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DECOMPOSITION FORMULAE FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS WITH THE GAUSS-KUMMER IDENTITY

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Abstract. In the theory of special functions, it is important to study some formulae describing hypergeometric functions with other hypergeometric functions. In this paper, we give some methods to obtain a lot of decomposition formulae for generalized hypergeometric functions.

1. Introduction

The generalized hypergeometric function

$$ {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p\\ \beta_1,\beta_2,\ldots,\beta_q\end{matrix};x\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n}\frac{x^n}{n!} \tag{1.1}$$

(see Section 2.1 for the precise definition) plays an important role in not only mathematics but also applied mathematics, physics, engineering and so on, and has been studying by many mathematicians such as Euler, Gauss, Kummer and so on. In the theory of special functions, it is important to study some relations among such important special functions. A decomposition formula for a hypergeometric function is the one which describes the hypergeometric function with a summation of other hypergeometric functions, such as (1.2) below. See Sections 2.2 and 3 for the details. It was started to study by Burchnall and Chaundy in 1940 for Appell's double hypergeometric functions ([1] and [2]). Nowadays, it has been studying for various special functions by many mathematicians (see [3], [4], [5], [6], [7], [8], [9] and so on). In particular, Choi-Hasanov gave a formula of an analytic continuation of the Clausen hypergeometric function ${}_3F_2$ as an application of their decomposition formula [5].

The aim of this paper is to study similar type of decomposition formulae to Choi-Hasanov's ones [5] proved by using the theory of symbolic operators. We give a simpler (essentially same but not to use symbolic operators) proof and

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find new decomposition formulae by modifying some calculations. An example of our main results is the following:

$$\begin{aligned} &{}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1\end{matrix};x\right] \\ &= (1-x)^{-\alpha_1} + \sum_{i=1}^{\infty}\frac{(\alpha_2-\beta_1)_i(\alpha_1)_1}{(\alpha_2+1)_i}x\,{}_3F_2\left[\begin{matrix}\alpha_1+1,\alpha_2+1,i+1\\ \alpha_2+i+1,2\end{matrix};x\right]\end{aligned} \tag{1.2}$$

(in the case of $p=2, q=1$ in Theorem 3.1 (i)). Note that although we focus only on the generalized hypergeometric functions in this paper, we could obtain a lot of decomposition formulae for various special functions by our methods (see Section 3.3).

2. Preliminaries

2.1. Generalized hypergeometric functions

Definition 2.1. For $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$, we define the Pochhammer symbol $(\alpha)_n$ by

$$(\alpha)_n = \begin{cases} 1 & (n=0), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \geq 1). \end{cases} \tag{2.1}$$

Note that $(\alpha)_n$ can be rewritten by using the Gamma function Γ as

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{2.2}$$

Definition 2.2. For $p,q \in \mathbb{Z}_{\geq 0}$, $\alpha_1,\ldots,\alpha_p \in \mathbb{C}$, $\beta_1,\ldots,\beta_q \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, we define the generalized hypergeometric function ${}_pF_q$ by

$${}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p\\ \beta_1,\beta_2,\ldots,\beta_q\end{matrix};x\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n}\frac{x^n}{n!}. \tag{2.3}$$

In this paper, by using multi-indexes $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_p) \in (\mathbb{C})^p$, $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_q) \in (\mathbb{C}\setminus\mathbb{Z}_{\leq 0})^q$ we describe it as

$${}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] = \sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha})_n}{(\boldsymbol{\beta})_n}\frac{x^n}{n!}. \tag{2.4}$$

The series (2.3) converges absolutely on $\mathbb{C}$ if $p \leq q$, on $\{x \in \mathbb{C} \mid |x| < 1\}$ if $p = q+1$ and diverges on $\mathbb{C}\setminus\{0\}$ if $p > q+1$. Moreover we could also see that the series (2.3) converges absolutely on $\{x \in \mathbb{C} \mid |x| = 1\}$ if $p = q+1$ and $\Re(\beta_1+\cdots+\beta_q-\alpha_1-\cdots-\alpha_p) > 0$.

In particular, ${}_2F_1\left[\begin{smallmatrix}\alpha_1,\alpha_2\\ \beta_1\end{smallmatrix};x\right]$ is called the Gauss hypergeometric function. It is well-known that the value ${}_2F_1\left[\begin{smallmatrix}\alpha_1,\alpha_2\\ \beta_1\end{smallmatrix};1\right]$ of the Gauss hypergeometric function at $x=1$ is described by using the Gamma function.

Theorem 2.3 (Gauss-Kummer [10]). *For* $\beta_1 \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ *and* $\Re(\beta_1-\alpha_1-\alpha_2) > 0$, *we have*

$$
{}_2F_1\left[\begin{matrix}\alpha_1,\alpha_2\\ \beta_1\end{matrix};1\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n}{(\beta_1)_n n!} = \frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)}. \tag{2.5}
$$

As a corollary of the Gauss-Kummer identity, we obtain the following formulae, which play important roles in this paper.

Corollary 2.4. *Let* n *be a non-negative integer.*

(i) *For* $b \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ *and* $\Re(a) > 0$, *we have*

$$
\frac{(a)_n}{(b)_n} = \frac{\Gamma(b)\Gamma(a+n)}{\Gamma(a)\Gamma(b+n)} = {}_2F_1\left[\begin{matrix}b-a,-n\\ b\end{matrix};1\right] = \sum_{k=0}^{\infty}\frac{(b-a)_k(-n)_k}{(b)_k k!}. \tag{2.6}
$$

(ii) *For* $a \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ *and* $\Re(b) > 0$, *we have*

$$
\frac{(a)_n}{(b)_n} = \frac{\Gamma(a+n)\Gamma(b)}{\Gamma(b+n)\Gamma(a)} = {}_2F_1\left[\begin{matrix}a-b,n\\ a+n\end{matrix};1\right] = \sum_{k=0}^{\infty}\frac{(a-b)_k(n)_k}{(a+n)_k k!}. \tag{2.7}
$$

(iii) *For* $b \in \mathbb{C}\setminus\mathbb{Z}$ *and* $\Re(a) < 1$, *we have*

$$
\frac{(a)_n}{(b)_n} = \frac{\Gamma(1-b-n)\Gamma(1-a)}{\Gamma(1-a-n)\Gamma(1-b)} = {}_2F_1\left[\begin{matrix}a-b,-n\\ 1-b-n\end{matrix};1\right] = \sum_{k=0}^{\infty}\frac{(a-b)_k(-n)_k}{(1-b-n)_k k!}. \tag{2.8}
$$

2.2. Choi-Hasanov's decomposition formulae

In this subsection, let us introduce some results of Choi-Hasanov for the generalized hypergeometric function ${}_pF_q$. In this paper, for a multi-index $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_p)$ and an integer $i \in \mathbb{Z}$, we use the conventions $\boldsymbol{\alpha}' = (\alpha_1,\dots,\alpha_{p-1})$, $\boldsymbol{\alpha}'' = (\alpha_1,\dots,\alpha_{p-2})$ and $\boldsymbol{\alpha}+\boldsymbol{i} = (\alpha_1+i,\dots,\alpha_p+i)$.

