0, \quad F\xi = 0. \tag{2.12}$$

Now applying ϕ to (2.9) and using (2.1), (2.9) and (2.10), we have

$$P^2 = -I - tF + \eta \otimes \xi, \quad FP + fF = 0. \tag{2.13}$$

Similarly, applying ϕ to (2.10) and using (2.1), (2.9) and (2.10), we find

$$Pt + tf = 0, \quad f^2 = -I - Ft. \tag{2.14}$$

On the other hand, from (2.2) and (2.3), it follows that

$$\bar{\nabla}_X \xi = \phi X = \nabla_X \xi + h(X, \xi)$$

for any vector field X tangent to M, and this combined with (2.3), (2.9) and (2.10) implies

$$\nabla_X \xi = PX, \quad FX = h(X, \xi), \quad A_V \xi = -tV \tag{2.15}$$

where V is a vector field normal to M.

Differentiating (2.9) covariantly along M and using (2.2) and (2.3), we can easily obtain

$$(\nabla_X P)Y = -g(X,Y)\xi + \eta(Y)X + A_{FY}X + th(X,Y), \tag{2.16}$$

$$(\nabla_X F)Y = fh(X,Y) - h(X,PY) \tag{2.17}$$

for any tangent vector fields X, Y, where we have defined $(\nabla_X P)Y$ and $(\nabla_X F)Y$ respectively by

$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y), \quad (\nabla_X F)Y = \nabla^{\perp}_X(FY) - F(\nabla_X Y).$$

Similarly, for any vector field X tangent to M and any vector field V normal to M, we have from (2.10)

$$(\nabla_X t)V = A_{fV}X - PA_V X, \quad (\nabla_X f)V = -FA_V X - h(X, tV) \tag{2.18}$$

with the aid of (2.2) and (2.3), where we have defined $(\nabla_X t)V$ and $(\nabla_X f)V$ respectively by

$$(\nabla_X t)Y = \nabla_X(tV) - t(\nabla^{\perp}_X V), \quad (\nabla_X f)V = \nabla^{\perp}_X(fV) - f(\nabla^{\perp}_X V)$$

(cf. [1], [8]).

3. Contact CR submanifolds in Sasakian manifolds

Let $\bar{M}$ be a real $(2m+1)$-dimensional Sasakian manifold with structure tensors $(\phi,\ \xi,\ \eta,\ g)$.

Definition 1. Let M be a real $(n+1)$-dimensional submanifold isometrically immersed in $\bar{M}$ tangent to the structure vector field ξ. Then M is called a *contact CR submanifold* (or *semi-invariant submanifold* (cf. [2])) of $\bar{M}$ if there exist two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M satisfying the following conditions:

(a) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \mathrm{Span}\{\xi\}$,
where $\mathcal{D}, \mathcal{D}^{\perp}$ and $\mathrm{Span}\{\xi\}$ are mutually orthogonal to each other,

(b) the distribution $\mathcal{D}$ is invariant by ϕ, that is, $\phi\mathcal{D}_x = \mathcal{D}_x$ for each $x \in M$, and

(c) the distribution $\mathcal{D}^{\perp}$ is anti-invariant by ϕ, that is, $\phi\mathcal{D}^{\perp}_x \subset T_x M^{\perp}$ for each $x \in M$.

It is well known (cf. [1], [3]) that for a contact CR submanifold of a Sasakian manifold the following relations are established

$$FP = 0, \quad fF = 0, \quad tf = 0, \quad Pt = 0. \tag{3.1}$$

Remark 1. Let M be a contact CR submanilfold of a Sasakian manifold $\bar{M}$. If $\dim\mathcal{D}^\perp = 0$ (resp. $\dim\mathcal{D} = 0$), then M is an invariant(resp. anti-invariant) submanifold of $\bar{M}$. If $\dim\mathcal{D}^\perp = \dim TM^\perp$, then M is a *generic submanifold* of $\bar{M}$. In particular, a contact CR submanifold is said to be *proper* if neither $\dim\mathcal{D} = 0$ nor $\dim\mathcal{D}^\perp = 0$.

