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A GENERAL SOLUTION OF A SPACE-TIME FRACTIONAL ANOMALOUS DIFFUSION PROBLEM USING THE SERIES OF BILATERAL EIGEN-FUNCTIONS

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Abstract. In the present paper, we consider an anomalous diffusion problem in two dimensional space involving Caputo time and Riesz-Feller fractional derivatives and then solve it by using a series involving bilateral eigen-functions. Also, we obtain a numerical approximation formula of this problem and discuss some of its particular cases.

1. Introduction

By a set of axiom, definitions and methods of fractional calculus many processes in the nature are modelled (see Kilbas et al. [8], Miller and Ross [12], Samko et al. [15] and Podlubny [14]). One of these processes is an anomalous diffusion which is a phenomenon that occurs in complex and non-homogeneous mediums.

The anomalous diffusion may be based on generalized diffusion equation which contains fractional order space and/or time derivatives (see Mainardi et al. [9]). Metzler and Klafter [11] and Turski et al. [18] presented the occurrence of the anomalous diffusion from the physical point of view and also explained the effects of fractional derivatives in space and/or time to diffusion propagation. Agrawal [1, 2] applied an analytical technique by using eigen-functions for a fractional diffusion-wave system.

Mathai, Saxena and Haubold [7, 10] investigated the solution of a unified fractional reaction diffusion equation associated with Caputo derivative as the time-derivative and Riesz-Feller fractional derivative (see Ciesielski et al. [3]) as the space-derivative. They have derived its solution by the application of the Laplace and Fourier transforms in a compact and closed form in terms of the H-function.

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Riesz introduced the pseudo-differential operator ${}_xI_0^\alpha$ whose symbol is $|k|^{-\alpha}$, well defined for any positive α with the exclusion of odd integer numbers, then was called Riesz Potential. The Riesz fractional derivative ${}_xD_0^\alpha = -{}_xI_0^\alpha$ is defined by

$$ {}_xD_0^\alpha = \begin{cases} -|k|^\alpha, & (1.1a) \\ -(k^2)^{\alpha/2}, & (1.1b) \\ -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}. & (1.1c) \end{cases} $$

In addition, Feller [4] generalized the Riesz fractional derivative to include the skewness parameter θ of the strictly stable densities. Feller showed that the pseudo-differential operator D_θ^α is an inverse to the Feller potential, which is a linear combination of two Riemann-Liouville (or Weyl) integrals:

$$ {}_xI_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} f(\xi)\, d\xi \tag{1.2} $$

and

$$ {}_xI_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} (\xi-x)^{\alpha-1} f(\xi)\, d\xi, \tag{1.3} $$

where $\alpha > 0$. By these definitions given in (1.2) and (1.3), the Feller potential can be defined as

$$ {}_xI_\theta^\alpha f(x) = c_+(\alpha,\theta)\, {}_xI_+^\alpha f(x) + c_-(\alpha,\theta)\, {}_xI_-^\alpha f(x), \tag{1.4} $$

where the real parameters α and θ are always restricted as follows $0 < \alpha \le 2$, $\alpha \ne 1$, $|\theta| \le \min\{\alpha, 2-\alpha\}$, and also the coefficients

$$ c_+(\alpha,\theta) = \frac{\sin\left(\frac{(\alpha-\theta)\pi}{2}\right)}{\sin(\alpha\pi)}, \; c_-(\alpha,\theta) = \frac{\sin\left(\frac{(\alpha+\theta)\pi}{2}\right)}{\sin(\alpha\pi)}. \tag{1.5} $$

Using the Feller potential given in (1.4) along with (1.5), Gorenflo and Mainardi [5, 6] defined the Riesz-Feller derivative

$$ \frac{\partial^\alpha f(x)}{\partial |x|_\theta^\alpha} = -{}_xI_\theta^{-\alpha} f(x) = -\left[c_+(\alpha,\theta)\, {}_xD_+^\alpha f(x) + c_-(\alpha,\theta)\, {}_xD_-^\alpha f(x)\right], \tag{1.6} $$

where ${}_xD_\pm^\alpha$ are Weyl fractional derivatives defined as

$$ {}_xD_\pm^\alpha f(x) = \begin{cases} \pm\frac{d}{dx}\left[{}_xI_\pm^{1-\alpha} f(x)\right], \; 0 < \alpha < 1, & (1.7a) \\ \frac{d^2}{dx^2}\left[{}_xI_\pm^{2-\alpha} f(x)\right], \; 1 < \alpha \le 2. & (1.7b) \end{cases} $$

The Caputo fractional derivative is defined as

$$ \frac{\partial^\beta u(t)}{\partial t^\beta} = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-q)^{n-\beta-1} \left(\frac{d}{dq}\right)^n u(q)\, dq, \tag{1.8} $$

provided that $0 < \beta \leq n$, $n \in \mathbb{N}$ (the set of natural numbers) (see Mainardi et al. [9]).