Theorem 2.5 (Theorem 1 of [5]). *Let* p *and* q *be natural numbers,* $\boldsymbol{\alpha} \in (\mathbb{C})^p$ *and* $\boldsymbol{\beta} \in (\mathbb{C}\setminus\mathbb{Z}_{\leq 0})^q$.

(i) *For* $p \geq 2$, $q \geq 1$ *and* $\Re(\alpha_p) > 0$, *we have*

$$
{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] = \sum_{i=0}^{\infty}\frac{(-1)^i(\boldsymbol{\alpha}')_i(\beta_q-\alpha_p)_i}{(\boldsymbol{\beta})_i i!}x^i\,{}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}'+\boldsymbol{i}\\ \boldsymbol{\beta}'+\boldsymbol{i}\end{matrix};x\right]. \tag{2.9}
$$

(ii) *For* $p \geq 3$, $q \geq 2$, $\Re(\alpha_p) > 0$ *and* $\Re(\alpha_{p-1}) > 0$, *we have*

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right]
&= \sum_{i,j=0}^{\infty}\frac{(\boldsymbol{\alpha}'')_{i+j}(\alpha_{p-1})_i(\beta_{q-1}-\alpha_{p-1})_j(\beta_q-\alpha_p)_i}{(\boldsymbol{\beta}')_{i+j}(\beta_q)_i i! j!}(-x)^{i+j}\,{}_{p-2}F_{q-2}\left[\begin{matrix}\boldsymbol{\alpha}''+\boldsymbol{i}+\boldsymbol{j}\\ \boldsymbol{\beta}''+\boldsymbol{i}+\boldsymbol{j}\end{matrix};x\right].
\end{aligned} \tag{2.10}
$$

Note that by using Theorem 2.5(i) and the analytic continuation of the Gauss hypergeometric function ${}_2F_1$ Choi-Hasanov gave a formula of an analytic continuation of the Clausen hypergeometric function ${}_3F_2$. See [5, (4.6)] for the details.

3. Main results

3.1. A proof of Theorem 2.5

Choi-Hasanov proved Theorem 2.5 by using the theory of symbolic operators. We shall prove Theorem 2.5 without symbolic operators for the reader's convenience, although our method is essentially same as that of them. Note that it is enough to consider only the range where series converge absolutely.

By applying (2.6) to $\frac{(\alpha_p)_n}{(\beta_q)_n}$, we have

$$ {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] = \sum_{n=0}^{\infty}\frac{(\alpha_p)_n}{(\beta_q)_n}\frac{(\boldsymbol{\alpha'})_n}{(\boldsymbol{\beta'})_n}\frac{x^n}{n!} \tag{3.1}$$

$$ = \sum_{i=0}^{\infty}\frac{(\beta_q-\alpha_p)_i}{(\beta_q)_i i!}\sum_{n=0}^{\infty}\frac{(\boldsymbol{\alpha'})_n(-n)_i}{(\boldsymbol{\beta'})_n}\frac{x^n}{n!} \tag{3.2}$$

$$ = \sum_{i=0}^{\infty}\frac{(\beta_q-\alpha_p)_i}{(\beta_q)_i i!}\sum_{n=i}^{\infty}\frac{(\boldsymbol{\alpha'})_n(-1)^i n!}{(\boldsymbol{\beta'})_n(n-i)!}\frac{x^n}{n!} \tag{3.3}$$

$$ = \sum_{i=0}^{\infty}\frac{(-1)^i(\beta_q-\alpha_p)_i}{(\beta_q)_i i!}\sum_{m=0}^{\infty}\frac{(\boldsymbol{\alpha'})_{m+i}}{(\boldsymbol{\beta'})_{m+i}}\frac{x^{m+i}}{m!} \tag{3.4}$$

$$ = \sum_{i=0}^{\infty}\frac{(-1)^i(\boldsymbol{\alpha'})_i(\beta_q-\alpha_p)_i}{(\boldsymbol{\beta})_i i!}x^i\sum_{m=0}^{\infty}\frac{(\boldsymbol{\alpha'}+\boldsymbol{i})_m}{(\boldsymbol{\beta'}+\boldsymbol{i})_m}\frac{x^m}{m!}. \tag{3.5}$$

By rewriting the last equation with the generalized hypergeometric functions, we obtain (2.9). By applying (2.9) to (2.9), we obtain (2.10). This completes the proof of Theorem 2.5.

3.2. New decomposition formulae

In this section, let us introduce our main results, which are similar type of decomposition formulae to those of Choi-Hasanov. We obtain new formulae by applying (2.7) and (2.8) instead of (2.6).

Theorem 3.1. *Let $p \geq 2$, $q \geq 1$ $(p,q \in \mathbb{N})$, $\boldsymbol{\alpha} \in (\mathbb{C})^p$ and $\boldsymbol{\beta} \in (\mathbb{C}\setminus\mathbb{Z}_{\leq 0})^q$.*

(i) *For $\alpha_p \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ and $\Re(\beta_q) > 0$, we have*

$$ {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right] \tag{3.6}$$
$$ = \sum_{i=1}^{\infty}\frac{(\alpha_p-\beta_q)_i(\boldsymbol{\alpha'})_1}{(\alpha_p+1)_i(\boldsymbol{\beta'})_1}x\,{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{1}, i+1\\ \boldsymbol{\beta'}+\boldsymbol{1}, \alpha_p+i+1, 2\end{matrix};x\right].$$