For a contact CR submanifold M of $\bar{M}$, (2.2) and (2.3) give

$$\bar{\nabla}_X(\phi Z) = -A_{\phi Z}X + \nabla^\perp_X \phi Z,$$

and

$$\bar{\nabla}_X(\phi Z) = -g(X,Z)\xi + \eta(Z)X + \phi(\nabla_X Z) + \phi h(X,Z) \tag{3.2}$$

for X, Z tangent to M Thus we obtain

$$\phi(\nabla_X Z) + \phi h(X,Z) - g(X,Z)\xi = -A_{\phi Z}X + \nabla^\perp_X \phi Z \tag{3.3}$$

for X tangent to M and $Z \in \mathcal{D}^\perp$.

From now on we shall give some basic lemmas for later use.

Lemma 3.1. *Let M be a contact CR submanifold of $\bar{M}$. Then we have*

$$g(\nabla_Y Z, X) = g(\phi A_{\phi Z}Y, X), \tag{3.4}$$

$$A_{FZ}W = A_{FW}Z, \tag{3.5}$$

$$A_{FN}X = -A_N PX \tag{3.6}$$

for any vector field Y tangent to M, $X \in \mathcal{D}$, $Z, W \in \mathcal{D}^\perp$, and $N \in \nu$.

Proof. Applying ϕ to (3.3), we obtain

$$\nabla_Y Z + h(Y,Z) - \eta(\nabla_Y Z)\xi = \phi A_{\phi Z}Y - \phi\nabla^\perp_Y \phi Z,$$

and consequently

$$g(\nabla_Y Z + h(Y,Z) - \eta(\nabla_Y Z)\xi, X) = g(\phi A_{\phi Z}Y, X) + g(\nabla^\perp_Y \phi Z, \phi X).$$

Thus we have (3.4).

Next, we will show (3.5). For $Z, W \in \mathcal{D}^\perp$, $PZ = PW = 0$ and hence, for any vector field Y tangent to M

$$g((\nabla_Y P)Z, W) = g(\nabla_Y(PZ), W) - g(P(\nabla_Y Z), W) = 0.$$

Therefore, (2.16) implies

$$\begin{aligned} 0 &= g((\nabla_Y P)Z, W) \\ &= g(-g(Y,Z)\xi + \eta(Z)Y + A_{\phi Z}Y + th(Y,Z), W) \\ &= g(Y, A_{FZ}W) - g(h(Y,Z), FW), \end{aligned}$$

which then implies (3.5).

On the other hand, it clear that from (2.3) and (3.2)

$$\begin{aligned} g(h(PX,Y),N) &= g(\bar{\nabla}_Y(PX), N) - g(\nabla_Y PX, N) \\ &= g(P(\nabla_Y X), N) + g(Fh(Y,X), N) \\ &= -g(h(Y,X), FN), \end{aligned}$$

that is,

$$g(A_N PX, Y) = -g(A_{FN}X, Y),$$

which yields (3.6). □

Lemma 3.2. *Let M be as in* Lemma 3.1. *Then for $Z, W \in \mathcal{D}^\perp$ we have*

$$\nabla^\perp_W \phi Z - \nabla^\perp_Z \phi W \in \phi\mathcal{D}^\perp. \tag{3.7}$$

Proof. For $N \in \nu$ and $Z, W \in \mathcal{D}^\perp$, it follows from (2.2) and (2.3) that

$$\begin{aligned} g(A_{\phi N}Z, W) &= g(-A_N Z + \nabla^\perp_Z N, \phi W) \\ &= g(\nabla^\perp_Z N, \phi W) \\ &= -g(N, \nabla^\perp_Z \phi W), \end{aligned}$$

and consequently

$$g(N, \nabla^\perp_W \phi Z - \nabla^\perp_Z \phi W) = -g(A_{\phi N}W, Z) + g(A_{\phi N}Z, W) = 0.$$

Thus we obtain (3.7). □

From Lemma 3.1 and 3.2 we have some fundamental lemmas without proof

Lemma 3.3. ([1], [3]) *The anti-invariant distribution $\mathcal{D}^\perp$ of a contact CR submanifold in a Sasakian manifold is integrable.*

