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CONVERGENCE OF MODIFIED MULTI-STEP ITERATIVE FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

Juan Xiao, Lei Deng, and Ming-ge Yang

Abstract. In a uniformly convex Banach space, we introduce a iterative scheme for a finite family of asymptotically quasi-nonexpansive mappings and utilize a new inequality to prove several convergence results for the iterative sequence. The results generalize and unify many important known results of relevant scholars.

1. Introduction and preliminaries

Let E be a Banach space, K be a nonempty closed convex subset of E, T be a self-mapping of K and $F(T)$ denotes the set of fixed points of T.

Definition 1.1 ([8])**.** T is called asymptotically nonexpansive mapping if there exists a sequence $u_n \in [0,\infty)$, $\lim_{n\to\infty} u_n = 0$, such that $\|T^n x - T^n y\| \leq (1+u_n)\|x-y\|$ for all $n \in \mathbb{N}$ and $x,\ y \in K$.

Definition 1.2 ([16])**.** T is called asymptotically quasi-nonexpansive mapping if there exists a sequence $u_n \in [0,\infty)$, $\lim_{n\to\infty} u_n = 0$, such that $\|T^n x - p\| \leq (1+u_n)\|x-p\|$ for all $n \in \mathbb{N}$, and $x \in K, p \in F(T)$, where $F(T) \neq \varnothing$.

Definition 1.3 ([15])**.** E is said to satisfy Opial's condition if for any sequence $x_n \in E$, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x.

Definition 1.4 ([12])**.** A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be semi-closed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

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Definition 1.5 ([6])**.** A finite family $\{T_i\}_{i=1}^m$ of self-mappings of K is said to satisfy condition (B) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0,\ f(r) > 0$ for all $r \in (0,\infty)$, such that $\max_{1\le i\le m} \|x - T_i x\| \geq f(d(x,F))$ for all $x \in K$ where $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $d(x,F) = \inf_{x^*\in F} \|x - x^*\|$.

In [1], [3] and [6], authors introduced several *Modified multi-step* iterations weakly and strongly converges to a common fixed point for a finite family of nonexpansive mappings or asymptotically quasi-nonexpansive mappings.

In [13], the author introduced the *Modified multi-step* iteration and proved that the sequence $\{x_n\}$ defined by (1) converges to a common fixed point of a continuous and strongly pseudocontractive operator in Banach spaces.

$$
(1)\qquad \begin{cases} x_1 \in E, \\ x_{n+1} = (1-\beta_n^1)x_n + \beta_n^1 T y_n^1, \\ y_n^i = (1-\beta_n^{i+1})x_n + \beta_n^{i+1} T y_n^{i+1},\ i = 1,2,\ldots,m-2, \\ y_n^{m-1} = (1-\beta_n^m)x_n + \beta_n^m T x_n,\ m \geq 2, \end{cases}
$$

where the sequences $\{\beta_n^i\}_{n=1}^\infty \subseteq [0,1],\ i = 1,2,\ldots,m$ satisfy certain conditions.

In this paper, we generalize the iterative processes (1) to the following iterative process:

Let K be a nonempty closed convex subset of a Banach space E and $T_1, T_2, \ldots, T_m : K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings. Then, the iterative sequence $\{x_n\}$ defined by the iterative scheme:

$$
(2)\qquad \begin{cases} x_j \in E\ (j = 1,2,\ldots,r), \\ x_{n+1} = (1-\beta_n^1)x_{n-q_1} + \beta_n^1 T_1^n y_n^1, \\ y_n^i = (1-\beta_n^{i+1})x_{n-q_{i+1}} + \beta_n^{i+1} T_{i+1}^n y_n^{i+1},\ i = 1,2,\ldots,m-2, \\ y_n^{m-1} = (1-\beta_n^m)x_{n-q_m} + \beta_n^m T_m^n x_n,\ m \geq 2,\ n \geq r, \end{cases}
$$

where $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ with $\delta \in (0,1)$, $\{q_i\}_{i=1}^m$ is a nonnegative integer sequence in $[0,r]$ and $r,\ m \in \mathbb{N}$ are fixed numbers.

Remark 1. In (2), taking $r = 1$, $\{T_i^n\}_{i=1}^m = T$ and $\{q_i\}_{i=1}^m = 0$ for all $n \geq 1$, then we get (1). So the sequence $\{x_n\}$ defined by (2) extend the sequence $\{x_n\}$ defined by (1).