(ii) ([5, Theorem 1 (3.2)]) *For $\beta_q \in \mathbb{C}\setminus\mathbb{Z}$ and $\Re(\alpha_p) < 1$, we have*

$$ {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right] = \sum_{i=1}^{\infty}\frac{(\alpha_p-\beta_q)_i(\boldsymbol{\alpha'})_i}{(\boldsymbol{\beta})_i i!}x^i\,{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha'}+\boldsymbol{i}, \beta_q\\ \boldsymbol{\beta}+\boldsymbol{i}\end{matrix};x\right]. \tag{3.7}$$

Proof. (i) By applying (2.7) to $\frac{(\alpha_p)_n}{(\beta_q)_n}$, we have

(3.8)
$$(\text{LHS}) = \sum_{i=0}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha'})_n (n)_i}{(\boldsymbol{\beta'})_n (\alpha_p + n)_i} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha'})_n}{(\boldsymbol{\beta'})_n} \frac{x^n}{n!}$$

(3.9)
$$= \sum_{i=1}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha'})_n (n)_i}{(\boldsymbol{\beta'})_n (\alpha_p + n)_i} \frac{x^n}{n!}$$

(3.10)
$$= \sum_{i=1}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{m=0}^{\infty} \frac{(\boldsymbol{\alpha'})_{m+1} (m+1)_i}{(\boldsymbol{\beta'})_{m+1} (\alpha_p + m + 1)_i} \frac{x^{m+1}}{(m+1)!}$$

(3.11)
$$= \sum_{i=1}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{m=0}^{\infty} \frac{(\boldsymbol{\alpha'})_1 (\boldsymbol{\alpha'} + \boldsymbol{1})_m (\alpha_p + 1)_m (i+1)_m i!}{(\boldsymbol{\beta'})_1 (\boldsymbol{\beta'} + \boldsymbol{1})_m (\alpha_p + 1)_i (\alpha_p + i + 1)_m (m+1)!} \frac{x^{m+1}}{m!}.$$

By rewriting the last equation with the generalized hypergeometric functions, we obtain (3.6).

(ii) By applying (2.8) to $\frac{(\alpha_p)_n}{(\beta_q)_n}$, we have

$$(3.12)\quad {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}; x\right] = \sum_{i=0}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{n=0}^{\infty} \frac{(\boldsymbol{\alpha'})_n (-n)_i}{(\boldsymbol{\beta'})_n (1 - \beta_q - n)_i} \frac{x^n}{n!}$$

$$(3.13)\quad = \sum_{i=0}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{n=i}^{\infty} \frac{(\boldsymbol{\alpha'})_n (-1)^i}{(\boldsymbol{\beta'})_n (1 - \beta_q - n)_i} \frac{x^n}{(n-i)!}$$

$$(3.14)\quad = \sum_{i=0}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{m=0}^{\infty} \frac{(\boldsymbol{\alpha'})_{m+i} (-1)^i}{(\boldsymbol{\beta'})_{m+i} (1 - \beta_q - m - i)_i} \frac{x^{m+i}}{m!}$$

$$(3.15)\quad = \sum_{i=0}^{\infty} \frac{(\alpha_p - \beta_q)_i}{i!} \sum_{m=0}^{\infty} \frac{(\boldsymbol{\alpha'})_i (\boldsymbol{\alpha'} + \boldsymbol{i})_m (\beta_q)_m}{(\boldsymbol{\beta'})_i (\boldsymbol{\beta'} + \boldsymbol{i})_m (\beta_q)_i (\beta_q + i)_m} \frac{x^{m+i}}{m!}.$$

By rewriting the last equation with the generalized hypergeometric functions, we obtain (3.7). □

Remark 3.2. Theorem 3.1(ii) is essentially same as [5, Theorem 1 (3.2)].

By changing the order of summation in Theorem 2.5(i) and Theorem 3.1(ii), we obtain other representations of formulae.

Corollary 3.3. *Let* $p \geq 2,\ q \geq 1$ $(p, q \in \mathbb{N})$, $\boldsymbol{\alpha} \in (\mathbb{C})^p$ *and* $\boldsymbol{\beta} \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$.

(i) *For* $\Re(\alpha_p) > 0$, *we have*

$$(3.16)\quad {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}; x\right] = \sum_{m=0}^{\infty} \frac{(\boldsymbol{\alpha'})_m}{(\boldsymbol{\beta'})_m m!} x^m {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha'} + \boldsymbol{m}, \beta_q - \alpha_p\\ \boldsymbol{\beta'} + \boldsymbol{m}, \beta_q\end{matrix}; -x\right].$$

(ii) *For $\beta_q \in \mathbb{C} \setminus \mathbb{Z}$ and $\Re(\alpha_p) < 1$, we have*

$$
{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] = \sum_{m=0}^{\infty} \frac{(\boldsymbol{\alpha}')_m}{(\boldsymbol{\beta}')_m m!} x^m \, {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}'+\boldsymbol{m}, \alpha_p-\beta_q\\ \boldsymbol{\beta}+\boldsymbol{m}\end{matrix};x\right]. \tag{3.17}
$$

Proof. Since (3.4) can be rewritten as

$$
\sum_{m=0}^{\infty} \frac{(\boldsymbol{\alpha}')_m}{(\boldsymbol{\beta}')_m m!} x^m \sum_{i=0}^{\infty} \frac{(-1)^i(\beta_q-\alpha_p)_i(\boldsymbol{\alpha}'+\boldsymbol{m})_i}{(\beta_q)_i(\boldsymbol{\beta}'+\boldsymbol{m})_i} \frac{x^i}{i!}, \tag{3.18}
$$

we obtain (3.16). (3.17) is obtained similarly from (3.14). □

By applying (2.6), (2.7) and (2.8) twice, we obtain the following new formulae, which are different types from Choi-Hasanov's ones (for example Theorem 2.5(ii)).