For the invariant distribution $\mathcal{D}$ we have

Lemma 3.4. ([1], [3]) *Let M be as in* Lemma 3.1. *Then $\mathcal{D} \oplus \mathrm{Span}\{\xi\}$ is integrable if and only if*

$$g(h(X, \phi Y), \phi Z) = g(h(\phi X, Y), \phi Z) \tag{3.8}$$

for $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Lemma 3.5. ([1], [3]) *For a contact CR submanifold M in a Sasakian manifold $\bar{M}$, the leaf $M^\perp$ of $\mathcal{D}^\perp$ is totally geodesic in M if and only if*

$$g(h(\mathcal{D}, \mathcal{D}^\perp), \phi\mathcal{D}^\perp) = 0. \tag{3.9}$$

4. Proof of Theorem A

A Sasakian space form $\bar{M}(c)$ is a Sasakian manifold of constant ϕ-sectional curvature c. The curvature tensor of a Sasakian space form $\bar{M}(c)$ is given by

(4.1)

$$\begin{aligned} \bar{R}(X, Y)Z = {} & \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ & - \frac{c-1}{4}\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi \\ & - g(X, Z)\eta(Y)\xi - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z\}, \end{aligned}$$

for any $X, Y, Z \in TM$.

According to Lemma 3.3, we can see that every contact CR submanifold M of a Sasakian manifold is foliated by anti-invariant submanifolds. Now we shall study the problem when a contact CR submanifold M is a Riemannian product of an invariant submanifold and an anti-invariant submanifold(for the definition, cf. [9]).

Definition 2. A contact CR submanifold M of a Sasakian manifold $\bar{M}$ is called a *contact CR product* if the distribution $\mathcal{D} \oplus \mathrm{Span}\{\xi\}$ is integrable and M is locally a Riemannain product $M^{\top} \times M^{\perp}$, where $M^{\top}$ and $M^{\perp}$ are leafs of $\mathcal{D} \oplus \mathrm{Span}\{\xi\}$ and $\mathcal{D}^{\perp}$, respectively.

First we give a characterization of contact CR product as follows.

Lemma 4.1. ([1]) *A contact CR submanifold M of a Sasakian manifold $\bar{M}$ is a contact CR product if and only if*

$$\nabla_Y X \in \mathcal{D} \oplus \mathrm{Span}\{\xi\} \tag{4.2}$$

for $Y \in TM$ and $X \in \mathcal{D}$.

From Lemma 4.1 we have the following lemma.

Lemma 4.2. ([1], [2]) *Let M be a contact CR submanifold of a Sasakian manifold $\bar{M}$. Then the following assertions are equivalent to each other:*

(i) *M is a contact CR product* ;

(ii) the fundamental tensors of Weingarten satisfy

$$A_{\phi\mathcal{D}^{\perp}}\mathcal{D} = 0; \tag{4.3}$$

(iii) the second fundamental form of M satisfies

$$th(X, Y) = 0, \text{ for } X \in \mathcal{D},\ Y \in TM; \tag{4.4}$$

(iv) the second fundamental form of M satisfies

$$h(\phi X, Y) = \phi h(X, Y), \text{ for } X \in \mathcal{D},\ Y \in TM. \tag{4.5}$$

On the other hand, the ϕ-holomorphic bisectional curvature of $X \wedge Z$ is defined by $\bar{H}_B(X, Z) = g(\bar{R}(X, \phi X)\phi Z, Z)$ for any unit vector fields $X, Z \in TM$(for the definition, cf. [6])

Lemma 4.3. *Let M be a contact CR product of a Sasakian manifold $\bar{M}$. Then for any unit vectors $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$ we have*

$$\bar{H}_B(X, Z) = 2\|h(X, Z)\|^2 - 2g(X, X)g(Z, Z),$$

where $\bar{H}_B(X, Z)$ is the ϕ-holomorphic bisectional curvature of $X \wedge Z$.