Motivated by above work, we consider the space-time fractional anomalous diffusion problem

$$\frac{\partial^{\beta} u(x,y,t)}{\partial t^{\beta}} = \frac{\partial^{\alpha} u(x,y,t)}{\partial |x|_{\theta_1}^{\alpha}} + \frac{\partial^{\mu} u(x,y,t)}{\partial |y|_{\theta_2}^{\mu}}, \tag{1.9a}$$

$$u(x,y,0) = u_0(x,y), \tag{1.9b}$$

and

$$\lim_{x,y\to\pm\infty} u(x,y,t) = 0, \tag{1.9c}$$

where $x, y \in \mathbb{R}$ (the set of real numbers); β, α, μ are real parameters restricted as $0 < \beta \leq 1$, $0 < \alpha \leq 1$, $1 < \mu \leq 2$; the skewness parameters θ_1 $(\theta_1 \leq \min\{\alpha, 1-\alpha\})$ and θ_2 $(\theta_2 \leq \min\{\mu, 2-\mu\})$ of the asymmetry of the probability distribution of a real-valued random variable among the x and y co-ordinate axes.

We assume that the solution of above problem (1.9a, b, c) is the series involving bilateral eigen-functions

$$u(x,y,t) = \sum_{n=1}^{\infty} u_n(t)\, \psi_n(x)\, \phi_n(y) \tag{1.10}$$

particularly, setting $u_n(t) = \gamma_n t^n$, $\{\gamma_n\}_{n=1}^{\infty}$ are independent of x, y and t, the sets of functions $\{\psi_n(x)\}_{n=1}^{\infty}$ and $\{\phi_n(x)\}_{n=1}^{\infty}$ are different, then $u(x,y,t)$ becomes a bilateral generating function (see Srivastava and Manocha [16, p. 79] and Srivastava and Panda [17]).

The eigen-functions $\psi_n(x)$ satisfy the eigen-value problem:

$$\frac{d}{dx}\psi_n(x) = i\lambda_n \psi_n(x), \tag{1.11a}$$

where

$$i = \sqrt{(-1)} \text{ and } \lambda_n \in \mathbb{R},\ n \in \mathbb{N}. \tag{1.11b}$$

The eigen-functions $\phi_n(y)$ satisfy the eigen-value problem:

$$\frac{d^2}{dy^2}\phi_n(y) = -{\lambda'_n}^2 \phi_n(y), \tag{1.12a}$$

where

$$\lambda'_n \in \mathbb{R},\ n \in \mathbb{N}. \tag{1.12b}$$

Before going to obtain the solution of above problem (1.9a, b, c), we present following theorems:

Theorem A. *If the eigen-functions* $\psi_n(x)\,(n \in \mathbb{N}, x \in \mathbb{R})$ *satisfy the eigen-value problem* (1.11a), (1.11b) *and* ψ_n *satisfies* $\psi_n(x+(-r)) = \psi_n(x) . \psi_n(-r)$, *then, for* $0 < \alpha < 1$ *and* $\theta_1 \leq \min\{\alpha,\ 1-\alpha\}$, *we have*

$$\frac{\partial^x \psi_n(x)}{\partial |x|^{\alpha}_{\theta_1}} = -c_1(\lambda_n, \alpha, \theta_1)\, \psi_n(x), \tag{1.13}$$

where

$$c_1(\lambda_n, \alpha, \theta_1) = i\lambda_n \left[c_+(\alpha, \theta_1)\, A_n(\alpha) - c_-(\alpha, \theta_1)\, A_n'(\alpha)\right], \tag{1.14}$$

$$A_n(\alpha) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} r^{-\alpha} \psi_n(-r)\, dr \tag{1.15}$$

and

$$A_n'(\alpha) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} (r)^{-\alpha}\, \psi_n(r)\, dr. \tag{1.16}$$

Also $c_+(\alpha, \theta_1)$ *and* $c_-(\alpha, \theta_1)$ *are found by* (1.5).