Theorem 3.4. *Let $p \geq 3$, $q \geq 2$ $(p, q \in \mathbb{N})$, $\boldsymbol{\alpha} \in (\mathbb{C})^p$ and $\boldsymbol{\beta} \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$.*

(i) *For $\Re(\alpha_p) > 0$ and $\Re(\alpha_{p-1}) > 0$, we have*

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}'', \alpha_p\\ \boldsymbol{\beta}'', \beta_q\end{matrix};x\right] + {}_{p-2}F_{q-2}\left[\begin{matrix}\boldsymbol{\alpha}''\\ \boldsymbol{\beta}''\end{matrix};x\right] \\
&= \sum_{1\leq j\leq i} \left(\frac{(\beta_q-\alpha_p)_i(\beta_{q-1}-\alpha_{p-1})_j}{(\beta_q)_i(\beta_{q-1})_j} + \frac{(\beta_q-\alpha_p)_j(\beta_{q-1}-\alpha_{p-1})_i}{(\beta_q)_j(\beta_{q-1})_i} \right) \\
&\qquad \times \frac{(-1)^{i+j}(\boldsymbol{\alpha}'')_i}{(\boldsymbol{\beta}'')_i j!(i-j)!} x^i \, {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}''+\boldsymbol{i}, i+1\\ \boldsymbol{\beta}''+\boldsymbol{i}, i-j+1\end{matrix};x\right] \\
&\quad - \sum_{i=1}^{\infty} \frac{(\beta_q-\alpha_p)_i(\beta_{q-1}-\alpha_{p-1})_i(\boldsymbol{\alpha}'')_i}{(\boldsymbol{\beta})_i i!} x^i \, {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}''+\boldsymbol{i}, i+1\\ \boldsymbol{\beta}''+\boldsymbol{i}, 1\end{matrix};x\right].
\end{aligned} \tag{3.19}
$$

(ii) *For $\alpha_{p-1} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $\Re(\alpha_p) > 0$ and $\Re(\beta_{q-1}) > 0$, we have*

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}'', \alpha_p\\ \boldsymbol{\beta}'', \beta_q\end{matrix};x\right] + {}_{p-2}F_{q-2}\left[\begin{matrix}\boldsymbol{\alpha}''\\ \boldsymbol{\beta}''\end{matrix};x\right] \\
&= \sum_{i,j=1}^{\infty} \frac{(-1)^i(\beta_q-\alpha_p)_i(\alpha_{p-1}-\beta_{q-1})_j(\boldsymbol{\alpha}'')_i(i)_j}{(\alpha_{p-1}+i)_j(\beta_q)_i(\boldsymbol{\beta}'')_i i! j!} x^i \, {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}'+\boldsymbol{i}, i+j\\ \boldsymbol{\beta}''+\boldsymbol{i}, \alpha_{p-1}+i+j, i\end{matrix};x\right].
\end{aligned} \tag{3.20}
$$

(iii) *For $\beta_{q-1} \in \mathbb{C} \setminus \mathbb{Z}$, $\Re(\alpha_p) > 0$ and $\Re(\alpha_{p-1}) < 1$, we have*

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}'\\ \boldsymbol{\beta}'\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha}'', \alpha_p\\ \boldsymbol{\beta}'', \beta_q\end{matrix};x\right] + {}_{p-2}F_{q-2}\left[\begin{matrix}\boldsymbol{\alpha}''\\ \boldsymbol{\beta}''\end{matrix};x\right] \\
&= \sum_{1\leq j\leq i} \frac{(-1)^i(\beta_q-\alpha_p)_i(\alpha_{p-1}-\beta_{q-1})_j(\boldsymbol{\alpha}'')_i(\beta_{q-1})_{i-j}}{(\boldsymbol{\beta})_i j!(i-j)!} x^i \, {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}''+\boldsymbol{i}, \beta_{q-1}+i-j, i+1\\ \boldsymbol{\beta}'+\boldsymbol{i}, i-j+1\end{matrix};x\right] \\
&\quad + \sum_{1\leq i<j} \frac{(-1)^i(\beta_q-\alpha_p)_i(\alpha_{p-1}-\beta_{q-1})_j(\boldsymbol{\alpha}'')_j}{(\beta_q)_i(\boldsymbol{\beta}')_j i!(j-i)!} x^j \, {}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}''+\boldsymbol{j}, \beta_{q-1}, j+1\\ \boldsymbol{\beta}'+\boldsymbol{j}, j-i+1\end{matrix};x\right].
\end{aligned} \tag{3.21}
$$

(iv) *For* $\alpha_p, \alpha_{p-1} \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$*,* $\Re(\beta_q) > 0$ *and* $\Re(\beta_{q-1}) > 0$*, we have*

(3.22)

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha''},\alpha_p\\ \boldsymbol{\beta''},\beta_q\end{matrix};x\right] + {}_{p-2}F_{q-2}\left[\begin{matrix}\boldsymbol{\alpha''}\\ \boldsymbol{\beta''}\end{matrix};x\right]\\
&= \sum_{i,j=1}^{\infty} \frac{(\alpha_p-\beta_q)_i(\alpha_{p-1}-\beta_{q-1})_j(\boldsymbol{\alpha''})_1}{(\alpha_p+1)_i(\alpha_{p-1}+1)_j(\boldsymbol{\beta''})_1} x\, {}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha}+\mathbf{1}, i+1, j+1\\ \boldsymbol{\beta''}+\mathbf{1}, \alpha_p+i+1, \alpha_{p-1}+j+1, 1, 2\end{matrix};x\right].
\end{aligned}
$$

(v) *For* $\alpha_p \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$*,* $\beta_{q-1} \in \mathbb{C} \setminus \mathbb{Z}$*,* $\Re(\beta_q) > 0$ *and* $\Re(\alpha_{p-1}) < 1$*, we have*

(3.23)

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha''},\alpha_p\\ \boldsymbol{\beta''},\beta_q\end{matrix};x\right] + {}_{p-2}F_{q-2}\left[\begin{matrix}\boldsymbol{\alpha''}\\ \boldsymbol{\beta''}\end{matrix};x\right]\\
&= \sum_{i,j=1}^{\infty} \frac{(\alpha_p-\beta_q)_i(\alpha_{p-1}-\beta_{q-1})_j(j)_i(\boldsymbol{\alpha''})_j}{(\alpha_p+j)_i(\boldsymbol{\beta'})_j i!j!} x^j\, {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha''}+\boldsymbol{j}, \beta_{q-1}, \alpha_p+j, i+j\\ \boldsymbol{\beta'}+\boldsymbol{j}, \alpha_p+i+j, j\end{matrix};x\right].
\end{aligned}
$$

(vi) *For* $\beta_q, \beta_{q-1} \in \mathbb{C} \setminus \mathbb{Z}$*,* $\Re(\alpha_p) < 1$ *and* $\Re(\alpha_{p-1}) < 1$*, we have*

