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# CERTAIN CLASS OF CONTACT $CR$-SUBMANIFOLDS OF A SASAKIAN SPACE FORM

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Abstract. In this paper we investigate $(n+1)(n \geq 3)$-dimensional contact $CR$-submanifolds $M$ of $(n-1)$ contact $CR$-dimension in a complete simply connected Sasakian space form of constant $\phi$-holomorphic sectional curvature $c \neq -3$ which satisfy the condition $h(FX, Y)+h(X, FY) = 0$ for any vector fields $X, Y$ tangent to $M$, where $h$ and $F$ denote the second fundamental form and a skew-symmetric endomorphism (defined by (2.3)) acting on tangent space of $M$, respectively.

## 1. Introduction

Let $S^{2m+1}$ be a $(2m+1)$-unit sphere in the complex $(m+1)$-space $\mathbb{C}^{m+1}$, i.e.,

$$S^{2m+1} := \{(z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1} \mid \sum_{j=1}^{m+1} |z_j|^2 = 1\}.$$

For any point $z \in S^{2m+1}$ we put $\xi = Jz$, where $J$ denotes the complex structure of $\mathbb{C}^{m+1}$. Denoting by $\pi$ the orthogonal projection : $T_z\mathbb{C}^{m+1} \to T_zS^{2m+1}$ and putting $\phi = \pi \circ J$, we can see that the set $(\phi, \xi, \eta, g)$ defines a Sasakian structure on $S^{2m+1}$, where $g$ is the standard metric on $S^{2m+1}$ induced from that of $\mathbb{C}^{m+1}$ and $\eta$ is a 1-form dual to $\xi$. Hence $S^{2m+1}$ can be considered as a Sasakian manifold of constant curvature 1 (cf. [2, 4, 5, 6, 7, 8, 9]).

Let $M$ be an $(n+1)$-dimensional submanifold tangent to the structure vector field $\xi$ of $S^{2m+1}$ and denote by $\mathcal{D}_x$ the $\phi$-invariant subspace $T_xM \cap \phi T_xM$ of the tangent space $T_xM$ of $M$ at $x \in M$. Then $\xi$ cannot be contained in $\mathcal{D}_x$ at any point $x \in M$ (cf. [7]). Thus the assumption $\dim\mathcal{D}_x^\perp$ being constant and equal to 2 at each point $x \in M$ yields that $M$ can be dealt with a contact $CR$-submanifold in the sense of Yano-Kon (cf. [4, 7, 8, 9]), where $\mathcal{D}_x^\perp$ denotes the complementary orthogonal subspace to $\mathcal{D}_x$ in $T_xM$. In fact, if there exists a non-zero vector $U$ which is orthogonal to $\xi$ and contained in $\mathcal{D}_x^\perp$, then $N := \phi U$

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must be normal to $M$. In particular we can easily see that real hypersurfaces tangent to $\xi$ of $S^{2m+1}$ are typical examples of such submanifolds.

In this point of view, the present authors investigated $(n+1)$-dimensional contact $CR$-submanifolds of $(n-1)$ contact $CR$-dimension in $S^{2m+1}$, namely, those with $\dim\mathcal{D}_x = n-1$ at each point $x$ in $M$ (cf. [4, 5, 8]) and proved the following (cf. [4]).

**Theorem K-P** ([4]). *Let $M$ be an $(n+1)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension in a $(2m+1)$-unit sphere $S^{2m+1}$. If for any vector fields $X, Y$ tangent to $M$, the equality given in* (3.1) *holds on $M$, then $M$ is locally isometric to*

$$S^{2n_1+1}(r_1) \times S^{2n_2+1}(r_2) \quad (r_1^2 + r_2^2 = 1)$$

*for some integers $n_1$, $n_2$ with $n_1 + n_2 = (n-1)/2$.*

In this paper we study $(n+1)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension in a Sasakian space form and determine such submanifolds in a complete simply connected Sasakian space form (cf. [9, Theorem 5.5, p. 282]) of constant $\phi$-holomorphic sectional curvature $c \neq -3$ under assumption that the equality given in (3.1) holds on $M$, which gives an improvement of the above Theorem K-P.

All manifolds, submanifolds and geometric objects will be assumed to be connected, differentiable and of class $C^\infty$, and all maps also be of class $C^\infty$ if not stated otherwise.