Proof. Put $f(x) = \psi_n(x)$, $\theta = \theta_1$ in (1.6) and then use the (1.2), (1.3) and (1.7a) in it, we get

$$\begin{aligned} \frac{\partial^\alpha \psi_n(x)}{\partial |x|^{\alpha}_{\theta_1}} = &- \left[c_+(\alpha, \theta_1) \frac{\partial}{\partial x} \left\{\frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\psi_n(x+(-r))}{(r)^{\alpha}} dr\right\}\right. \\ &\left. - c_-(\alpha, \theta_1) \frac{\partial}{\partial x} \left\{\frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} \frac{\psi_n(x+r)}{(r)^{\alpha}} dr\right\}\right]. \end{aligned} \tag{1.17}$$

Now making an appeal to Theorem A in the integrands of the right-hand side of (1.17) with (1.11a), (1.11b), (1.14), (1.15) and (1.16), we find (1.13). □

Theorem B. *If the eigen-functions* $\phi_n(y)$, $n \in \mathbb{N}$, $y \in \mathbb{R}$ *satisfy the eigen-value problem* (1.12a), (1.12*b*) *and the operator* ϕ *is defined by the function* $\phi_n : \mathbb{R} \to \mathbb{R}$ *such that*

$$\phi_n(y) = \phi(ny), \quad n \in \mathbb{N}, \quad y \in \mathbb{R}. \tag{1.18}$$

Another operator Φ *is defined by the function* $\Phi_n : \mathbb{R} \to \mathbb{R}$ *such that*

$$\Phi_n(y) = \Phi(ny), \quad n \in \mathbb{N}, \quad y \in \mathbb{R}. \tag{1.19}$$

Also assume that the relation between $\phi_n(y)$ *and* $\Phi_n(y)$ *are given by*

$$\phi_n(\theta_n - y) = \Phi_n(y) \ \text{ and } \Phi_n(\theta_n - y) = \phi_n(y). \tag{1.20}$$

The addition formula for above operators is given by

$$\phi_n(y+(-r)) = \phi_n(y)\phi_n(\theta_n - (-r)) + \phi_n(\theta_n - y)\phi_n(-r) \tag{1.21}$$

then, for $1 \leq \mu < 2$ *and* $\theta_2 \leq \min\{\mu,\ 2-\mu\}$, *there exists*

$$\frac{\partial^\mu \phi_n(y)}{\partial |y|^{\mu}_{\theta_2}} = -c_2(\lambda_n', \mu, \theta_2, \theta_n)\phi_n(y) - c_3(\lambda_n', \mu, \theta_2)\phi_n(\theta_n - y), \tag{1.22}$$

where

$$c_2(\lambda'_n,\mu,\theta_2,\theta_n) = -(\lambda'_n)^2[c_+(\mu,\theta_2)B_1(\mu,\theta_n) + c_-(\mu,\theta_2)B_3(\mu,\theta_2)] \tag{1.23}$$

and

$$c_3(\lambda'_n,\mu,\theta_2) = -(\lambda'_n)^2[c_+(\mu,\theta_2)B_2(\mu) + c_-(\mu,\theta_2)B_4(\mu)] \tag{1.24}$$

and here

$$B_1\left(\mu,\theta_n\right) = \frac{1}{\Gamma\left(2-\mu\right)}\int_0^\infty \frac{\phi_n\left(\theta_n+\eta\right)}{\left(\eta\right)^{\mu-1}}d\eta, \tag{1.25}$$

$$B_2\left(\mu\right) = \frac{1}{\Gamma\left(2-\mu\right)}\int_0^\infty \frac{\phi_n\left(-\eta\right)}{\left(\eta\right)^{\mu-1}}d\eta, \tag{1.26}$$

$$B_3\left(\mu,\theta_n\right) = \frac{1}{\Gamma\left(2-\mu\right)}\int_0^\infty \frac{\phi_n\left(\theta_n-\eta\right)}{\left(\eta\right)^{\mu-1}}d\eta, \tag{1.27}$$

$$B_4\left(\mu\right) = \frac{1}{\Gamma\left(2-\mu\right)}\int_0^\infty \frac{\phi_n\left(\eta\right)}{\left(\eta\right)^{\mu-1}}d\eta. \tag{1.28}$$