(3.24)

$$
\begin{aligned}
&{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha''},\alpha_p\\ \boldsymbol{\beta''},\beta_q\end{matrix};x\right] + {}_{p-2}F_{q-2}\left[\begin{matrix}\boldsymbol{\alpha''}\\ \boldsymbol{\beta''}\end{matrix};x\right]\\
&= \sum_{1\leq j\leq i} \frac{(\alpha_p-\beta_q)_i(\alpha_{p-1}-\beta_{q-1})_j(\boldsymbol{\alpha''})_i(\beta_{q-1})_{i-j}}{(\boldsymbol{\beta})_i j!(i-j)!} x^i\, {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha''}+\boldsymbol{i}, \beta_q, \beta_{q-1}+i-j, i+1\\ \boldsymbol{\beta}+\boldsymbol{i}, i-j+1\end{matrix};x\right]\\
&\quad + \sum_{1\leq i<j} \frac{(\alpha_p-\beta_q)_i(\alpha_{p-1}-\beta_{q-1})_j(\boldsymbol{\alpha''})_j(\beta_q)_{j-i}}{(\boldsymbol{\beta})_j i!(j-i)!} x^j\, {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha''}+\boldsymbol{j}, \beta_{q-1}, \beta_q+j-i, j+1\\ \boldsymbol{\beta}+\boldsymbol{j}, j-i+1\end{matrix};x\right].
\end{aligned}
$$

Proof. By applying (i) (2.6) and (2.6), (ii) (2.6) and (2.7), (iii) (2.6) and (2.8), (iv) (2.7) and (2.7), (v) (2.7) and (2.8) and (vi) (2.8) and (2.8) to $\frac{(\alpha_{p-1})_n}{(\beta_{q-1})_n} \cdot \frac{(\alpha_p)_n}{(\beta_q)_n}$ in

$$
{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] = \sum_{n=0}^{\infty} \frac{(\alpha_{p-1})_n}{(\beta_{q-1})_n} \cdot \frac{(\alpha_p)_n}{(\beta_q)_n} \cdot \frac{(\boldsymbol{\alpha''})_n}{(\boldsymbol{\beta''})_n} \cdot \frac{x^n}{n!}, \tag{3.25}
$$

we obtain (i)-(vi) respectively. Since the calculation is similar to that of Theorem 3.1, we omit the details. □

In some special cases of Theorem 3.4, we obtain other representations of our decomposition formulae.

Corollary 3.5. *Let* $p \geq 2,\ q \geq 1$ $(p, q \in \mathbb{N})$*,* $\boldsymbol{\alpha} \in (\mathbb{C})^p$ *and* $\boldsymbol{\beta} \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^q$*.*

(i) *For* $\Re(\alpha_p) > 0$ *and* $\Re(\beta_q) > 0$*, we have*

$$
\begin{aligned}
&2\,{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right] - {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha},\alpha_p\\ \boldsymbol{\beta},\beta_q\end{matrix};x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right]\\
&= \sum_{1\leq j\leq i}\left(\frac{(\beta_q-\alpha_p)_i(\alpha_p-\beta_q)_j}{(\beta_q)_i(\alpha_p)_j} + \frac{(\beta_q-\alpha_p)_j(\alpha_p-\beta_q)_i}{(\beta_q)_j(\alpha_p)_i}\right)\\
&\qquad\qquad \times \frac{(-1)^{i+j}(\boldsymbol{\alpha})_i}{(\boldsymbol{\beta})_i j!(i-j)!} x^i\, {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{i}, i+1\\ \boldsymbol{\beta}+\boldsymbol{i}, i-j+1\end{matrix};x\right]
\end{aligned}
\tag{3.26}
$$

$$-\sum_{i=1}^{\infty}\frac{(\beta_q-\alpha_p)_i(\alpha_p-\beta_q)_i(\boldsymbol{\alpha'})_i}{(\beta_q)_i(\boldsymbol{\beta})_i i!}x^i{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{i},i+1\\ \boldsymbol{\beta}+\boldsymbol{i},1\end{matrix};x\right].$$

(ii) *For $\Re(\alpha_p)>0$, we have*

(3.27)
$$2{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right]-{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha},\alpha_p\\ \boldsymbol{\beta},\beta_q\end{matrix};x\right]-{}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right]$$
$$=\sum_{i,j=1}^{\infty}\frac{(-1)^i(\beta_q-\alpha_p)_i(\beta_q-\alpha_p)_j(\boldsymbol{\alpha})_i(i)_j}{(\beta_q+i)_j(\beta_q)_i(\boldsymbol{\beta})_i i!j!}x^i{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{i},i+j\\ \boldsymbol{\beta'}+\boldsymbol{i},\beta_q+i+j,i\end{matrix};x\right].$$

(iii) *For $\alpha_p\in\mathbb{C}\setminus\mathbb{Z}$, $\Re(\alpha_p)>0$ and $\Re(\beta_q)<1$, we have*

(3.28)
$$2{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right]-{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha},\alpha_p\\ \boldsymbol{\beta},\beta_q\end{matrix};x\right]-{}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right]$$
$$=\sum_{1\le j\le i}\frac{(-1)^i(\beta_q-\alpha_p)_i(\beta_q-\alpha_p)_j(\alpha_p)_{i-j}(\boldsymbol{\alpha'})_i}{(\beta_q)_i(\boldsymbol{\beta})_i j!(i-j)!}x^i{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha'}+\boldsymbol{i},\alpha_p+i-j,i+1\\ \boldsymbol{\beta}+\boldsymbol{i},i-j+1\end{matrix};x\right]$$
$$+\sum_{1\le i<j}\frac{(-1)^i(\beta_q-\alpha_p)_i(\beta_q-\alpha_p)_j(\boldsymbol{\alpha'})_j}{(\beta_q)_i(\boldsymbol{\beta})_j i!(j-i)!}x^j{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha'}+\boldsymbol{j},\alpha_p,j+1\\ \boldsymbol{\beta}+\boldsymbol{j},j-i+1\end{matrix};x\right].$$

(iv) *For $\Re(\alpha_p)>0$ and $\Re(\beta_q)>0$, we have*

(3.29)
$$2{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right]-{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha},\alpha_p\\ \boldsymbol{\beta},\beta_q\end{matrix};x\right]-{}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right]$$
$$=\sum_{i,j=1}^{\infty}\frac{(\alpha_p-\beta_q)_i(\beta_q-\alpha_p)_j(\boldsymbol{\alpha})_1}{(\alpha_p+1)_i(\beta_q+1)_j(\boldsymbol{\beta})_1}x{}_{p+3}F_{q+3}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{1},\alpha_p+1,i+1,j+1\\ \boldsymbol{\beta'}+\boldsymbol{1},\alpha_p+i+1,\beta_q+j+1,1,2\end{matrix};x\right].$$