Remark 2. The class of asymptotically quasi-nonexpansive mappings is a generalization of the class of nonexpansive mappings and asymptotically nonexpansive mappings.

In a uniformly convex Banach spaces, we introduce a new inequality and prove that the sequence $\{x_n\}$ defined by (2) weakly and strongly converges to a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings, finally we obtain several corollaries. Our results generalize and unify the corresponding results of relevant scholars [1, 2, 3, 4, 5, 6, 10, 13, 20].

2. Lemmas

We need the following lemmas to prove our main results.

Lemma 1 ([1])**.** *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{l_n\}_{n=1}^{\infty}$ *be two nonnegative real sequences satisfying*

$$a_{n+1} \leq (1+l_n)a_n, \quad \forall\ n \in \mathbb{N}.$$

where $\sum_{n=1}^{\infty} l_n < +\infty$*, then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2. *Let* $\{a_n\}$ *and* $\{l_n^{(i)}\}_{i=0}^{m-1}$ *be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq l_n^{(0)} a_n + \sum_{i=1}^{m} l_n^{(i)} a_{n-q_i}, \quad (n \geq r) \tag{3}$$

where $\{q_i\}_{i=1}^{m}$ *is a nonnegative integer sequence in* $[0, r]$ *and* $r, m \in \mathbb{N}$ *are fixed numbers. If* $\liminf_{n\to\infty} l_n^{(0)} > 0$ *and* $\sum_{n=1}^{\infty} c_n < \infty$*, where*

$$c_n = \begin{cases} 0, & \sum\limits_{i=0}^{m} l_n^{(i)} \leq 1; \\ \sum\limits_{i=0}^{m} l_n^{(i)} - 1, & \sum\limits_{i=0}^{m} l_n^{(i)} > 1, \end{cases}$$

then $\lim_{n\to\infty} a_n$ *exists.*

Proof. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence defined by

$$b_n = \begin{cases} a_n, & n = 1, 2, \dots, r; \\ \max\{a_{n-r},\ a_{n-r+1}, \dots, a_n\}, & n = r+1,\ r+2, \dots. \end{cases}$$

Using (3), we obtain

$$\begin{aligned} a_{n+1} &\leq l_n^{(0)} a_n + \sum_{i=1}^{m} l_n^{(i)} a_{n-q_i} \leq l_n^{(0)} b_n + \sum_{i=1}^{m} l_n^{(i)} b_n \\ &\leq (1+c_n) b_n,\ n = r+1, r+2, \dots. \end{aligned}$$

Thus,

$$\begin{aligned} b_{n+1} &= \max\{a_{n-r+1}, a_{n-r+2}, \dots, a_{n+1}\} \\ &\leq \max\{a_{n-r}, a_{n-r+1}, a_{n-r+2}, \dots, a_n, a_{n+1}\} \\ &\leq \max\{b_n, a_{n+1}\} \\ &\leq (1+c_n) b_n,\ n = r+1, r+2, \dots. \end{aligned}$$

Since $\sum_{n=1}^{\infty} c_n < \infty$, by Lemma 1, we obtain $\lim_{n\to\infty} b_n$ exists.

Suppose $\lim_{n\to\infty} b_n = a$. Next, we will prove $\{a_n\}_{n=1}^{\infty}$ also converges to a.

By the definition of $\{b_n\}_{n=1}^{\infty}$, we get $a_n \leq b_n$ for all $n \in \mathbb{N}$. If $a = 0$, then it is easy to obtain $\{a_n\}$ also converges to 0. If $a \neq 0$, suppose that $\{a_n\}_{n=1}^{\infty}$ does not converge to a. Since $\lim_{n\to\infty} b_n = a$, it is easy to find $\xi > 0$ such that for all $j > 0$, there exists $n_j > j$ satisfying

$$a_{n_j} < a - \xi. \tag{4}$$

As $\liminf_{n\to\infty} l_n^{(0)} > 0$, then there exist $\theta \in (0,1)$ and $N > 0$ such that for all $n \geq N$

$$l_n^{(0)} \geq \theta. \tag{5}$$

Let $\varepsilon = \min\left\{\frac{(1-\theta)\theta^r\xi}{2-\theta-\theta^r},\ 2a\right\}$. By $\lim_{n\to\infty} b_n = a$ and $\lim_{n\to\infty} c_n = 0$, then there exists $N_\varepsilon > N$ such that for all $n \geq N_\varepsilon$

$$a - \frac{\varepsilon}{4} < b_n < a + \frac{\varepsilon}{4}\ ,\quad c_n < \frac{\varepsilon}{2a} \leq 1. \tag{6}$$