## 2. Fundamental properties of contact $CR$-submanifolds

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure $(\phi, \xi, \eta, g)$. Then, by definition, it follows that

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2.1}$$

for any vector fields $X$, $Y$ tangent to $\overline{M}$ (cf. [1, 9]).

Let $M$ be an $(n+1)$-dimensional submanifold tangent to the structure vector field $\xi$ of $\overline{M}$. If the $\phi$-invariant subspace $\mathcal{D}_x$ has constant dimension for any $x \in M$, then $M$ is called a *contact $CR$-submanifold* and the constant is called *contact $CR$-dimension* of $M$ (cf. [4, 7, 8]).

From now on we assume that $M$ is a contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension in $\overline{M}$, where $n-1$ must be even. Then, as was already mentioned in $S^{2m+1}$, the structure vector $\xi$ is always contained in $\mathcal{D}_x^\perp$ and $\phi\mathcal{D}_x^\perp \subset T_xM^\perp$ at any point $x \in M$. Further, by definition $\dim\mathcal{D}_x^\perp = 2$ at any point $x \in M$, and so there exists a unit vector field $U$ contained in $\mathcal{D}^\perp$ which is orthogonal to $\xi$. Since $\phi\mathcal{D}^\perp \subset TM^\perp$, $\phi U$ is a unit normal vector field to $M$, which will be denoted by $N$, that is,

$$N := \phi U. \tag{2.2}$$

Moreover, it is clear that $\phi TM \subset TM \oplus \mathrm{Span}\{N\}$. Hence we have, for any tangent vector field $X$ and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\dots,p}$ ($N_1 := N$, $p := 2m-n$) of normal vectors to $M$, the following decomposition in tangential and normal components:

$$\phi X = FX + u(X)N, \tag{2.3}$$

$$\phi N_\alpha = PN_\alpha, \quad \alpha = 2,\dots,p. \tag{2.4}$$

It is easily shown that $F$ is a skew-symmetric linear endomorphism acting on $T_xM$. Since the structure vector field $\xi$ is tangent to $M$, (2.1), (2.2) and (2.3) imply

$$F\xi = 0,\ FU = 0,\ g(U,X) = u(X),\ u(\xi) = g(U,\xi) = 0,\ u(U) = 1. \tag{2.5}$$

Next, applying $\phi$ to (2.3) and using (2.1), (2.2), (2.3) and (2.5), we also have

$$F^2X = -X + \eta(X)\xi + u(X)U, \quad u(FX) = 0. \tag{2.6}$$

On the other hand, it is clear from (2.1) and (2.5) that

$$\phi N = -U, \tag{2.7}$$

which combined with (2.4) yields the existence of a local orthonormal basis $\{N, N_a, N_{a^*}\}_{a=1,\dots,q}$ of normal vectors to $M$ such that

$$N_{a^*} := \phi N_a, \quad a = 1,\dots,q := p/2. \tag{2.8}$$

We denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connection on $\overline{M}$ and $M$, respectively, and by $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of $M$. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.9}$$

$$\overline{\nabla}_X N = -AX + \nabla^\perp_X N = -AX + \sum_{a=1}^{q}\{s_a(X)N_a + s_{a^*}(X)N_{a^*}\}, \tag{2.10$_1$}$$

$$\overline{\nabla}_X N_a = -A_aX - s_a(X)N + \sum_{b=1}^{q}\{s_{ab}(X)N_b + s_{ab^*}(X)N_{b^*}\}, \tag{2.10$_2$}$$

$$\overline{\nabla}_X N_{a^*} = -A_{a^*}X - s_{a^*}(X)N + \sum_{b=1}^{q}\{s_{a^*b}(X)N_b + s_{a^*b^*}(X)N_{b^*}\} \tag{2.10$_3$}$$

for any vector fields $X, Y$ tangent to $M$, where $s$'s are coefficients of the normal connection $\nabla^\perp$. Here and in the sequel $h$ denotes the second fundamental form and $A, A_a, A_{a^*}$ the shape operators corresponding to the normals $N, N_a, N_{a^*}$, respectively. They are related by

$$h(X,Y) = g(AX,Y)N + \sum_{a=1}^{q}\{g(A_aX,Y)N_a + g(A_{a^*}X,Y)N_{a^*}\}. \tag{2.11}$$