(v) *For $\alpha_p\in\mathbb{C}\setminus\mathbb{Z}$ and $0<\Re(\beta_q)<1$, we have*

(3.30)
$$2{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right]-{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha},\alpha_p\\ \boldsymbol{\beta},\beta_q\end{matrix};x\right]-{}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right]$$
$$=\sum_{i,j=1}^{\infty}\frac{(\alpha_p-\beta_q)_i(\beta_q-\alpha_p)_j(\boldsymbol{\alpha'})_j(j)_i}{(\alpha_p+j)_i(\boldsymbol{\beta})_j i!j!}x^j{}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{j},\alpha_p,i+j\\ \boldsymbol{\beta}+\boldsymbol{j},\alpha_p+i+j,j\end{matrix};x\right].$$

(vi) *For α_p, $\beta_q\in\mathbb{C}\setminus\mathbb{Z}$, $\Re(\alpha_p)<1$ and $\Re(\beta_q)<1$, we have*

(3.31)
$$2{}_pF_q\left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix};x\right]-{}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha},\alpha_p\\ \boldsymbol{\beta},\beta_q\end{matrix};x\right]-{}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix};x\right]$$
$$=\sum_{1\le j\le i}\frac{(\alpha_p-\beta_q)_i(\beta_p-\alpha_q)_j(\alpha_p)_{i-j}(\boldsymbol{\alpha'})_i}{(\beta_q)_i(\boldsymbol{\beta})_i j!(i-j)!}x^i{}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha'}+\boldsymbol{i},\alpha_p+i-j,\beta_q,i+1\\ \boldsymbol{\beta}+\boldsymbol{i},\beta_q+i,i-j+1\end{matrix};x\right]$$
$$+\sum_{1\le i<j}\frac{(\alpha_p-\beta_q)_i(\beta_q-\alpha_p)_j(\beta_q)_{j-i}(\boldsymbol{\alpha'})_j}{(\beta_q)_j(\boldsymbol{\beta})_j i!(j-i)!}x^j{}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha'}+\boldsymbol{j},\alpha_p,\beta_q+j-i,j+1\\ \boldsymbol{\beta}+\boldsymbol{j},\beta_q+j,j-i+1\end{matrix};x\right].$$

(vii) *For $\alpha_p \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and $\Re(\beta_q) > 0$, we have*

$$2 {}_pF_q \left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}; x\right] - {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha}, \alpha_p\\ \boldsymbol{\beta}, \beta_q\end{matrix}; x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix}; x\right] \tag{3.32}$$
$$= \sum_{i,j=1}^{\infty} \frac{(-1)^i(\alpha_p-\beta_q)_i(\alpha_p-\beta_q)_j(\boldsymbol{\alpha'})_i(i)_j}{(\alpha_p+i)_j(\boldsymbol{\beta})_i i! j!} x^i {}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{i}, \alpha_p+i, i+j\\ \boldsymbol{\beta}+\boldsymbol{i}, \alpha_p+i+j, i\end{matrix}; x\right].$$

(viii) *For $\alpha_p \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $\beta_q \in \mathbb{C} \setminus \mathbb{Z}$, $\Re(\alpha_p) < 1$ and $\Re(\beta_q) > 0$, we have*

(3.33)

$$2 {}_pF_q \left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}; x\right] - {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha}, \alpha_p\\ \boldsymbol{\beta}, \beta_q\end{matrix}; x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix}; x\right]$$
$$= \sum_{1\leq j\leq i} \frac{(-1)^i(\alpha_p-\beta_q)_i(\alpha_p-\beta_q)_j(\beta_q)_{i-j}(\boldsymbol{\alpha'})_i}{(\beta_q)_i(\boldsymbol{\beta})_i j!(i-j)!} x^i {}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{i}, \beta_q+i-j, i+1\\ \boldsymbol{\beta}+\boldsymbol{i}, \beta_q+i, i-j+1\end{matrix}; x\right]$$
$$+ \sum_{1\leq i< j} \frac{(-1)^i(\alpha_p-\beta_q)_i(\alpha_p-\beta_q)_j(\boldsymbol{\alpha})_j}{(\alpha_p)_i(\beta_q)_j(\boldsymbol{\beta})_j i!(j-i)!} x^j {}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{j}, \beta_q, j+1\\ \boldsymbol{\beta}+\boldsymbol{j}, \beta_q+j, j-i+1\end{matrix}; x\right].$$

(ix) *For $\beta_q \in \mathbb{C} \setminus \mathbb{Z}$ and $0 < \Re(\alpha_p) < 1$, we have*

$$2 {}_pF_q \left[\begin{matrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{matrix}; x\right] - {}_{p+1}F_{q+1}\left[\begin{matrix}\boldsymbol{\alpha}, \alpha_p\\ \boldsymbol{\beta}, \beta_q\end{matrix}; x\right] - {}_{p-1}F_{q-1}\left[\begin{matrix}\boldsymbol{\alpha'}\\ \boldsymbol{\beta'}\end{matrix}; x\right] \tag{3.34}$$
$$= \sum_{i,j=1}^{\infty} \frac{(\beta_q-\alpha_p)_i(\alpha_p-\beta_q)_j(\boldsymbol{\alpha})_j(j)_i}{(\beta_q)_j(\beta_q+j)_i(\boldsymbol{\beta})_j i! j!} x^j {}_{p+2}F_{q+2}\left[\begin{matrix}\boldsymbol{\alpha}+\boldsymbol{j}, \beta_q, i+j\\ \boldsymbol{\beta}+\boldsymbol{j}, \beta_q+i+j, j\end{matrix}; x\right].$$

Proof. By applying Theorem 3.4 to ${}_pF_q\left[\begin{smallmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{smallmatrix}; x\right] = {}_{p+2}F_{q+2}\left[\begin{smallmatrix}\boldsymbol{\alpha}, \beta_q, \alpha_p\\ \boldsymbol{\beta}, \alpha_p, \beta_q\end{smallmatrix}; x\right]$, we obtain (i)-(v), respectively. And by applying Theorem 3.4 (ii), (iii), (v) to ${}_pF_q\left[\begin{smallmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{smallmatrix}; x\right] = {}_{p+2}F_{q+2}\left[\begin{smallmatrix}\boldsymbol{\alpha}, \alpha_p, \beta_q\\ \boldsymbol{\beta}, \beta_q, \alpha_p\end{smallmatrix}; x\right]$, we obtain (vii)-(ix), respectively. □

3.3. Decomposition formulae for Kampé de Fériet functions

Our methods can be applied to not only generalized hypergeometric functions but also other special functions. As a simple example of it, in this subsection we give some decomposition formulae for Kampé de Fériet's double hypergeometric functions.