By (4), there exists $n_0 \geq N_\varepsilon + 2r + 1 (n_0 - r \geq N_\varepsilon + r + 1)$, such that $a_{n_0-r} < a - \xi$. From (3), (5) and (6), we have

$$\begin{aligned}
a_{n_0-r+1} &\leq l_{n_0-r}^{(0)} a_{n_0-r} + \sum_{i=1}^{m} l_{n_0-r}^{(i)} a_{n_0-r-q_i} \\
&< l_{n_0-r}^{(0)}(a-\xi) + \sum_{i=1}^{m} l_{n_0-r}^{(i)} \left(a + \frac{\varepsilon}{4}\right) \\
&< (1 + c_{n_0-r})a - l_{n_0-r}^{(0)}\xi + (1 + c_{n_0-r})\frac{\varepsilon}{4} \\
&< a - \theta\xi + \varepsilon. \qquad (7)
\end{aligned}$$

Further, by (7), we obtain

$$\begin{aligned}
a_{n_0-r+2} &\leq l_{n_0-r+1}^{(0)} a_{n_0-r+1} + \sum_{i=1}^{m} l_{n_0-r+1}^{(i)} a_{n_0-r+1-q_i} \\
&< l_{n_0-r+1}^{(0)}(a - \theta\xi + \varepsilon) + \sum_{i=1}^{m} l_{n_0-r+1}^{(i)} \left(a + \frac{\varepsilon}{4}\right) \\
&< (1 + c_{n_0-r+1})a - l_{n_0-r+1}^{(0)}(\theta\xi - \varepsilon) + (1 + c_{n_0-r+1})\frac{\varepsilon}{4} \\
&< a - \theta^2\xi + (\theta+1)\varepsilon.
\end{aligned}$$

Continuously, we can prove that

$$a_{n_0-r+i} < a - \theta^i\xi + \left(\frac{1-\theta^i}{1-\theta}\right)\varepsilon,\ i = 0, 1, \ldots, r. \tag{8}$$

It follows from (8) and $\varepsilon \leq \dfrac{(1-\theta)\theta^r\xi}{2-\theta-\theta^r}$ that

$$\begin{aligned}
a_{n_0-r+i} &< a - \theta^i\xi + \left(\frac{1-\theta^i}{1-\theta}\right)\varepsilon \\
&\leq a - \theta^i \frac{2-\theta-\theta^r}{(1-\theta)\theta^r}\varepsilon + \left(\frac{1-\theta^i}{1-\theta}\right)\varepsilon \\
&= a - \frac{2-\theta-\theta^{r-i}}{(1-\theta)\theta^{r-i}}\varepsilon
\end{aligned}$$

$$\begin{aligned}
&= a - \frac{(1-\theta)\theta^{r-i} + (2-\theta)(1-\theta^{r-i})}{(1-\theta)\theta^{r-i}}\varepsilon \\
&\le a - \varepsilon, \ i = 0, 1, \ldots, r.
\end{aligned}$$

By the definition of $\{b_n\}$, we have $b_{n_0} < a - \varepsilon$, which contradicts $b_{n_0} > a - \frac{\varepsilon}{4}$. Thus $\{a_n\}_{n=1}^{\infty}$ converges to a. □

Lemma 3. *Let E be a Banach space, K be a nonempty closed convex subset of E, $\{T_i\}_{i=1}^m$ be a finite family of asymptotically quasi-nonexpansive selfmappings of K with sequences $\{u_n^i\}_{i=1}^m \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n^i < +\infty$. Let $\{x_n\}$ be the sequence as defined by (2) satisfying $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ with $\delta \in (0,1)$. Then $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|y_n^i - x^*\|$ are existent and equal for all $x^* \in F$ and $i = 1,\ 2, \ldots, m-1$.*

Proof. Let $x^* \in F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$, $v_n = \max_{1\le i\le m} u_n^i$ for each n. Since $\sum_{n=1}^{\infty} u_n^i < +\infty$ for each i, therefore $\sum_{n=1}^{\infty} v_n < +\infty$. Put

$$\beta_n^{(i)} = \begin{cases} \beta_n^1\beta_n^2\cdots\beta_n^m, & i = 0; \\ \beta_n^1\beta_n^2\cdots(1-\beta_n^i), & i = 1, 2, \ldots, m. \end{cases}$$

