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LIOUVILLE TYPE THEOREM FOR p-HARMONIC MAPS II

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Abstract. Let M be a complete Riemannian manifold and let N be a Riemannian manifold of non-positive sectional curvature. Assume that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ at all $x \in M$ and $\mathrm{Vol}(M)$ is infinite, where $\mu_0 > 0$ is the infimum of the spectrum of the Laplacian acting on L^2-functions on M. Then any p-harmonic map $\phi : M \to N$ of finite p-energy is constant. Also, we study Liouville type theorem for p-harmonic morphism.

1. Introduction

Let (M, g) and (N, h) be smooth Riemannian manifolds and let $\phi : M \to N$ be a smooth map. For a compact domain $\Omega \subset M$, the *p-energy* $E_p(\phi; \Omega)$ of ϕ over Ω is defined by

$$E_p(\phi; \Omega) = \frac{1}{p}\int_\Omega |d\phi|^p \mu_M, \tag{1.1}$$

where the differential $d\phi$ is a section of the bundle $T^*M \otimes \phi^{-1}TN \to M$ and $\phi^{-1}TN$ denotes the pull-back bundle via the map ϕ. The bundle $T^*M \otimes \phi^{-1}TN \to M$ carries the connection ∇ induced by the Levi-Civita connections on M and N. A map $\phi : M \to N$ is called *p-harmonic* if the p-tension field $\tau_p(\phi) = 0$, which is defined by

$$\tau_p(\phi) = \mathrm{tr}_g \nabla(|d\phi|^{p-2} d\phi), \tag{1.2}$$

where tr_g denote the trace with respect to the metric g. A p-harmonic map ϕ is a critical point of the energy functional defined by (1.1) on any compact domain $\Omega \subset M$. When $p = 2$, p-harmonic maps are well-known to be harmonic maps. Several studies are given for harmonic maps (see [5], [6], [7], [8], [10], [11], [12], [13], [14], [16]). Let μ_0 be the infimum of the spectrum of the Laplacian Δ_M acting on L^2-functions on M and Ric^M be the Ricci tensor of M.

Recently, D. J. Moon, H. Liu and S. D. Jung [9] proved the following theorem for p-harmonic maps.

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Theorem 1.1 ([9]). *Let M be a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ for all x and $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0. Let N be a complete Riemannian manifold of non-positive sectional curvature. Then any p-harmonic map $\phi : M \to N$ of $E_p(\phi) < \infty$ is constant.*

In Theorem 1.1, the condition $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0 is essential. In this paper, we prove Theorem 1.1 when the volumn is infinite, i.e., $\mathrm{Vol}(M) = \infty$ instead of $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point. Then we have the following theorem.

Theorem A *Let M be a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ for all x and* Vol(M) *is infinite. Let N be a complete Riemannian manifold of non-positive sectional curvature. Then any p-harmonic map $\phi : M \to N$ of $E_p(\phi) < \infty$ is constant.*

A map $\phi : (M, g) \to (N, h)$ is a *p-harmonic morphism* if it pulls back (local) p-harmonic functions on N to (local) p-harmonic functions on M, i.e., for any function $f : V \subset N \to \mathbb{R}$ if $\tau_p(f) = 0$, then $\tau_p(f \circ \phi) = 0$. It is well-known [4, 8] that a non-constant map is a p-harmonic morphism if and only if it is a horizontally weakly conformal p-harmonic map. A *horizontally weakly conformal* map $\phi : (M, g) \to (N, h)$ generalizes the notion of a Riemannian submersion in that for any $x \in M$ at which $d\phi_x \neq 0$, the restriction $d\phi_x|_{H_x} : H_x \to T_{\phi(x)}N$ is conformal and surjective, where the horizontal space H_x is the orthogonal complement of $V_x = \mathrm{Ker}(d\phi_x)$ in T_xM. Trivially, if we put $C_\phi = \{x \in M \mid d\phi_x = 0\}$, then there exists a function $\lambda : M \backslash C_\phi \to \mathbb{R}^+$ such that

$$h(d\phi(X), d\phi(Y)) = \lambda^2 g(X, Y) \quad \forall X, Y \in H. \tag{1.3}$$