From now on we specialize to the case of an ambient Sasakian manifold $\overline{M}$, that is,

$$\overline{\nabla}_X \xi = \phi X, \tag{2.12}$$

$$(\overline{\nabla}_X \phi)Y = -g(X,Y)\xi + \eta(Y)X. \tag{2.13}$$

Since the structure vector $\xi$ is tangent to $M$, we can easily verify from (2.1), (2.3), (2.7), (2.8), $(2.10)_2$–$(2.10)_3$ and (2.13) that

$$A_a X = -FA_{a^*}X + s_{a^*}(X)U, \quad A_{a^*}X = FA_aX - s_a(X)U, \tag{2.14}$$

$$s_a(X) = -u(A_{a^*}X), \quad s_{a^*}(X) = u(A_aX). \tag{2.15}$$

Since $F$ is skew-symmetric, (2.14) implies

$$(2.16)_1 \qquad g((FA_a + A_aF)X, Y) = s_a(X)u(Y) - s_a(Y)u(X),$$

$$(2.16)_2 \qquad g((FA_{a^*} + A_{a^*}F)X, Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X).$$

Differentiating (2.3) and (2.7) covariantly along $M$ and comparing the tangential with normal parts, we have

$$(\nabla_Y F) = -g(Y,X)\xi + \eta(X)Y - g(AY,X)U + u(X)AY, \tag{2.17}$$

$$(\nabla_Y u)X = g(FAY, X), \tag{2.18}$$

$$\nabla_X U = FAX, \tag{2.19}$$

where we have used (2.3), (2.7), (2.8), (2.9), (2.11) and (2.13). On the other hand, since the structure vector $\xi$ is tangent to $M$, we get

$$\phi X = \overline{\nabla}_X \xi = \nabla_X \xi + h(X,\xi),$$

which together with (2.3), (2.9), (2.11) and (2.12) gives

$$\nabla_X \xi = FX, \tag{2.20}$$

$$g(A\xi, X) = u(X), \quad \text{i.e.,} \quad A\xi = U, \tag{2.21}$$

$$A_a\xi = 0, \quad A_{a^*}\xi = 0, \quad a = 2, \ldots, q. \tag{2.22}$$

If the ambient manifold $\overline{M}$ is a Sasakian space form $\overline{M}(c)$, i.e., a Sasakian manifold of constant $\phi$-holomorphic sectional curvature $c$, then its curvature tensor $\overline{R}$ satisfies

$$\begin{aligned} \overline{R}(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\} \end{aligned} \tag{2.23}$$

for any vector fields $X, Y, Z$ tangent to $\overline{M}$. In this case taking account of (2.3) and (2.4) we obtain that the equation of Codazzi implies

$$(2.24)_1 \qquad (\nabla_X A)Y - (\nabla_Y A)X$$

$$= \frac{c-1}{4}\{u(X)FY - u(Y)FX - 2g(FX,Y)U\}$$
$$+ \sum_{a=1}^{q}\{s_a(X)A_aY - s_a(Y)A_aX + s_{a^*}(X)A_{a^*}Y - s_{a^*}(Y)A_{a^*}X\},$$

$(2.24)_2$
$$(\nabla_X A_a)Y - (\nabla_Y A_a)X$$
$$= s_a(Y)AX - s_a(X)AY$$
$$+ \sum_{b=1}^{q}\{s_{ab}(X)A_bY - s_{ab}(Y)A_bX + s_{ab^*}(X)A_{b^*}Y - s_{ab^*}(Y)A_{b^*}X\},$$

$(2.24)_3$
$$(\nabla_X A_{a^*})Y - (\nabla_Y A_{a^*})X$$
$$= s_{a^*}(Y)AX - s_{a^*}(X)AY$$
$$+ \sum_{b=1}^{q}\{s_{a^*b}(X)A_bY - s_{a^*b}(Y)A_bX + s_{a^*b^*}(X)A_{b^*}Y$$
$$- s_{a^*b^*}(Y)A_{b^*}X\}$$

for any vector fields $X, Y$ tangent to $M$(cf. [1, 2, 9]).

## 3. Main results

In this section we let $M$ be an $(n+1)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension immersed in a Sasakian manifold $\overline{M}$ and let us use the same notation as stated in the previous section.