It follows from (2), we obtain that for any $i = 1, 2, \ldots, m-2$

$$\begin{aligned}
\|y_n^i - x^*\| &= \|(1-\beta_n^{i+1})x_{n-q_{i+1}} + \beta_n^{i+1} T_{i+1}^n y_n^{i+1} - x^*\| \\
&\le (1-\beta_n^{i+1})\|x_{n-q_{i+1}} - x^*\| + \beta_n^{i+1}(1+u_n^{i+1})\|y_n^{i+1} - x^*\| \\
&\le (1+v_n)\left((1-\beta_n^{i+1})\|x_{n-q_{i+1}} - x^*\| + \beta_n^{i+1}\|y_n^{i+1} - x^*\|\right), \qquad (9)
\end{aligned}$$

and for $i = m-1$, we have

$$\begin{aligned}
\|y_n^{m-1} - x^*\| &= \|(1-\beta_n^m)x_{n-q_m} + \beta_n^m T_m^n x_n - x^*\| \\
&\le (1-\beta_n^m)\|x_{n-q_m} - x^*\| + \beta_n^m(1+u_n^m)\|x_n - x^*\| \\
&\le (1+v_n)\left((1-\beta_n^m)\|x_{n-q_m} - x^*\| + \beta_n^m\|x_n - x^*\|\right). \qquad (10)
\end{aligned}$$

Then, from (2), (9) and (10), we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\beta_n^1)\|x_{n-q_1} - x^*\| + \beta_n^1\|T_1^n y_n^1 - x^*\| \\
&\le (1+v_n)\left((1-\beta_n^1)\|x_{n-q_1} - x^*\| + \beta_n^1\|y_n^1 - x^*\|\right) \\
&\le (1+v_n)^2\big((1-\beta_n^1)\|x_{n-q_1} - x^*\| + \beta_n^1(1-\beta_n^2)\|x_{n-q_2} - x^*\| \\
&\quad + \beta_n^1\beta_n^2\|y_n^2 - x^*\|\big) \\
&\ \ \vdots \\
&\le (1+v_n)^{m-1}\Bigg(\sum_{i=1}^{m-1} \beta_n^{(i)}\|x_{n-q_i} - x^*\| \\
&\quad + \beta_n^1\beta_n^2\cdots\beta_n^{m-1}\|y_n^{m-1} - x^*\|\Bigg) \qquad (11)
\end{aligned}$$

$$\leq (1+v_n)^m \left(\beta_n^{(0)} \|x_n - x^*\| + \sum_{i=1}^{m} \beta_n^{(i)} \|x_{n-q_i} - x^*\| \right)$$
$$= l_n^{(0)} \|x_n - x^*\| + \sum_{i=1}^{m} l_n^{(i)} \|x_{n-q_i} - x^*\|,$$

where $l_n^{(i)} = (1+v_n)^m \beta_n^{(i)}$. By the definition of $\{\beta_n^{(i)}\}$, we obtain $l_n^{(0)} = (1+v_n)^m \beta_n^{(0)} \geq \delta^m$ for some $\delta \in (0,1)$, then we have $\liminf_{n\to\infty} l_n^{(0)} > 0$. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence defined by

$$c_n = \begin{cases} 0, & \sum\limits_{i=0}^{m} l_n^{(i)} \leq 1; \\ \sum\limits_{i=0}^{m} l_n^{(i)} - 1, & \sum\limits_{i=0}^{m} l_n^{(i)} > 1. \end{cases}$$

Since $\sum_{i=0}^{m} \beta_n^{(i)} = 1$ and $\sum_{n=1}^{\infty} v_n < +\infty$, then

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{m} l_n^{(i)} - 1 \right) \tag{12}$$
$$= \sum_{n=1}^{\infty} \big((1+v_n)^m - 1 \big)$$
$$= \sum_{n=1}^{\infty} (C_m^1 v_n + C_m^2 v_n^2 + \cdots + C_m^m v_n^m) < +\infty.$$

It follows from Lemma 2 together with (11) and (12), we obtain

$$\lim_{n\to\infty} \|x_n - x^*\|$$

exists for all $x^* \in F$. Moreover, $\lim_{n\to\infty} \|x_n - x^*\|$ exists together with (11), it is easy to see that $\lim_{n\to\infty} \|y_n^i - x^*\|$ also exist for all $i = 1, 2, \ldots, m-1$ and $\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n^i - x^*\|$. □