Proof. Let M be a contact CR product of $\bar{M}$. By (2.5) and Codazzi equation (2.7) we obtain

$$\begin{aligned} g(\bar{R}(X,\phi X)Z,\phi Z) &= g((\nabla_X h)(\phi X, Z) - (\nabla_{\phi X} h)(X,Z), \phi Z) \\ &= g(\nabla^{\perp}_X h(\phi X, Z) - h(\nabla_X(\phi X), Z) - h(\phi X, \nabla_X Z) \\ &\quad - \nabla^{\perp}_{\phi X} h(X,Z) + h(\nabla_{\phi X} X, Z) + h(X, \nabla_{\phi X} Z), \phi Z). \end{aligned}$$

Hence, it is clear from (2.1), (4.3) and (4.5) that

$$\begin{aligned} g(\bar{R}(X,\phi X)Z,\phi Z) &= g(\nabla^{\perp}_X h(\phi X, Z) - \nabla^{\perp}_{\phi X} h(X,Z), \phi Z) \\ &\quad - g(h(\nabla_X(\phi X), Z), \phi Z) + g(h(\nabla_{\phi X} X, Z), \phi Z). \end{aligned} \tag{4.6}$$

First of all, from (3.3) and (4.3) it follows that

$$\begin{aligned} &g(\nabla^{\perp}_X h(\phi X, Z) - \nabla^{\perp}_{\phi X} h(X,Z), \phi Z) \\ &\quad = -g(h(\phi X, Z), \nabla^{\perp}_X \phi Z) + g(h(X,Z), \nabla^{\perp}_{\phi X} \phi Z) \\ &\quad = -g(h(\phi X, Z), \phi(\nabla_X Z) + \phi h(X,Z)) + g(h(X,Z), \phi(\nabla_{\phi X} Z) + \phi h(\phi X, Z)). \end{aligned}$$

On the other hand, we get $\nabla_X Z \in \mathcal{D}^{\perp}$ for $Z \in \mathcal{D}^{\perp}$ since $M^{\perp}$ is totally geodesic, that is, $\phi \nabla_X Z \in \phi \mathcal{D}^{\perp}$ and thus from (2.1), (3.9) and (4.5) it follows thta

$$g(\nabla^{\perp}_X h(\phi X, Z) - \nabla^{\perp}_{\phi X} h(X,Z), \phi Z) = -2\|h(X,Z)\|^2.$$

Next, from (2.15), (2.16), (4.4) and (4.5) we find

$$\begin{aligned} g(h(\nabla_X(\phi X), Z), \phi Z) &= g(h((\nabla_X P)X + P(\nabla_X X), Z), \phi Z) \\ &= -g(X,X)g(h(\xi, Z), \phi Z) + g(h(P(\nabla_X X), Z), \phi Z) \\ &= -g(X,X)g(FZ, \phi Z) \\ &= -g(X,X)g(Z,Z) \end{aligned}$$

where $\phi X = PX \in \mathcal{D}$ for $X \in \mathcal{D} \oplus \mathrm{Span}\{\xi\}$ and $\phi Z = FZ$ for $Z \in \mathcal{D}^{\perp}$. On the other hand, we can put

$$\nabla_{\phi X} X = (\nabla_{\phi X} X)_{\mathcal{D}} + \alpha \xi,$$

where $(\nabla_{\phi X} X)_{\mathcal{D}}$ denotes the $\mathcal{D}$-component of $\nabla_{\phi X} X$. In fact

$$\alpha = g(\nabla_{\phi X} X, \xi) = -g(\nabla_{\phi X} \xi, X) = -g(P(\phi X), X) = g(X,X)$$

and consequently

$$\begin{aligned} g(h(\nabla_{\phi X} X, Z), \phi Z) &= g(h((\nabla_{\phi X} X)_{\mathcal{D}}, Z), \phi Z) + g(h(g(X,X)\xi, Z), \phi Z) \\ &= g(X,X)g(h(\xi, Z), \phi Z) \\ &= g(X,X)g(\phi Z, \phi Z) \\ &= g(X,X)g(Z,Z). \end{aligned}$$