Definition 3.6. For $\widetilde{p}, p_x, p_y, \widetilde{q}, q_x, q_y \in \mathbb{Z}_{\geq 0}$, $(\widetilde{\boldsymbol{\alpha}}) \in (\mathbb{C})^{\widetilde{p}}$, $(\boldsymbol{\alpha_x}) \in (\mathbb{C})^{p_x}$, $(\boldsymbol{\alpha_y}) \in (\mathbb{C})^{p_y}$, $(\widetilde{\boldsymbol{\beta}}) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{\widetilde{q}}$, $(\boldsymbol{\beta_x}) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{q_x}$, $(\boldsymbol{\beta_y}) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{q_y}$, we define the Kampé de Fériet function $F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}$ by

$$F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}; \boldsymbol{\alpha_x}; \boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}}; \boldsymbol{\beta_x}; \boldsymbol{\beta_y};\end{matrix} x, y\right] = \sum_{m,n=0}^{\infty} \frac{(\widetilde{\boldsymbol{\alpha}})_{m+n}(\boldsymbol{\alpha_x})_m(\boldsymbol{\alpha_y})_n}{(\widetilde{\boldsymbol{\beta}})_{m+n}(\boldsymbol{\beta_x})_m(\boldsymbol{\beta_y})_n} \frac{x^m y^n}{m!n!}. \tag{3.35}$$

See [11] for the details of this function. Similarly to Theorems 2.5 and 3.1, we obtain the following decomposition formulae for Kampé de Fériet functions. Note that some of them seem to be obtained by using symbolic operators and others are new formulae.

Theorem 3.7. (i) $p_x \geq 2$, $q_x \geq 1$ *and* $\Re(\alpha_{x,p_x}) > 0$, *we have*

$$F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}};\boldsymbol{\alpha_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}};\boldsymbol{\beta_x};\boldsymbol{\beta_y};\end{matrix}x,y\right] = \sum_{i=0}^{\infty}\frac{(-1)^i(\widetilde{\boldsymbol{\alpha}})_i(\boldsymbol{\alpha'_x})_i(\beta_{x,q_x}-\alpha_{x,p_x})_i}{(\widetilde{\boldsymbol{\beta}})_i(\boldsymbol{\beta_x})_i i!}x^i F^{\widetilde{p},p_x-1,p_y}_{\widetilde{q},q_x-1,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}+\boldsymbol{i};\boldsymbol{\alpha'_x}+\boldsymbol{i};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}}+\boldsymbol{i};\boldsymbol{\beta'_x}+\boldsymbol{i};\boldsymbol{\beta_y};\end{matrix}x,y\right]. \tag{3.36}$$

(ii) $p_x \geq 2$, $q_x \geq 1$, $\alpha_{x,p_x} \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ *and* $\Re(\beta_{x,q_x}) > 0$, *we have*

$$F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}};\boldsymbol{\alpha_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}};\boldsymbol{\beta_x};\boldsymbol{\beta_y};\end{matrix}x,y\right] - F^{\widetilde{p},p_x-1,p_y}_{\widetilde{q},q_x-1,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}};\boldsymbol{\alpha'_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}};\boldsymbol{\beta'_x};\boldsymbol{\beta_y};\end{matrix}x,y\right] = \sum_{i=1}^{\infty}\frac{(\widetilde{\boldsymbol{\alpha}})_1(\boldsymbol{\alpha'_x})_1(\alpha_{x,p_x}-\beta_{x,q_x})_i}{(\widetilde{\boldsymbol{\beta}})_1(\boldsymbol{\beta'_x})_1(\alpha_{x,p_x}+1)_i}xF^{\widetilde{p},p_x+1,p_y}_{\widetilde{q},q_x+1,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}+\boldsymbol{1};\boldsymbol{\alpha_x}+\boldsymbol{1},i+1;\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}}+\boldsymbol{1};\boldsymbol{\beta'_x}+\boldsymbol{1},\alpha_{x,p_x}+i+1,2;\boldsymbol{\beta_y};\end{matrix}x,y\right]. \tag{3.37}$$

(iii) $p_x \geq 2$, $q_x \geq 1$, $\beta_{x,q_x} \in \mathbb{C}\setminus\mathbb{Z}$ *and* $\Re(\alpha_{x,p_x}) < 1$, *we have*

$$F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}};\boldsymbol{\alpha_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}};\boldsymbol{\beta_x};\boldsymbol{\beta_y};\end{matrix}x,y\right] = \sum_{i=0}^{\infty}\frac{(\widetilde{\boldsymbol{\alpha}})_i(\boldsymbol{\alpha'_x})_i(\alpha_{x,p_x}-\beta_{x,q_x})_i}{(\widetilde{\boldsymbol{\beta}})_i(\boldsymbol{\beta_x})_i i!}x^i F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}+\boldsymbol{i};\boldsymbol{\alpha'_x}+\boldsymbol{i},\beta_{x,q_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}}+\boldsymbol{i};\boldsymbol{\beta_x}+\boldsymbol{i};\boldsymbol{\beta_y};\end{matrix}x,y\right]. \tag{3.38}$$

(iv) $\widetilde{p} \geq 2$, $\widetilde{q} \geq 1$ *and* $\Re(\widetilde{\alpha}_{\widetilde{p}}) > 0$, *we have*

$$F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}};\boldsymbol{\alpha_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}};\boldsymbol{\beta_x};\boldsymbol{\beta_y};\end{matrix}x,y\right] = \sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\widetilde{\boldsymbol{\alpha}}')_{i+j}(\widetilde{\beta}_{\widetilde{q}}-\widetilde{\alpha}_{\widetilde{p}})_{i+j}(\boldsymbol{\alpha_x})_i(\boldsymbol{\alpha_y})_j}{(\widetilde{\boldsymbol{\beta}})_{i+j}(\boldsymbol{\beta_x})_i(\boldsymbol{\beta_y})_j i!j!}x^iy^j \times F^{\widetilde{p}-1,p_x,p_y}_{\widetilde{q}-1,q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}'+\boldsymbol{i}+\boldsymbol{j};\boldsymbol{\alpha_x}+\boldsymbol{i};\boldsymbol{\alpha_y}+\boldsymbol{j};\\ \widetilde{\boldsymbol{\beta}}'+\boldsymbol{i}+\boldsymbol{j};\boldsymbol{\beta_x}+\boldsymbol{i};\boldsymbol{\beta_y}+\boldsymbol{j};\end{matrix}x,y\right]. \tag{3.39}$$