Note that at the point $x \in C_\phi$ we can let $\lambda(x) = 0$ and obtain a continuous function $\lambda : M \to \mathbb{R}^+ \cup \{0\}$ which is called the *dilation* of a horizontally weakly conformal map ϕ. A non-constant horizontally weakly conformal map ϕ is said to be *horizontally homothetic* if the gradient of $\lambda^2(x)$ is vertical, meaning that $X(\lambda^2) = 0$ for any horizontal vector field X on M. In 2008, D. J. Moon, H. Liu and S. D. Jung [9] also proved the following.

Theorem 1.2 ([9]). *Let M be a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ for all x and $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point x_0. Let N be a complete Riemannian manifold of non-positive scalar curvature. Then any p-harmonic morphism $\phi : M \to N$ of $E_p(\phi) < \infty$ is constant.*

In this paper, we prove Theorem 1.2 under the condition $\mathrm{Vol}(M) = \infty$ instead of $Ric^M > -\frac{4(p-1)}{p^2}\mu_0$ at some point.

Theorem B *Let M be a complete Riemannian manifold such that $Ric^M \geq -\frac{4(p-1)}{p^2}\mu_0$ for all x and the volume* Vol(M) *is infinite. Let N be a complete*

Riemannian manifold of non-positive scalar curvature. Then any p-harmonic morphism $\phi : M \to N$ of $E_p(\phi) < \infty$ is constant.

2. The Weitzenböck formula

First, we recall the Weitzenböck formula. Let (M^m, g) and (N^n, h) be Riemannian manifolds with $\dim M = m \geq n = \dim N$. Let $\phi : M \to N$ be a smooth map and $E = \phi^{-1}TN$ be the induced bundle over M. Then E has a naturally induced metric connection $\nabla \equiv \phi^{-1}\nabla^N$ and $d\phi$ is a cross section of $\mathrm{Hom}(TM, E)$ over M. Since $\mathrm{Hom}(TM, E)$ is canonically identified with $T^*M \otimes E$, $d\phi$ is regarded as an E-valued 1-form. Let $d_\nabla : A^r(E) \to A^{r+1}(E)$ be an anti-derivation and δ_∇ the formal adjoint of d_∇, where $A^r(E)$ is the space of E-valued r-forms with an inner product $\langle \cdot, \cdot \rangle$ on M. Let $\{e_i\}_{i=1,\ldots,m}$ be local orthonormal frame field on M and let $\{\omega^i\}$ be its dual coframe field. Locally, the operators d_∇ and δ_∇ are expressed by

$$d_\nabla = \sum_{j=1}^{m} \omega^j \wedge \nabla_{e_j} \quad \text{and} \quad \delta_\nabla = -\sum_{j=1}^{m} i(e_j)\nabla_{e_j},$$

respectively, where $i(X)$ is the interior product. The Laplacian Δ on $A^*(E)$ is defined by

$$\Delta = d_\nabla \delta_\nabla + \delta_\nabla d_\nabla. \tag{2.1}$$

Then we have the following Weitzenböck formula.

Lemma 2.1 (cf. [6], [7])**.** *Let $\phi : (M^m, g) \to (N^n, h)$ be an arbitrary smooth map. Then the Weitzenböck formula is given by*

$$-\frac{1}{2}\Delta^M |d\phi|^{2p-2} = |\nabla(|d\phi|^{p-2}d\phi)|^2 - \langle |d\phi|^{p-2}d\phi, \Delta(|d\phi|^{p-2}d\phi)\rangle + F(\phi), \tag{2.2}$$

where

$$\begin{aligned} F(\phi) =& |d\phi|^{2p-4} \sum_{k=1}^{m} h(d\phi(Ric^M(e_k)), d\phi(e_k)) \\ &- |d\phi|^{2p-4} \sum_{k,j=1}^{m} h(R^N(d\phi(e_j), d\phi(e_k))d\phi(e_k), d\phi(e_j)). \end{aligned} \tag{2.3}$$

Let $\phi : (M, g) \to (N, h)$ be a p-harmonic map. Then, from (1.2)

$$\delta_\nabla(|d\phi|^{p-2}d\phi) = 0. \tag{2.4}$$

Then we have the following lemma.