We assume that the equality

$$h(FX,Y) + h(X,FY) = 0 \tag{3.1}$$

holds on $M$ for any vector fields $X, Y$ tangent to $M$. Then by means of (2.11) the condition (3.1) is equivalent to

$(3.2)_1$
$$FA = AF,$$

$(3.2)_2$
$$FA_a = A_aF, \quad FA_{a^*} = A_{a^*}F$$

for all $a = 1, \ldots, q$. Moreover, the last two equations combined with $(2.16)_1$ and $(2.16)_2$ yield

$(3.3)_1$
$$2g((FA_a)X,Y) = s_a(X)u(Y) - s_a(Y)u(X),$$

$(3.3)_2$
$$2g((FA_{a^*})X,Y) = s_{a^*}(X)u(Y) - s_{a^*}(Y)u(X),$$

from which, putting $Y = X$ into (3.3) and (3.4), respectively, and using (2.5), we obtain

$$s_a(X) = s_a(U)u(X), \quad s_{a^*}(X) = s_{a^*}(U)u(X), \quad a = 1, \ldots, q. \tag{3.4}$$

Hence, it is clear from $(3.2)_2$ and $(3.3)_{1-2}$ that

$$FA_a = A_aF = 0, \quad FA_{a^*} = A_{a^*}F = 0. \tag{3.5}$$

As a direct consequence of $(3.2)_1$ and (3.5), it follows from (2.5), (2.6), (2.15), (2.21) and (2.22) that

$$AU = \lambda U + \xi, \quad \lambda := u(AU), \tag{3.6}$$

$$A_a X = s_{a^*}(X)U, \quad A_{a^*}X = -s_a(X)U. \tag{3.7}$$

Substituting (3.4) and (3.7) into $(2.24)_1$, we have

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c-1}{4}\{u(X)FY - u(Y)FX - 2g(FX,Y)U\}. \tag{3.8}$$

Now we prepare some lemmas for later use.

**Lemma 3.1.** *Let $M$ be an $(n+1)(n \geq 3)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension immersed in a Sasakian space form $\overline{M}(c)$ $(c \neq -3)$. If for any vector fields $X, Y$ tangent to $M$, the equality* (3.1) *holds on $M$, then*

$$s_a = 0, \quad s_{a^*} = 0, \quad a = 1, \dots q,$$

*namely, the distinguished normal vector field $N$ is parallel with respect to the normal connection. Moreover,*

$$A_a = 0, \quad A_{a^*} = 0, \quad a = 1, \dots q.$$

*Proof.* Since the ambient manifold is a Sasakian space form, applying $F$ to the both sides of $(2.24)_2$ and using (3.4)–(3.5), we have

$$F((\nabla_X A_a)Y - (\nabla_Y A_a)X) = s_a(U)u(Y)FAX - s_a(U)u(X)FAY. \tag{3.9}$$

On the other hand, differentiating $FA_a = 0$ covariantly along $M$ and making use of (2.17), (2.22), $(3.4)_1$ and (3.9), we can easily obtain

$$F(\nabla_X A_a)Y = s_{a^*}(U)u(X)u(Y)\xi + s_{a^*}(U)u(AX)u(Y)U - s_{a^*}(U)u(Y)AX,$$

from which together with (3.6), we get

$$\begin{aligned} &F((\nabla_X A_a)Y - (\nabla_Y A_a)X) \\ &= s_{a^*}(U)\{\eta(X)u(Y) - \eta(Y)u(X)\}U - s_{a^*}(U)\{u(Y)AX - u(X)AY\}. \end{aligned} \tag{3.10}$$

Comparing (3.10) with (3.9), it is clear that

$$\begin{aligned} &s_a(U)\{u(Y)FAX - u(X)FAY\} \\ &= s_{a^*}(U)\{\eta(X)u(Y) - u(X)\eta(Y)\}U - s_{a^*}(U)\{u(Y)AX - u(X)AY\}, \end{aligned}$$

from which, putting $Y = U$ into the last equation and taking account of (2.5) and (3.6), it follows that

$$s_a(U)g(FAX,Y) = s_{a^*}(U)\{\eta(X)u(Y) + \eta(Y)u(X) + \lambda u(X)u(Y) - g(AX,Y)\}$$

and consequently

$$s_a(U)\{g(FAX,Y) - g(FAY,X)\} = 2s_a(U)g(FAX,Y) = 0 \tag{3.11}$$

with the aid of the fact that $F$ is skew-symmetric and $(3.2)_1$.