Lemma 4 ([17]). *Let $p > 1,\ r > 0$ be two fixed numbers and E be a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - \omega_p(\lambda) g(\|x-y\|)$ for all $x, y \in B_r(0) = \{x \in E : \|x\| \leq r\}$, and $\lambda \in [0,1]$, where $\omega_p(\lambda) = \lambda(1-\lambda)^p + (1-\lambda)\lambda^p$.*

Lemma 5 ([14]). *Let E be a uniformly convex Banach space, K be a nonempty closed subset of E, and $T : K \to K$ an asymptotically nonexpansive mapping. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\} \subset K$, if $\{x_n\}$ converges weakly to $p \in K$ and $\{(I-T)x_n\}$ converges strongly to 0, then $p \in F(T)$.*

Lemma 6 ([15]). *Let E be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in E. Let $u, v \in E$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and*

$\lim_{n\to\infty} \|x_n - v\|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *are subsequences of* $\{x_n\}$ *which converge weakly to* u *and* v*, respectively, then* $u = v$.

3. Main results

In this section, we will prove our main theorems.

Theorem 1. *Let* E *be a Banach space,* K *be a nonempty closed convex subset of* E*,* $\{T_i\}_{i=1}^m$ *and* $\{x_n\}$ *as taken in Lemma 3. Then the sequence* $\{x_n\}_{n=1}^\infty$ *be given by* (2) *converges strongly to a common fixed point of* $\{T_i\}_{i=1}^m$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$*, where* $d(x_n, F) = \inf_{p\in F} \|x_n - p\|$.

Proof. Necessity is obvious. We only prove the sufficiency. Suppose

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

As proved in Lemma 3, we obtain $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. This further implies that $\lim_{n\to\infty} d(x_n, F)$ exists.

By the fact that $\liminf_{n\to\infty} d(x_n, F) = 0$, we obtain $\lim_{n\to\infty} d(x_n, F) = 0$, that is

$$\lim_{n\to\infty} d(x_n, F) = \lim_{n\to\infty} \inf_{x^*\in F} \|x_n - x^*\| = 0.$$

It implies that

$$\inf_{x^*\in F} \lim_{n\to\infty} \|x_n - x^*\| = 0.$$

So for any given $\varepsilon > 0$, there exist $p \in F$ and $N > 0$ such that

$$\|x_n - p\| < \frac{\varepsilon}{2}$$

for all $n > N$. This shows that

$$\|x_{n+m} - x_n\| \le \|x_{n+m} - p\| + \|x_n - p\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n > N$ and $m \ge 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since E is a complete, we can obtain $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = x'$. There exists $N' > 0$ such that

$$\|x_n - x'\| < \frac{\varepsilon}{2}$$

for all $n > N'$. Let $N^* = \max\{N, N'\}$. For all $n \ge N^*$, we have

$$\begin{aligned}\|T_i x' - x'\| &\le \|T_i x' - T_i p\| + \|p - x'\| \\ &\le (1 + v_1)\|p - x'\| + \|p - x'\| \\ &\le (2 + v_1)(\|x_n - p\| + \|x' - x_n\|) \\ &\le (2 + v_1)\varepsilon\end{aligned}$$

for any $i = 1, 2, \ldots, m$. By the arbitrariness of ε, it gets $\|T_i x' - x'\| = 0$ for any $i = 1, 2, \ldots, m$. So the sequence $\{a_n\}_{n=1}^\infty$ converges strongly to $x' \in F$. □

Theorem 2. *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E, $\{T_i\}_{i=1}^m$ be a family of asymptotically quasi-nonexpansive self-mappings of K with sequences $\{u_n^i\}_{i=1}^m \subset [0,\infty)$ such that $\sum_{n=1}^\infty u_n^i < +\infty$. If the iteration sequence $\{x_n\}$ is defined by (2) satisfying $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ with $\delta \in (0,1)$, then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$.*