Lemma 2.2 ([9])**.** *Let M be a complete Riemannian manifold such that for some constant $C \geq 0$, $Ric^M \geq -C$ at all $x \in M$ and let N be a Riemannian* *manifold of non-positive sectional curvature. If $\phi : (M, g) \to (N, h)$ is a p-harmonic map, then*

$$|d\phi|\Delta^M |d\phi|^{p-1} - G_p(\phi) \le -|d\phi|^{p-2} \sum_{i=1}^m h(d\phi(Ric^M(e_i)), d\phi(e_i)) \le C|d\phi|^p,$$

where $G_p(\phi) = \langle d\phi, \delta_\nabla d_\nabla(|d\phi|^{p-2} d\phi)\rangle$. If ϕ is harmonic, then $G_2(\phi) = 0$.

Let x_0 be a point of M and fix it. We choose a Lipschitz continuous function ω_ℓ on M such that $0 \le \omega_\ell(y) \le 1$ for any $y \in M$, $\omega_\ell \equiv 1$ on $B(x_0, \ell)$, supp $\omega_\ell \subset B(x_0, 2\ell)$, $\lim_{\ell\to\infty} \omega_\ell = 1$ and $|d\omega_\ell| \le \tilde{C}/\ell$ for some constant $\tilde{C} > 0$, where $\ell \in \mathbb{R}_+$ and $B(x_0, \ell)$ is the Riemannian open ball with radius ℓ.

Lemma 2.3. *Let M and N be complete Riemannian manifolds. For any smooth map $\phi : (M, g) \to (N, h)$, we have*

$$\begin{aligned} |\int_{B(2l)} \omega_l^2 G_p(\phi)| &\le A_1 \int_M \omega_\ell |d\omega_\ell||d\phi|^{\frac{p}{2}} |d|d\phi|^{\frac{p}{2}}| \\ &\le A_1 \Big(\int_M |d\omega_\ell|^2 |d\phi|^p\Big)^{\frac12} \Big(\int_M \omega_\ell^2 |d|d\phi|^{\frac{p}{2}}|^2\Big)^{\frac12}, \end{aligned}$$

where $A_1 = \frac{4(p-2)}{p} b^2$ for some constant b. In particular, if $\phi : M \to N$ satisfies $E_p(\phi) < \infty$ and $\int_M |d|d\phi|^{\frac{p}{2}}|^2 < \infty$, then

$$\int_M \omega_l^2 G_p(\phi) \to 0 \ (l \to \infty).$$

Proof. It is well-known [10] that for a function f on M and for some constant $b > 0$,

$$|d_\nabla(f d\phi)| \le b|df||d\phi|. \tag{2.5}$$

By the Schwartz's inequality with (2.5), we have

$$\begin{aligned} |\int_{B(2l)} \langle \omega_\ell^2 d\phi, \delta_\nabla d_\nabla(|d\phi|^{p-2} d\phi)\rangle| &= |\int_{B(2l)} \langle d_\nabla(\omega_\ell^2 d\phi), d_\nabla(|d\phi|^{p-2} d\phi)\rangle| \\ &\le \int_{B(2l)} |d_\nabla(\omega_\ell^2 d\phi)||d_\nabla(|d\phi|^{p-2} d\phi)| \\ &\le 2b^2 \int_{B(2l)} |\omega_\ell d\omega_\ell||d|d\phi|^{p-2}||d\phi|^2 \\ &\le A_1 \int_{B(2l)} \omega_\ell |d\omega_\ell||d\phi|^{\frac{p}{2}} |d|d\phi|^{\frac{p}{2}}|, \end{aligned}$$