Finally, from (4.6) we have

$$g(\bar{R}(X,\phi X)Z,\phi Z) = -2\|h(X,Z)\|^2 + g(X,X)g(Z,Z) + g(X,X)g(Z,Z)$$

which implies

$$\bar{H}_B(X,Z) = 2\|h(X,Z)\|^2 - 2g(X,X)g(Z,Z).$$

□

Proof of Theorem A. Now we suppose that $\bar{H}_B(X,Z) < -2$ for $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. Then Lemma 4.3 yields

$$\|h(X,Z)\|^2 < 0,$$

which is a contradiction if $\dim\mathcal{D} \neq 0$ and $\dim\mathcal{D}^\perp \neq 0$. Thus we can see that $\dim\mathcal{D} = 0$ or $\dim\mathcal{D}^\perp = 0$ and consequently Theorem A is established. □

Corollary 4.4. *Let $\bar{M}$ be a Sasakian manifold with $\bar{H}_B > -2$, and M a proper contact CR product in $\bar{M}$. Then* (1) *M is not a generic submanifold, and* (2) $h(\mathcal{D},\mathcal{D}^\perp) \neq 0$; hence M is not totally geodesic in $\bar{M}$.

Proof. Suppose that M is a generic submanifold. Since $\dim\mathcal{D}^\perp = \dim TM^\perp$, $\dim\nu = 0$. On the other hand, from (4.3) we have

$$g(h(X,Z),\phi W) = g(A_{\phi W}X,Z) = 0$$

for $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$, that is, $h(X,Z) \in \nu$. Since $\dim\nu = 0$, $h(X,Z) = 0$. It is a contradiction to $\bar{H}_B > -2$. Hence M is not a generic submanifold.

Since $\bar{M}$ is a Sasakian manifold with $\bar{H}_B > -2$, $\dim\mathcal{D} \neq 0$ or $\dim\mathcal{D}^\perp \neq 0$. Therefore $h(\mathcal{D},\mathcal{D}^\perp) \neq 0$. Hence h is non-zero tensor, M is not totally geodesic in $\bar{M}$. □

Finally, we suppose that M is a contact CR product in a Sasakian space form $\bar{M}(c)$. By using the curvature tensor of $\bar{M}(c)$, we find

$$\begin{aligned}\bar{R}(X,\phi X)Z &= \frac{1}{4}(1-c)\{2g(\phi X,\phi X)\phi Z\}\\ &= \frac{1}{2}(1-c)g(X,X)\phi Z\end{aligned}$$

and consequently,

$$\begin{aligned}g(\bar{R}(X,\phi X)Z,\phi Z) &= \frac{1}{2}(1-c)g(X,X)g(\phi Z,\phi Z)\\ &= \frac{1}{2}(1-c)g(X,X)g(Z,Z),\end{aligned}$$

which implies

$$\frac{1}{2}(1-c)g(X,X)g(Z,Z) = -\bar{H}_B(X,Z).$$

Thus for any unit vector fields X, Z tangent to M

$$c = 2\bar{H}_B(X,Z) + 1,$$

which together with Theorem A yields the following.

Corollary 4.5. ([1], [2]) *There exists no proper contact CR product in Sasakian space form $\bar{M}(c)$ with $c < -3$.*

5. Mixed foliate contact CR submanifolds of a Sasakian manifold

For a contact CR submanifold M of a Sasakian manifold $\bar{M}$, the distribution $\mathcal{D}^\perp$ is completely integrable(cf. [1], [3]). Integrability of the distribution $\mathcal{D} \oplus \mathrm{Span}\{\xi\}$ is provided in Lemma 3.4.

Definition 3. A contact CR submanifold is said to be *mixed foliate* if
(d) the distribution $\mathcal{D} \oplus \mathrm{Span}\{\xi\}$ is integrable, and
(e) $h(X, Y) = 0$ for any vector fields $X \in \mathcal{D},\ Y \in \mathcal{D}^\perp$.

Now we prepare some lemmas concerning mixed foliate contact CR submanifolds of a Sasakian manifold for later use.

Lemma 5.1. *For a mixed foliate contact CR submanifold of a Sasakian manifold,*

$$A_V X \in \mathcal{D},\ X \in \mathcal{D} \quad ; \quad A_V X \in \mathcal{D}^\perp \oplus \mathrm{Span}\{\xi\},\ X \in \mathcal{D}^\perp \tag{5.1}$$

for any vector field V normal to $\mathcal{D}$.