(v) $\widetilde{p} \geq 2$, $\widetilde{q} \geq 1$, $\widetilde{\alpha}_{\widetilde{p}} \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ *and* $\Re(\widetilde{\beta}_{\widetilde{q}}) > 0$, *we have*

$$\begin{aligned}&F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}};\boldsymbol{\alpha_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}};\boldsymbol{\beta_x};\boldsymbol{\beta_y};\end{matrix}x,y\right] - F^{\widetilde{p}-1,p_x,p_y}_{\widetilde{q}-1,q_x,q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}';\boldsymbol{\alpha_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}}';\boldsymbol{\beta_x};\boldsymbol{\beta_y};\end{matrix}x,y\right] - {}_{\widetilde{p}+p_x}F_{\widetilde{q}+q_x}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}},\boldsymbol{\alpha_x}\\ \widetilde{\boldsymbol{\beta}},\boldsymbol{\beta_x}\end{matrix};x\right]\\ &-{}_{\widetilde{p}+p_y}F_{\widetilde{q}+q_y}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}},\boldsymbol{\alpha_y}\\ \widetilde{\boldsymbol{\beta}},\boldsymbol{\beta_y}\end{matrix};y\right] + {}_{\widetilde{p}+p_x-1}F_{\widetilde{q}+q_x-1}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}',\boldsymbol{\alpha_x}\\ \widetilde{\boldsymbol{\beta}}',\boldsymbol{\beta_x}\end{matrix};x\right] + {}_{\widetilde{p}+p_y-1}F_{\widetilde{q}+q_y-1}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}',\boldsymbol{\alpha_y}\\ \widetilde{\boldsymbol{\beta}}',\boldsymbol{\beta_y}\end{matrix};y\right]\\ &=\sum_{i=1}^{\infty}\frac{(\widetilde{\boldsymbol{\alpha}}')_2(\boldsymbol{\alpha_x})_1(\boldsymbol{\alpha_y})_1(\widetilde{\alpha}_{\widetilde{p}}-\widetilde{\beta}_{\widetilde{q}})_i(i+1)}{(\widetilde{\boldsymbol{\beta}}')_2(\boldsymbol{\beta_x})_1(\boldsymbol{\beta_y})_1(\widetilde{\alpha}_{\widetilde{p}}+2)_i}xy\\ &\qquad\times F^{\widetilde{p}+1,p_x+1,p_y+1}_{\widetilde{q}+1,q_x+1,q_y+1}\left[\begin{matrix}\widetilde{\boldsymbol{\alpha}}+\boldsymbol{2},i+2;\boldsymbol{\alpha_x}+\boldsymbol{1},1;\boldsymbol{\alpha_y}+\boldsymbol{1},1;\\ \widetilde{\boldsymbol{\beta}}'+\boldsymbol{2},\widetilde{\alpha}_{\widetilde{p}}+i+2,2;\boldsymbol{\beta_x}+\boldsymbol{1},2;\boldsymbol{\beta_y}+\boldsymbol{1},2;\end{matrix}x,y\right].\end{aligned} \tag{3.40}$$

(vi) $\widetilde{p} \geq 2$, $\widetilde{q} \geq 1$, $\beta_{\widetilde{q}} \in \mathbb{C} \setminus \mathbb{Z}$ *and* $\Re(\widetilde{\alpha}_{\widetilde{p}}) < 1$, *we have*

(3.41)

$$F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\begin{bmatrix}\widetilde{\boldsymbol{\alpha}};\boldsymbol{\alpha_x};\boldsymbol{\alpha_y};\\ \widetilde{\boldsymbol{\beta}};\boldsymbol{\beta_x};\boldsymbol{\beta_y};\end{bmatrix}x,y$$
$$= \sum_{i,j=0}^{\infty} \frac{(\widetilde{\boldsymbol{\alpha}}')_{i+j}(\widetilde{\alpha}_{\widetilde{p}}-\widetilde{\beta}_{\widetilde{q}})_{i+j}(\boldsymbol{\alpha_x})_i(\boldsymbol{\alpha_y})_j}{(\widetilde{\boldsymbol{\beta}})_{i+j}(\boldsymbol{\beta_x})_i(\boldsymbol{\beta_y})_j i!j!} x^i y^j F^{\widetilde{p},p_x,p_y}_{\widetilde{q},q_x,q_y}\begin{bmatrix}\widetilde{\boldsymbol{\alpha}}'+\boldsymbol{i}+\boldsymbol{j},\widetilde{\beta}_{\widetilde{q}};\boldsymbol{\alpha_x}+\boldsymbol{i};\boldsymbol{\alpha_y}+\boldsymbol{j};\\ \widetilde{\boldsymbol{\beta}}+\boldsymbol{i}+\boldsymbol{j};\boldsymbol{\beta_x}+\boldsymbol{i};\boldsymbol{\beta_y}+\boldsymbol{j};\end{bmatrix}x,y.$$

Proof. By applying (i) (2.6), (ii) (2.7) and (iii) (2.8) to $\frac{(\alpha_{x,p_x})_m}{(\beta_{x,q_x})_m}$, we obtain (i), (ii), (iii), respectively. By applying (iv) (2.6), (v) (2.7) and (vi) (2.8) to $\frac{(\widetilde{\alpha}_{\widetilde{p}})_{m+n}}{(\widetilde{\beta}_{\widetilde{q}})_{m+n}}$, we obtain (iv), (v), (vi), respectively. Since the calculations are similar to that of Theorems 2.5 and 3.1, we omit the details. □

Remark 3.8. By applying (2.6), (2.7) and (2.8) repeatedly and moreover by applying both Choi-Hasanov's method (applying a decomposition formula to other ones) and our methods (applying (2.6), (2.7) and (2.8) repeatedly, a specialization and a changing the order of summation), we could obtain more decomposition formulae for generalized hypergeometric functions, Kampé de Fériet functions and various special functions.

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