where $A_1 = \frac{4(p-2)}{p} b^2$. By the Hölder inequality, we have

$$\int_{B(2l)} \omega_\ell |d\omega_\ell||d\phi|^{\frac{p}{2}} |d|d\phi|^{\frac{p}{2}}| \le \Big(\int_{B(2l)} |d\omega_\ell|^2 |d\phi|^p\Big)^{\frac12} \Big(\int_{B(2l)} \omega_\ell^2 |d|d\phi|^{\frac{p}{2}}|^2\Big)^{\frac12},$$

which completes the proof. $\square$

3. Proof of Theorem A

Let M be a complete Riemannian manifold such that $Ric^M \geq -C$, where $C = \frac{4(p-1)}{p^2}\mu_0$. From Lemma 2.2, if we multiply by ω_ℓ^2 and integrate by parts, we get

$$\begin{aligned}
&\int_M \langle \omega_\ell^2 |d\phi|, \Delta^M |d\phi|^{p-1}\rangle - \int_M \omega_\ell^2 G_p(\phi) \\
&\leq \ -\sum_{i=1}^m \int_M \omega_\ell^2 |d\phi|^{p-2} h(d\phi(Ric^M(e_i)), d\phi(e_i)) \\
&\leq C \int_M \omega_\ell^2 |d\phi|^p.
\end{aligned} \tag{3.1}$$

On the other hand, by using the Schwartz's inequality $|\langle V, W\rangle| \leq |V||W|$, we have

$$\begin{aligned}
&\int_M \langle \omega_\ell^2 |d\phi|, \Delta^M |d\phi|^{p-1}\rangle \\
&= A_2 \int_M \langle |d\phi|^{\frac{p}{2}} d\omega_\ell, \omega_\ell d|d\phi|^{\frac{p}{2}}\rangle + \frac{A_2}{p}\int_M \omega_\ell^2 |d|d\phi|^{\frac{p}{2}}|^2 \\
&\geq \ -A_2 \int_M \omega_\ell |d\phi|^{\frac{p}{2}} |d\omega_\ell||d|d\phi|^{\frac{p}{2}}| + \frac{A_2}{p}\int_M \omega_\ell^2 |d|d\phi|^{\frac{p}{2}}|^2,
\end{aligned} \tag{3.2}$$

where $A_2 = \frac{4(p-1)}{p}$. From Lemma 2.3 and (3.2), we have

$$\begin{aligned}
&\int_M \langle \omega_l^2 |d\phi|, \Delta^M |d\phi|^{p-1}\rangle - \int_M \omega_l^2 G_p(\phi) \\
&\geq \ -(A_1 + A_2)\int_M \omega_l |d\omega_l||d\phi|^{\frac{p}{2}} |d|d\phi|^{\frac{p}{2}}| + \frac{A_2}{p}\int_M |\omega_l d|d\phi|^{\frac{p}{2}}|^2 \\
&\geq \ -\frac{1}{2\epsilon}(A_1 + A_2)\int_M |d\omega_l|^2 |d\phi|^p + \Big(\frac{A_2}{p} - \frac{\epsilon}{2}(A_1 + A_2)\Big)\int_M \omega_l^2 |d|d\phi|^{\frac{p}{2}}|^2,
\end{aligned}$$

where $0 < \epsilon < \frac{2A_2}{p(A_1+A_2)}$. From (3.1), if we let $l \to \infty$, then

$$\Big(\frac{A_2}{p} - \frac{\epsilon}{2}(A_1 + A_2)\Big)\int_M |d|d\phi|^{\frac{p}{2}}|^2 \leq C\int_M |d\phi|^p.$$

And if we let $\epsilon \to 0$, then

$$\frac{A_2}{p}\int_M |d|d\phi|^{\frac{p}{2}}|^2 \leq C\int_M |d\phi|^p. \tag{3.3}$$

Hence $d|d\phi|^{\frac{p}{2}} \in L^2$. Hence, from Lemma 2.3, if we let $\ell \to \infty$, then

$$\int_M \omega_\ell |d\omega_\ell||d\phi|^{\frac{p}{2}} |d|d\phi|^{\frac{p}{2}}| \to 0 \quad \text{and} \quad \int_M \omega_l^2 G_p(\phi) \to 0. \tag{3.4}$$