Proof. Put $x = y$, $\alpha = \mu$ ($1 < \mu \leq 2$), $\theta = \theta_2$ $(\theta_2 \leq \min\{\mu,\ 2-\mu\})$ and $f\left(y\right) = \phi_n\left(y\right)$ in (1.6) and use the (1.2), (1.3) and (1.7b) in it, we get

$$\begin{aligned}\frac{\partial^\mu \phi_n\left(y\right)}{\partial\left|y\right|^\mu_{\theta_2}} = -&\left[c_+\left(\mu,\theta_2\right)\frac{\partial^2}{\partial y^2}\left\{\frac{1}{\Gamma\left(2-\mu\right)}\int_0^\infty \frac{\phi_n\left(y+\left(-\eta\right)\right)}{\left(\eta\right)^{\mu-1}}d\eta\right\}\right.\\ &\left.+\ c_-\left(\mu,\theta_2\right)\frac{\partial^2}{\partial y^2}\left\{\frac{1}{\Gamma\left(2-\mu\right)}\int_0^\infty \frac{\phi_n\left(y+\eta\right)}{\left(\eta\right)^{\mu-1}}d\eta\right\}\right].\end{aligned} \tag{1.29}$$

Now making an appeal to the addition formula (1.21) in integrands of the right-hand side of (1.29) and the eigen-value problem (1.12a) and (1.12b) in it, we get

$$\begin{aligned}\frac{\partial^\mu \phi_n(y)}{\partial|y|^\mu_{\theta_2}} &= (\lambda'_n)^2[c_+(\mu,\theta_2)B_1(\mu,\theta_n) + c_-(\mu,\theta_2)B_3(\mu,\theta_n)]\phi_n(y)\\ &+ c_+(\mu,\theta_2)B_2(\mu) + c_-(\mu,\theta_2)B_4\ (\mu)\}\phi_n(\theta_n - y).\end{aligned} \tag{1.30}$$

Finally, making an appeal to the (1.23), (1.28) in (1.30), we get relation (1.22).

Note. Partcularly, put $\phi_n = \sin n$, $\Phi_n = \cos n$ and $\theta_n = \frac{\pi}{2n}$ in (1.20), and with help of the definitions given in (1.18) and (1.19), we get following trigonometrical relations $\sin\left(\frac{\pi}{2} - ny\right) = \cos ny$ and $\cos\left(\frac{\pi}{2} - ny\right) = \sin ny$ and the addition formulae are

$$\begin{aligned}\sin\left(ny - nr\right) &= \sin ny\cos nr - \cos ny\sin nr;\\ \cos\left(ny - nr\right) &= \cos ny\cos nr + \sin ny\sin nr.\end{aligned}$$

□

2. Solution of anomalous diffusion problem

In this section, we obtain the solution of the space-time fractional anomalous diffusion problem given in the (1.9a, b, c).

We make an appeal to (1.9a), (1.10), (1.13) and (1.22), to get

$$\frac{\partial^\beta u_n(t)}{\partial t^\beta}\phi_n(y)+u_n(t)\,c_1(\lambda_n,\alpha,\theta_1)\,\phi_n(y)+u_n(t)\,c_2(\lambda'_n,\mu,\theta_2,\theta_n)\,\phi_n(y)+u_n(t)\,c_3(\lambda'_n,\mu,\theta_2)\,\phi_n(\theta_n-y)=0. \tag{2.1}$$

Dividing by $u_n(t)\,\phi_n(y)$ in (2.1), we get

$$\begin{aligned}\frac{1}{u_n(t)}\frac{\partial^\beta u_n(t)}{\partial t^\beta} &= -\left[c_1(\lambda_n,\alpha,\theta_1)+c_2(\lambda'_n,\mu,\theta_2)+c_3(\lambda'_n,\mu,\theta_2)\frac{\phi_n(\theta_n-y)}{\phi_n(y)}\right]\\ &= -\,\xi_n \text{ (any constant).}\end{aligned} \tag{2.2}$$

Therefore, from the (2.2), we get

$$\frac{\partial^\beta u_n(t)}{\partial t^\beta}=-\xi_n u_n(t),\ 0<\beta<1,\ n\in\mathbb{N}, \tag{2.3}$$

and

$$c_1(\lambda_n,\alpha,\theta_1)\phi_n(y)+c_2(\lambda'_n,\mu,\theta_2,\theta_n)\phi_n(y)+c_3(\lambda'_n,\mu,\theta_2)\phi_n(\theta_n-y)-\xi_n\phi_n(y)=0. \tag{2.4}$$