Proof. For $X \in \mathcal{D}$ and $Y \in \mathcal{D}^\perp$, the condition (e) implies

$$g(A_V X, Y) = g(h(X, Y), V) = 0,$$

which together with $g(A_V X, \xi) = g(h(X, \xi), V) = g(FX, V) = 0$ yields the first assertion of (5.1). The second assertion of (5.1) can be also derived from the same reason. □

Lemma 5.2. *For a mixed foliate contact CR submanifold M of a Sasakian manifold,*

$$A_{FX} P + P A_{FX} = 0,\ X \in \mathcal{D}^\perp. \tag{5.2}$$

Proof. It is clear that, for any vector field X tangent to M,

$$PX (= \nabla_X \xi) \in \mathcal{D} \tag{5.3}$$

by means of $PY = 0$ for any $Y \in \mathcal{D}^\perp$. Moreover, from (2.14), (2.15) and (3.1) we get

$$P A_{FZ} \xi = -PtFZ = tfFZ = 0. \tag{5.4}$$

In order to prove (5.2), we first notice that

$$g(A_{FZ} PX, Y) + g(P A_{FZ} X, Y) = 0,\ X, Y \in \mathcal{D},\ Z \in \mathcal{D}^\perp \tag{5.5}$$

because of the condition (d) and (3.8). On the other hand, it follows from (5.1) and (5.3) that $A_V PX \in \mathcal{D}$ and $P A_V X \in \mathcal{D}$ for $X \in \mathcal{D}$. Hence we have

$$g(A_V PX, Y) + g(P A_V X, Y) = 0,\ X \in \mathcal{D},\ Y \in \mathcal{D}^\perp,$$

which together with (5.4) and (5.5) yields

$$A_{FZ} PX + P A_{FZ} X = 0,\ X \in \mathcal{D}. \tag{5.6}$$

But, for $X \in \mathcal{D}^\perp$, it is clear that $A_V PX = 0$ and $PA_V X = 0$ because of (5.1), thus from which together with (5.4) and (5.6) we have (5.2). □

Lemma 5.3. *For a mixed foliate contact CR submanifold M of a Sasakian manifold,*

$$\nabla_X Y \in \mathcal{D}^\perp,\ X,\ Y \in \mathcal{D}^\perp \quad ; \quad \nabla_X Y \in \mathcal{D},\ X \in \mathcal{D}^\perp,\ Y \in \mathcal{D} \tag{5.7}$$

$$\nabla^\perp_X \phi Y \in \phi\mathcal{D}^\perp, \quad \nabla^\perp_\xi \phi Y \in \phi\mathcal{D}^\perp,\ X \in \mathcal{D},\ Y \in \mathcal{D}^\perp. \tag{5.8}$$

Proof. For any vector field X tangent to M, it follows that

$$g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = -g(PX, Y) = 0,\ Y \in \mathcal{D}^\perp$$

because of (2.15) and (5.3). In order to prove the first equation of (5.7), it suffices to show that

$$g(\nabla_X Y, Z) = 0,\ X, Y \in \mathcal{D}^\perp,\ Z \in \mathcal{D}. \tag{5.9}$$

Since $\phi\mathcal{D}_x = \mathcal{D}_x$, there exists $W \in \mathcal{D}_x$ such that $Z = \phi W$. Thus, for $X, Y \in \mathcal{D}^\perp$, it follows that

$$g(\nabla_X Y, Z) = g(\nabla_X Y, \phi W) = -g(P\nabla_X Y, W) = g((\nabla_X P)Y, W)$$

because of $PY = 0$, from which together with (2.16) and (5.1), we can easily obtain (5.9).

Since

$$g(\nabla_X Y, \xi) = -g(PX, Y) = 0,\ X \in \mathcal{D}^\perp,\ Y \in \mathcal{D},$$

the second equation of (5.7) can be easily derived from the first equation of (5.7). In fact, for $X, Z \in \mathcal{D}^\perp$ and $Y \in \mathcal{D}$

$$g(\nabla_X Y, Z) = -g(\nabla_X Z, Y) = 0$$

because of the first equation of (5.7).