By the Rayleigh quotient theorem, i.e., $\frac{\int_M \langle df, df\rangle}{\int_M f^2} \geq \mu_0$ for any smooth function f such that $supp(f) \subset \Omega$, a compact domain, and the Hölder inequality, if we put $f = \omega_\ell |d\phi|^{\frac{p}{2}}$, then

$$\mu_0 \int_M |d\phi|^p \leq \int_M |d|d\phi|^{\frac{p}{2}}|^2. \tag{3.5}$$

From (3.3) and (3.5), we have

$$\mu_0 \int_M |d\phi|^p \leq \int_M |d|d\phi|^{\frac{p}{2}}|^2 \leq \mu_0 \int_M |d\phi|^p. \tag{3.6}$$

Since μ_0 is the infimum of the spectrum, from (3.6),

$$\Delta_M |d\phi|^{\frac{p}{2}} = \mu_0 |d\phi|^{\frac{p}{2}}, \tag{3.7}$$

which implies that $|d\phi|$ is constant by the maximum principle [15]. Since $\mathrm{Vol}(M)$ is infinite, $E_p(\phi) < \infty$ implies that $d\phi = 0$, i.e., ϕ is constant.

4. Proof of Theorem B

Let $\phi : (M^m, g) \to (N^n, h)$ $(m \geq n)$ be a p-harmonic morphism with dilation λ. Let $\{e_i\}_{i=1,\ldots,m}$ be a local orthonormal frame field on M such that $\{e_i\}_{i=1,\ldots,n} \in H_x$ and $\{e_i\}_{i=n+1,\ldots,m} \in V_x$. Then it is trivial from (1.3) that

$$|d\phi|^2 = n\lambda^2. \tag{4.1}$$

Moreover, it is easy to see that

$$\sum_{i=1}^{m} h(d\phi(Ric^M(e_i)), d\phi(e_i)) = \lambda^2 \mathrm{tr}_g(Ric^M|_H) \tag{4.2}$$

and

$$\sum_{i,j=1}^{m} h(R^N(d\phi(e_i), d\phi(e_j))d\phi(e_j), d\phi(e_i)) = \lambda^4 \mathrm{scal}_N \circ \phi, \tag{4.3}$$

where $Ric^M|_H$ is the Ricci tensor of M on the horizontal distribution H and scal_N is the scalar curvature of N. From (4.1), (4.2) and (4.3), we have the following lemma.

Lemma 4.1 ([9]). *Let $\phi : (M, g) \to (N, h)$ be a p-harmonic morphism with dilation λ. Then*

$$\begin{aligned} -\frac{1}{2} n \Delta^M \lambda^{2p-2} = {} & |\nabla(\lambda^{p-2} d\phi)|^2 - \langle d\phi, \delta_\nabla d_\nabla(\lambda^{p-2} d\phi)\rangle \\ & + \lambda^{2p} \mathrm{tr}_g(Ric^M|_H) - \lambda^{2p} \mathrm{scal}_N \circ \phi. \end{aligned} \tag{4.4}$$

Lemma 4.2 ([9]). *Let M be a complete Riemannian manifold such that $Ric^M \geq -C$ at all $x \in M$ and let N be a Riemannian manifold of non-positive scalar curvature. If $\phi : (M, g) \to (N, h)$ is a p-harmonic morphism, then*

$$n\lambda \Delta^M \lambda^{p-1} - \langle d\phi, \delta_\nabla d_\nabla(\lambda^{p-2} d\phi)\rangle \leq -\lambda^p \mathrm{tr}_g(Ric^M|_H) \leq nC\lambda^p. \tag{4.5}$$

Proof of Theorem B. Let us put $C = \frac{4(p-1)}{p^2}\mu_0$ in Lemma 4.2. By the same process as in the proof of Theorem A, we have

$$\Delta_M \lambda^{\frac{p}{2}} = \mu_0 \lambda^{\frac{p}{2}}. \tag{4.6}$$

By the maximum principle [15], λ is constant. Since $\mathrm{Vol}(M)$ is infinite, $\lambda = 0$, i.e., ϕ is constant. □

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