Now multiplying by $\frac{\overline{\phi_n(y)}}{\left[\int_{a'}^{b'}|\phi_n(y)|^2dy\right]^{1/2}}$ in (2.4) and then integrating the resulting identity with respect to y from a' to b', we get

$$\begin{aligned}&\frac{c_1(\lambda_n,\alpha,\theta_1)\int_{a'}^{b'}\phi_n(y)\overline{\phi_n(y)}dy}{\left[\int_{a'}^{b'}|\phi_n(y)|^2dy\right]^{1/2}}+\frac{c_2(\lambda'_n,\mu,\theta_2,\theta_n)\int_{a'}^{b'}\phi_n(y)\overline{\phi_n(y)}dy}{\left[\int_{a'}^{b'}|\phi_n(y)|^2dy\right]^{1/2}}\\ &+\frac{c_3(\lambda'_n,\mu,\theta_2)\int_{a'}^{b'}\phi_n(\theta_n-y)\overline{\phi_n(y)}dy}{\left[\int_{a'}^{b'}|\phi_n(y)|^2dy\right]^{1/2}}-\frac{\xi_n\int_{a'}^{b'}\phi_n(y)\overline{\phi_n(y)}}{\left[\int_{a'}^{b'}|\phi_n(y)|^2dy\right]^{1/2}}=0.\end{aligned} \tag{2.5}$$

Then, on applying the orthogonal property of eigen-functions in (2.5), we get

$$\xi_n=[c_1(\lambda_n,\alpha,\theta_1)+c_2(\lambda'_n,\mu,\theta_2,\theta_n)]. \tag{2.6}$$

Taking Laplace transform of (2.3), we get (see Kilbas et al. [8])

$$s^\beta u_n(s)-s^{\beta-1}u_n(0)+\xi_n u_n(s)=0. \tag{2.7}$$

The (2.7) gives us

$$u_n(s)=\frac{u_n(0)\,s^{\beta-1}}{s^\beta+\xi_n}. \tag{2.8}$$

The inverse Laplace transform of (2.8) is found by

$$u_n(t) = u_n(0)\ E_{\beta,\,1}\left(-\xi_n t^{\beta}\right) \tag{2.9}$$

(see also Kilbas et al. [8], Mathai, Saxena and Haubold [10]), where ξ_n is given in (2.6) and $E_{\beta,\,1}(\cdot)$ is well known Mittag-Leffler function (see Srivastava and Manocha [16]).

Now to find out the value of $u_n(0)$, we take the value of (1.10) at $t = 0$, and make an appeal to (1.9b) to get

$$u_0(x,y) = \sum_{n=1}^{\infty} u_n(0)\psi_n(x)\phi_n(y). \tag{2.10}$$

Then, multiply both the side of (2.10) by

$$\frac{\overline{\psi_m(x)}}{\left[\int_a^b |\psi_m(x)|^2\,dx\right]^{1/2}}\frac{\overline{\phi_m(y)}}{\left[\int_{a'}^{b'} |\phi_m(y)|^2\,dy\right]^{1/2}}$$

and integrate it with respect to x from a to b and then that with respect to y from a' to b', to get

$$\begin{aligned}&\frac{\int_a^b\int_{a'}^{b'} u_0(x,y)\,\overline{\psi_m(x)}\;\overline{\phi_m(y)}dxdy}{\left[\int_a^b |\psi_m(x)|^2\,dx\right]^{1/2}\left[\int_{a'}^{b'} |\phi_m(y)|^2\,dy\right]^{1/2}}\\ &= \sum_{n=0}^{\infty} u_n(0)\,\frac{\int_a^b \psi_n(x)\,\overline{\psi_m(x)}\,dx\int_{a'}^{b'}\phi_n(x)\,\overline{\phi_m(y)}dy}{\left[\int_a^b |\psi_m(x)|^2\,dx\right]^{1/2}\left[\int_{a'}^{b'} |\phi_m(y)|^2\,dy\right]^{1/2}}.\end{aligned} \tag{2.11}$$

Now in (2.11) making an appeal to the orthogonal property of eigen-functions, we get

$$u_n(0) = \frac{1}{\left[\int_a^b |\psi_n(x)|^2\,dx\right]\left[\int_{a'}^{b'} |\phi_n(y)|^2\,dy\right]}\int_a^b\int_{a'}^{b'} u_0(x,y)\,\overline{\psi_n(x)}\;\overline{\phi_n(y)}dxdy. \tag{2.12}$$

Therefore, with the aid of (2.9) and (2.12), we get a sequence of integrals, $n \in \mathbb{N}$

$$\begin{aligned}u_n(t) =& \left[\frac{1}{\left[\int_a^b |\psi_n(x)|^2\,dx\right]\left[\int_{a'}^{b'} |\phi_n(y)|^2\,dy\right]}\int_a^b\int_{a'}^{b'} u_0(x,y)\,\overline{\psi_n(x)\phi_n(y)}dxdy\right]\\ &\times E_{\beta,1}\left(-\xi_n t^{\beta}\right),\end{aligned} \tag{2.13}$$

where ξ_n is given in the (2.6).