Next, we will prove (5.8). It is clear from the Weingarten formula (2.3) that

$$\bar\nabla_X \phi Y = -A_{\phi Y} X + \nabla^\perp_X \phi Y,\ X \in \mathcal{D},\ Y \in \mathcal{D}^\perp. \tag{5.10}$$

On the other hand, it follows from (2.3) that $\bar\nabla_X \phi Y = (\bar\nabla_X \phi)Y + \phi(\nabla_X Y + h(X, Y))$, thus from which together with (2.2) and the condition (e), we have

$$\bar\nabla_X \phi Y = \phi\nabla_X Y = \phi((\nabla_X Y)_{\mathcal{D}} + (\nabla_X Y)_{\mathcal{D}^\perp}),\ X \in \mathcal{D},\ Y \in \mathcal{D}^\perp. \tag{5.11}$$

Comparing (5.10) with (5.11), we find

$$\nabla^\perp_X \phi Y = \phi(\nabla_X Y)_{\mathcal{D}^\perp} \in \phi\mathcal{D}^\perp,$$

which completes the first equation of (5.8). The second equation of (5.8) can be similarly derived. □

6. Proof of Theorem B

In this section we specialize to the case of an ambient Sasakian space form $\bar{M}(c)$ and let M be a mixed foliate contact CR submanifold of $\bar{M}(c)$.
We first prove

Theorem 6.1. *If M is a mixed foliate proper contact CR submanifold of a Sasakian space form $\bar{M}(c)$, then $c \leq 1$.*

Proof. Let M be a mixed foliate proper contact CR submanifold of a Sasakian space form $\bar{M}(c)$. Then, for $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, we have

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = h(X, \nabla_Y Z) - h(Y, \nabla_X Z) \tag{6.1}$$

by means of the conditions (d) and (e).

If we take a vector field V normal to M such that $V = FZ$, i.e., $Z = -\phi V = -tV$, then we have $\nabla_Y Z = PA_V Y - t\nabla^\perp_Y V$ by means of (2.2), (2.3) and (5.1). Since $g(X, t\nabla^\perp_Y V) = 0$ and $g(\xi, t\nabla^\perp_Y V) = 0$, $t\nabla^\perp_Y V \in \mathcal{D}^\perp$ and consequently $h(X, t\nabla^\perp_Y V) = 0$ because of the condition (e). Thus (6.1) implies

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = h(X, PA_V Y) - h(Y, PA_V X),$$

which together with (3.8) and (5.2) yields

$$g((\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), V) = -2g(h(X, A_V PY), V) \tag{6.2}$$

On the other hand, (2.7) and (4.1) yield

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \frac{c-1}{2}\, g(X, PY)FZ. \tag{6.3}$$

Comparing (6.2) with (6.3), we have

$$\frac{c-1}{4} g(X, PY)g(V, V) = -g(h(X, A_V PY), V).$$

Putting $X = PY$ in this equation, we have

$$0 \leq g(A_V PY, A_V PY) = -\frac{c-1}{4}\, g(PY, PY)g(V, V),$$

which completes our assertion since M is proper. □

For $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, it follows from (6.1) and (6.3) that

$$-\frac{c-1}{2}\, g(PX, Y)FZ = h(X, \nabla_Y Z) - h(Y, \nabla_X Z), \tag{6.4}$$

from which, taking the inner product with $FW \in \phi\mathcal{D}^\perp$ and replacing X by PX, we have

$$\begin{aligned} \frac{c-1}{2} g(X, Y)g(Z, W) &= g(A_{FW}X, P\nabla_Y Z) - g(A_{FW}Y, \nabla_{PX} Z) \\ &= -g(A_{FW}X, A_{FZ}Y) - g(A_{FW}Y, \nabla_{PX} Z), \end{aligned} \tag{6.5}$$

where we have used $PZ = 0$, the condition (e), (2.16) and (5.2). On the other hand, for $Y \in \mathcal{D}$ and $W \in \mathcal{D}^\perp$, it is clear from (5.1) that $A_{FW}Y \in \mathcal{D}$ at each point $x \in M$. Thus, in order to compute $g(A_{FW}Y, \nabla_{PX}Z)$ more precisely, it suffices to consider only $\mathcal{D}$-component of $\nabla_{PX}Z$. In fact, (3.4) implies