Now we assume that $s_a(U) \neq 0$. Then it follows from (2.18), (2.19) and (3.11) that

$$FAX = 0, \quad \nabla_X U = 0, \quad \nabla_X u = 0. \tag{3.12}$$

Furthermore, it follows from (2.6), (2.21), (3.6) that the first equation of (3.12) implies

$$AX = \{\lambda u(X) + \eta(X)\}U + u(X)\xi. \tag{3.13}$$

Differentiating (3.13) covariantly along $M$ and using (2.20) and (3.12), we have

$$(\nabla_Y A)X = \{(Y\lambda)u(X) + g(X, FY)\}U + u(X)FY$$

and consequently

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = \{&(X\lambda)u(Y) - (Y\lambda)u(X) + 2g(FX, Y)\}U \\ &+ u(Y)FX - u(X)FY.\end{aligned}$$

Comparing the last equation with (3.8), we obtain

$$\frac{c+3}{4}\{u(X)FY - u(Y)FX - 2g(FX, Y)U\} = \{(X\lambda)u(Y) - (Y\lambda)u(X)\}U,$$

from which, taking inner product with $U$ and using (2.5), it is clear that

$$-\frac{c+3}{2}g(FX, Y) = (X\lambda)u(Y) - (Y\lambda)u(X). \tag{3.14}$$

Putting $Y = U$ into (3.14) and taking account of (2.5), we can see that $X\lambda = (U\lambda)u(X)$, which combined with (3.14) reduces to

$$(c+3)g(FX, Y) = 0.$$

Since $c \neq -3$, we have $FX = 0$, which is a contradiction because of $n \geq 3$. Hence $s_a(U) = 0$, which combined with (3.4) yields

$$s_a(X) = 0, \quad a = 1, \ldots, q \tag{3.15}$$

everywhere on $M$.

Next, the second equation of (3.7) combined with (3.15) turns out to be

$$A_{a^*} = 0, \quad a = 1, \ldots, q,$$

from which, making use of $(2.24)_3$ and (3.5), it follows that

$$s_{a^*}(U)\{u(Y)FAX - u(X)FAY\} = 0.$$

Putting $Y = U$ into the last equation and using (2.5), we have $s_{a^*}(U)FAX = 0$ and hence, by the same method as in the case of (3.15), we get

$$s_{a^*}(X) = 0, \quad a = 1, \ldots, q \tag{3.16}$$

everywhere on $M$. Hence it is clear from (3.7) and (3.16) that

$$A_a = 0, \quad a = 1, \ldots, q.$$

□

For the submanifold $M$ given in Lemma 3.1, we can easily see that its first normal space is contained in Span$\{N\}$ which is invariant under parallel translation with respect to the normal connection $\nabla^{\perp}$ because of Lemma 3.1. Thus we may apply Erbacher's reduction theorem ([3, p. 339]) for the submanifold $M$ in a unit sphere $S^{2m+1}$ and thus we have:

**Lemma 3.2** ([4]). *Let $M$ be an $(n+1)(n \geq 3)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension immersed in a unit sphere $S^{2m+1}$. If for any vector fields $X, Y$ tangent to $M$, the equality* (3.1) *holds on $M$, then there exists an $(n+2)$-dimensional totally geodesic unit sphere $S^{n+2}$ such that $M \subset S^{n+2}$.*

Differentiating (3.6) covariantly along $M$ and using (2.19) and (2.20), we have

$$(\nabla_X A)U + AFAX = (X\lambda)U + \lambda FAX + FX,$$

from which together with $(3.2)_1$, we obtain

$$\begin{aligned} &g((\nabla_X A)Y - (\nabla_Y A)X, U) + 2g(FAX, AY) \\ &= (X\lambda)u(Y) - (Y\lambda)u(X) + 2\lambda g(FAX, Y) + 2g(FX, Y). \end{aligned}$$

Substituting (3.8) into the last equation and taking account of (2.5) and $(3.2)_1$, we can easily verify that

$$\begin{aligned} &-\frac{c+3}{2}g(FX, Y) + 2g(FX, A^2Y) \\ &= (X\lambda)u(Y) - (Y\lambda)u(X) + 2\lambda g(FX, AY). \end{aligned} \tag{3.17}$$