Thus, the solution of that anomalous diffusion equation is given by

$$u(x,y,t) = \sum_{n=1}^{\infty} u_n(t)\,\psi_n(x)\,\phi_n(y),$$

where $u_n(t)$ is given in (2.13).

3. Numerical approximation formula

In this section, we obtain the numerical solution of the problem on applying Grünwald-Letnikov approximation for Caputo derivative.

The Riemann-Liouville derivative ${}_aD_t^\beta$ and Caputo derivative ${}_a^CD_t^\beta \equiv \frac{\partial^\beta}{\partial t^\beta}$ are related by (see Özdemir et al. [13])

$$(3.1)\qquad {}_aD_t^\beta u(t) = {}_a^CD_t^\beta u(t) + \sum_{r=0}^{m-1} \frac{d^r}{dt^r} u(t) \mid_{t=a} \frac{(t-a)^{r-\beta}}{\Gamma(r-\beta+1)},$$

where $m \in \mathbb{N},\ m-1 < \beta \le m,\ a \in \mathbb{R}$.

Note that, under the assumption $\mid \lim_{a\to-\infty} \frac{d^r}{dt^r} u(t) \mid_{t=a} < \infty$ for $r = 0, 1, 2, \ldots, m-1$, we have

$$(3.2)\qquad {}_{-\infty}D_t^\beta u(t) = {}_{-\infty}^CD_t^\beta u(t).$$

Since in our problem $0 < \beta < 1$ and $a = 0$, therefore (3.1) gives us

$$(3.3)\qquad {}_0^CD_t^\beta u(t) = {}_0D_t^\beta u(t) - u(0) \frac{t^{-\beta}}{\Gamma(1-\beta)}.$$

Thus in (3.3), use Grünwald-Letnikov formula (see Özdemir et al. [13])

$${}_0D_t^\beta u_n(t) \approx \frac{1}{h^\beta} \sum_{r=0}^{M} w_r^{(\beta)} u_n(hM - rh), M = \frac{t}{h},\ h \text{ is step size,}$$

we get approximation of Caputo derivative for $u_n(t)$ in the form

$$(3.4)\qquad {}_0^CD_t^\beta u_n(t) \approx \frac{1}{h^\beta} \sum_{r=0}^{M} w_r^{(\beta)} u_n(hM - rh) - u_n(0) \frac{(hM)^{-\beta}}{\Gamma(1-\beta)},$$

where $M = \frac{t}{h}$ represents the number of sub-time intervals, h is step size and $w_r^{(\beta)}$ are the coefficients of Günwald-Letnikov formula $\left(w_r^{(\beta)} = (-1)^r \binom{c\beta}{r}\right)$, particularly,

$$(3.5)\qquad w_0^{(\beta)} = 1, w_r^{(\beta)} = \left(1 - \frac{\beta+1}{r}\right) w_{r-1}^{(\beta)}.$$

Now making an appeal to (2.3) and (3.4), we get

$$(3.6)\qquad u_n(t) = \frac{1}{\left(h^\beta \xi_n - w_0^{(\beta)}\right)} \left[\sum_{r=1}^{M} w_r^{(\beta)} u_n(hM - rh) - u_n(0) \frac{(M)^{-\beta}}{\Gamma(1-\beta)}\right],$$

where ξ_n is given in (2.6), $w_0^{(\beta)}$ and $w_r^{(\beta)}$ are given in (3.5).