$$g(\nabla_{PX}Z, U) = g(A_{FZ}X, U)$$

for any $U \in \mathcal{D}$ and consequently

$$g(A_{FW}Y, \nabla_{PX}Z) = g(A_{FW}Y, A_{FZ}X),$$

which and (6.5) give

$$-\frac{c-1}{2}g(X,Y)g(Z,W) = g(A_{FW}X, A_{FZ}Y) + g(A_{FZ}X, A_{FW}Y) \tag{6.6}$$

for $X, Y \in \mathcal{D},\ Z, W \in \mathcal{D}^\perp$.

On the other hand, for $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$, it is clear from the condition (d) and (5.8) that

$$R^\perp(X,Y)FZ \in \phi\mathcal{D}^\perp,$$

which together with (2.8) and (4.1) implies

$$g([A_{FZ}, A_N]X, Y) = 0,\ X, Y \in \mathcal{D},\ Z \in \mathcal{D}^\perp,\ N \in (\phi\mathcal{D}^\perp)^\perp, \tag{6.7}$$

where $(\phi\mathcal{D}^\perp)^\perp$ denotes the orthogonal complement of $\phi\mathcal{D}^\perp \subset TM^\perp$.

Next, taking the inner product with $N \in (\phi\mathcal{D}^\perp)^\perp$ in (6.4) and replacing X by PX, we can obtain by the same method as in (6.6) that

$$g(A_NX, A_{FZ}Y) + g(A_{FZ}X, A_NY) = 0,\ X, Y \in \mathcal{D},\ Z \in \mathcal{D}^\perp,\ N \in (\phi\mathcal{D}^\perp)^\perp. \tag{6.8}$$

Combining (6.7) with (6.8) and using (5.1), we have

$$A_{FZ}A_NX = 0,\ X \in \mathcal{D},\ Z \in \mathcal{D}^\perp,\ N \in (\phi\mathcal{D}^\perp)^\perp \tag{6.9}$$

because of $A_{FZ}A_NX \in \mathcal{D}$. Substituting A_NX into (6.6) instead of X and using (3.5) and (6.9), we have

$$(c-1)g(h(X,Y),N)g(Z,W) = 0,\ X, Y \in \mathcal{D},\ Z, W \in \mathcal{D}^\perp,\ N \in (\phi\mathcal{D}^\perp)^\perp.$$

Thus we have

Lemma 6.2. *Let M be a mixed foliate proper contact CR submanifold of a Sasakian space form $\bar{M}(c)(c<1)$. Then*

$$h(X,Y) \in \phi\mathcal{D}^\perp,\ X, Y \in \mathcal{D}.$$

Lemma 6.3. *Let M be a mixed foliate proper contact CR submanifold of a Sasakian space form $\bar{M}(c)(c<1)$. If*

$$h(X,Y) \in \phi\mathcal{D}^\perp,\ X, Y \in \mathcal{D}^\perp,$$

then $T_xM \oplus \phi\mathcal{D}_x^\perp$ is the first osculating space $O_1(M) = T_xM \oplus \mathrm{Span}\{h(X,Y)|X, Y \in T_xM\}$ at any point $x \in M$.

Proof. Owing to Lemma 6.2 and our assumption, it suffices to show that

$$\phi\mathcal{D}_x^\perp \subset \{h(X,Y)|\ X,Y \in T_xM\}$$

at each point $x \in M$. Suppose that there exists a unit vector $\phi Z \in \phi\mathcal{D}_x^\perp$ such that

$$g(h(X,Y),\phi Z) = 0$$

for any $X,Y \in T_xM$. Then $A_{FZ}X = 0$ for any $X \in T_xM$, which and (6.6) yield

$$(c-1)g(X,Y)g(Z,W) = 0,\ X,Y \in \mathcal{D},\ Z,W \in \mathcal{D}^\perp.$$

Therefore if $c < 1$, then we have $g(X,X)g(Z,Z) = 0$, which is a contradiction since M is proper. □

Combining Lemma 6.3 and the theorem([5, Theorem 3.3, p.329]) provided by Funabashi, we have Theorem B.

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