Therefore by means of (3.17), we can prove:

**Lemma 3.3.** *Let $M$ be as in Lemma 3.1. If for any vector fields $X, Y$ tangent to $M$, the equality* (3.1) *holds on $M$, then the function $\lambda = u(AU)$ is locally constant.*

*Proof.* Putting $X = U$ into (3.17) and taking account of (2.5), we can obtain

$$X\lambda = \beta u(X), \quad \beta := U\lambda, \tag{3.18}$$

from which, differentiating covariantly along $M$ and using (2.18) and $(3.2)_1$, we get

$$(Y\beta)u(X) - (X\beta)u(Y) - 2\beta g(FAX, Y) = 0. \tag{3.19}$$

Putting $X = U$ into (3,19) and taking account of (2.5), we can see that $Y\beta = (U\beta)u(Y)$, which combined with (3.18) gives $\beta FAX = 0$. Hence, by the quite same method as in the proof of (3.15), we can easily verify $\beta = 0$, which together with (3.18) yields our assertion. □

**Lemma 3.4.** *Let $M$ be as in Lemma 3.1. If for any vector fields $X, Y$ tangent to $M$, the equality* (3.1) *holds on $M$, then*

$$\rho_1 := \frac{\lambda + \sqrt{\lambda^2 + 4}}{2}, \quad \rho_2 := \frac{\lambda - \sqrt{\lambda^2 + 4}}{2}$$

*are non-zero constant eigenvalues of the shape operator $A$.*

*Proof.* Let $\rho_1$ and $\rho_2$ be distinct solutions of the quadratic equation $\rho^2-\lambda\rho-1=0$. Then (3.6) implies

$$A(\rho_1 U+\xi)=\rho_1(\rho_1 U+\xi),\quad A(\rho_2 U+\xi)=\rho_2(\rho_2 U+\xi)$$

since $\lambda\rho_i+1=\rho_i^2,\ i=1,2.$ □

On the other hand, owing to Lemma 3.3 it follows from (3.17) that

$$-\frac{c+3}{4}g(FX,Y)+g(FX,A^2Y)=\lambda g(FX,AY). \tag{3.20}$$

Inserting $FX$ into (3.20) instead of $X$ and using (2.6) and (3.6), we can easily obtain

$$A^2X=\lambda AX+\frac{c+3}{4}X-\frac{c-1}{4}\{u(X)U+\eta(X)\xi\}. \tag{3.21}$$

If there exists an eigenvector $X\in\mathrm{Span}\{U,\xi\}^\perp$ corresponding to the eigenvalue $\rho_i$ in Lemma 3.4, then we have from (3.21)

$$\rho_i^2-\lambda\rho_i-\frac{c+3}{4}=0,$$

which combined with the fact that $\rho_i^2-\lambda\rho_i=1$ yields $c=1$.

Thus we have the main result.

**Theorem.** *Let $M$ be as in Lemma 3.1. If for any vector fields $X,Y$ tangent to $M$, the equality* (3.1) *holds on $M$ and if multiplicity of one of the eigenvalues $\rho_1$ and $\rho_2$ appeared in Lemma 3.4 is not less than* 2*, then $M$ is an $(n+1)(n\geq 3)$-dimensional unit sphere.*

When $c=1$, (3.21) reduces to $A^2X=\lambda AX+X$ and consequently $A$ has exactly two eigenvalues $\rho_1$ and $\rho_2$. If $n\geq 3$, multiplicity of one of the eigenvalues $\rho_1$ and $\rho_2$ is not less than 2.

Combining Theorem K-P stated in Section 1 with Lemma 3.2 and the above main Theorem, we have

**Corollary** ([4])**.** *Let $M$ be an $(n+1)(n\geq 3)$-dimensional contact $CR$-submanifold of $(n-1)$ contact $CR$-dimension immersed in a $(2m+1)$-unit sphere $S^{2m+1}$. If for any vector fields $X,Y$ tangent to $M$, the equality* (3.1) *holds on $M$, then $M$ is locally isometric to*

$$S^{2n_1+1}(r_1)\times S^{2n_2+1}(r_2)\quad (r_1^2+r_2^2=1)$$

*for some integers $n_1,n_2$ with $n_1+n_2=(n-1)/2$.*

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