Then, making an appeal to (1.10) and (3.6), we get the numerical approximation formula of anomalous diffusion (1.9a, b, c) in the form

(3.7)
$$u(x,y,t) = \sum_{n=1}^{\infty} \frac{1}{\left(h^{\beta}\xi_n - w_0^{(\beta)}\right)} \left[\sum_{r=1}^{M} w_r^{(\beta)} u_n\left(hM - rh\right) - u_n\left(0\right) \frac{\left(M\right)^{-\beta}}{\Gamma\left(1-\beta\right)}\right] \times \psi_n\left(x\right)\phi_n\left(y\right).$$

4. Particular cases

Set $\psi_{\mathrm{n}}(\mathrm{x}) = e^{inx}$ in Theorem A and $\phi_n(y) = \sin ny$, $\theta_{\mathrm{n}} = \frac{\pi}{2n}$ in Theorem B, then from Theorem A we get

$$A_n(\alpha) = (in)^{\alpha-1}, \; A_n'(\alpha) = (-in)^{\alpha-1} \text{ and } \lambda_n = n.$$

Therefore, we have

$$c_1(n,\alpha,\theta_1) = \left[(in)^{\alpha} c_+(\alpha,\theta_1) + (in)^{\alpha} c_-(\alpha,\theta_1)\right], \quad 0<\alpha<1, \; \theta_1 \le \min\{\alpha, \; 1-\alpha\}. \tag{4.1}$$

Again, from Theorem B we find $\lambda_n^{'} = n$,

$$B_1\left(\mu, \frac{\pi}{2n}\right) = (n)^{\mu-2}\cos\frac{\pi}{2}\left(2-\mu\right), \tag{4.2}$$

$$B_2\left(\mu\right) = (-n)^{\mu-2}\sin\frac{\pi}{2}\left(2-\mu\right), \tag{4.3}$$

$$B_3\left(\mu, \frac{\pi}{2n}\right) = (-n)^{\mu-2}\cos\frac{\pi}{2}\left(2-\mu\right), \tag{4.4}$$

$$B_4\left(\mu\right) = (n)^{^{\mu-2}}\sin\frac{\pi}{2}\left(2-\mu\right). \tag{4.5}$$

Therefore,

$$c_2\left(n,\mu,\theta_2,\frac{\pi}{2n}\right) = -\left(n\right)^{\mu}\cos\frac{\pi}{2}\left(2-\mu\right)\left[c_+\left(\mu,\theta_2\right) + (-1)^{\mu-2} c_-\left(\mu,\theta_2\right)\right] \tag{4.6}$$

and

$$c_3\left(n,\mu,\theta_2\right) = -\left(-n\right)^{\mu}\sin\frac{\pi}{2}\left(2-\mu\right)\left[(-1)^{\mu-2} c_+\left(\mu,\theta_2\right) - c_-\left(\mu,\theta_2\right)\right], \quad 1<\mu\le 2, \; \theta_2 \le \min\{\mu, \; 2-\mu\}. \tag{4.7}$$

Again, on using (2.6), (4.1), (4.6) and (4.7), we get

$$\xi_n = \left[(in)^{\alpha}\left\{c_+\left(\alpha,\theta_1\right) + c_-\left(\alpha,\theta_1\right)\right\} - (n)^{\mu} \times \cos\frac{\pi}{2}\left(2-\mu\right)\left\{c_+\left(\mu,\theta_2\right) + (-1)^{\mu-2} c_-\left(\mu,\theta_2\right)\right\}\right]. \tag{4.8}$$

Now set $u_0(x,y) = (\sin x)^v \sin ny$, such that $\mathrm{Re}(v) > -1,\ 0 \le x \le \pi,\ 0 \le y \le \pi$, in (2.13), we get

$$u_n(t) = \frac{e^{\frac{-in\pi}{2}}\ \Gamma(\nu+1)}{2^v \Gamma\left(1+\frac{\nu-n}{2}\right)\ \Gamma\left(1+\frac{\nu+n}{2}\right)} E_{\beta,1}\left(-\xi_n t^\beta\right). \tag{4.9}$$

Hence, the particular solution of the problem (1.9a, b, c) is found by

$$u(x,y,t) = \sum_{n=1}^{\infty} \frac{e^{in\left(x-\frac{\pi}{2}\right)}\ \Gamma(\nu+1)}{2^v \Gamma\left(1+\frac{\nu-n}{2}\right)\ \Gamma\left(1+\frac{\nu+n}{2}\right)} \sin ny E_{\beta,1}\left(-\xi_n t^\beta\right). \tag{4.10}$$

Also, using above particular cases given of (4.1) in (3.7), we get the approximation solution of problem (1.9a, b, c) in the form

(4.11)

$$u(x,y,t) = \sum_{n=1}^{\infty} \frac{1}{\left(h^\beta \xi_n - w_0^{(\beta)}\right)} \left[\sum_{r=1}^{M} w_r^{(\beta)} u_n(hM - rh) - u_n(0) \frac{(M)^{-\beta}}{\Gamma(1-\beta)}\right] \times e^{inx} \sin ny.$$

Example 1. $\alpha = 0.5,\ \mu = 1.5,\ \beta = 0.6,\ 0 \le x \le \pi,\ 0 \le y \le \pi,\ t = 5,\ \nu = 1.5$ then $\theta_1 = 0.5$ and $\theta_2 = 0.5$; $c_+(0.5, 0.5) = 0$ and $c_-(0.5, 0.5) = 1$; $c_+(1.5, 0.5) = 1$ and $c_-(1.5, 0.5) = 0$;

$$|\xi_n| = \sqrt{\frac{n(n-1)^2 + n}{2}}. \tag{4.12}$$

From (4.10) the real value of $u(x,y,t)$ for $|\xi_n|$ is given by

$$\mathrm{Re}u(x,y,t) = \sum_{n=1}^{\infty} \frac{\cos n\left(x-\frac{\pi}{2}\right)\ \Gamma(\nu+1)}{2^v \Gamma\left(1+\frac{\nu-n}{2}\right)\ \Gamma\left(1+\frac{\nu+n}{2}\right)} \sin ny E_{\beta,1}\left(-|\xi_n| t^\beta\right). \tag{4.13}$$

For all values given in (4.12) and the formula (4.13) ,we plot following graph of Re $u(x,y,t)$ with the help of Wolfram *Mathematica* 7, like Egg-Tray (see Figure 1).

The real value of $u(x,y,t)$ from approximation formula given in (4.11) is found by

(4.14)

$$Re\ u(x,y,t) = \sum_{n=1}^{\infty} \frac{1}{\left(h^\beta \xi_n - w_0^{(\beta)}\right)} \left[\sum_{r=1}^{M} w_r^{(\beta)} u_n(hM - rh) - u_n(0) \frac{(M)^{-\beta}}{\Gamma(1-\beta)}\right] \times \cos nx \sin ny.$$

For all values of Eqn. (4.12), $u_{10}(0) = u_{10} = -0.00193417$, at $t = 0$ and with the help of formula (4.14) and for $\beta = 1$ (standard diffusion) we note that all coefficients $w_r^{(\beta)}(r > 1)$ vanishing except $w_1^{(1)}(w_1^{(1)} = -1)$.

We plot following graph of Re $u(x,y,t)$ like Egg-Tray (see Figure 2).

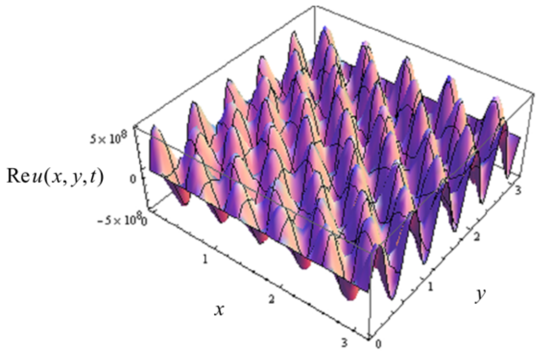

FIGURE 1. For $n = 1$ to 10, $0 \le x \le \pi,\ 0 \le y \le \pi$ and $t = 1$ to 5

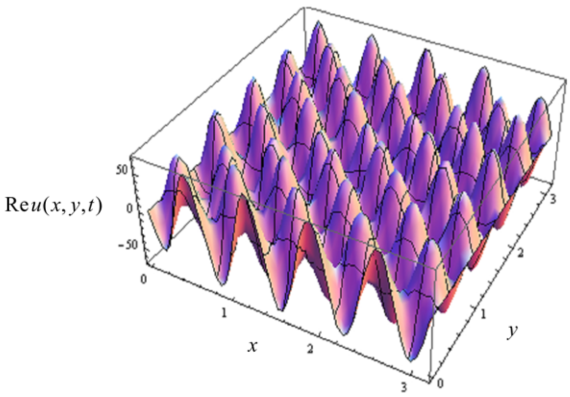

FIGURE 2. For $n = 1$ to 10, $0 \le x \le \pi$, $0 \le y \le \pi$ and $t = 1$ to 5

Concluding remarks. Both Figures 1 and 2 seems same and like Egg-Tray, for same values of x, y, t and n. By our formula we get larger values of $\mathrm{Re}u(x, y, t)$ than due to numerical approximation formula (4.14).

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