Proof. Let x^* be a common fixed point of $\{T_i\}_{i=1}^m$, $v_n = \max_{1\le i\le m} u_n^i$ for each n. Since $\sum_{n=1}^\infty u_n^i < +\infty$ for each i, therefore $\sum_{n=1}^\infty v_n < +\infty$. Since E is uniformly convex Banach space, from Lemma 4, let $p = 2$, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n^1)x_{n-q_1} + \beta_n^1 T_1^n y_n^1 - x^*\|^2 \\
&= \|(1-\beta_n^1)(x_{n-q_1} - x^*) + \beta_n^1 (T_1^n y_n^1 - x^*)\|^2 \\
&\le (1-\beta_n^1)\|x_{n-q_1} - x^*\|^2 + \beta_n^1 \|T_1^n y_n^1 - x^*\|^2 \\
&\quad - \left(\beta_n^1 (1-\beta_n^1)^2 + (1-\beta_n^1){\beta_n^1}^2\right) g\left(\|T_1^n y_n^1 - x_{n-q_1}\|\right) \\
&\le (1-\beta_n^1)\|x_{n-q_1} - x^*\|^2 + \beta_n^1 (1+v_n)\|y_n^1 - x^*\|^2 \\
&\quad - 2\delta^3 g\left(\|T_1^n y_n^1 - x_{n-q_1}\|\right), \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n^i - x^*\|^2 &= \|(1-\beta_n^{i+1})x_{n-q_{i+1}} + \beta_n^{i+1} T_{i+1}^n y_n^{i+1} - x^*\|^2 \\
&\le (1-\beta_n^{i+1})\|x_{n-q_{i+1}} - x^*\|^2 + \beta_n^{i+1}\|T_{i+1}^n y_n^{i+1} - x^*\|^2 \\
&\quad - \omega_2(\beta_n^{i+1}) g\left(\|T_{i+1}^n y_n^{i+1} - x_{n-q_{i+1}}\|\right) \\
&\le (1-\beta_n^{i+1})\|x_{n-q_{i+1}} - x^*\|^2 + \beta_n^{i+1}(1+v_n)\|y_n^{i+1} - x^*\|^2 \\
&\quad - \omega_2(\beta_n^{i+1}) g\left(\|T_{i+1}^n y_n^{i+1} - x_{n-q_{i+1}}\|\right) \\
&\le (1-\beta_n^{i+1})\|x_{n-q_{i+1}} - x^*\|^2 + \beta_n^{i+1}(1+v_n)\|y_n^{i+1} - x^*\|^2 \\
&\quad - 2\delta^3 g\left(\|T_{i+1}^n y_n^{i+1} - x_{n-q_{i+1}}\|\right) \qquad (14)
\end{aligned}$$

for any $i = 1, 2, \dots, m-2$ and

$$\begin{aligned}
\|y_n^{m-1} - x^*\|^2 &= \|(1-\beta_n^m)x_{n-q_m} + \beta_n^m T_m^n x_n - x^*\|^2 \\
&\le (1-\beta_n^m)\|x_{n-q_m} - x^*\|^2 + \beta_n^m (1+v_n)\|x_n - x^*\|^2 \\
&\quad - \omega_2(\beta_n^m) g\left(\|T_m^n x_n - x_{n-q_m}\|\right) \\
&\le (1-\beta_n^m)\|x_{n-q_m} - x^*\|^2 + \beta_n^m (1+v_n)\|x_n - x^*\|^2 \\
&\quad - 2\delta^3 g\left(\|T_m^n x_n - x_{n-q_m}\|\right). \qquad (15)
\end{aligned}$$

By (13), we have

$$\begin{aligned}
2\delta^3 g\left(\|T_1^n y_n^1 - x_{n-q_1}\|\right) &\le (1-\beta_n^1)(\|x_{n-q_1} - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\
&\quad + \beta_n^1 \left((1+v_n)\|y_n^1 - x^*\|^2 - \|x_{n+1} - x^*\|^2\right). \qquad (16)
\end{aligned}$$

Using Lemma 3, we have $\lim_{n\to\infty}\|x_n - x^*\|$ and $\lim_{n\to\infty}\|y_n^i - x^*\|$ are existent and equal. Hence, by (16) and $\lim_{n\to\infty} v_n = 0$, we obtain

$$g\left(\|T_1^n y_n^1 - x_{n-q_1}\|\right) \to 0 \quad (\text{as } n\to\infty).$$

But g is strictly increasing, continuous and $g(0)=0$. Therefore

$$\|T_1^n y_n^1 - x_{n-q_1}\| \to 0 \quad (\text{as } n\to\infty). \tag{17}$$

Further, similar to the computations above, using (14) and (15), we also can get for any $i = 1, 2, \ldots, m-2$

$$\|T_{i+1}^n y_n^{i+1} - x_{n-q_{i+1}}\| \to 0 \quad (\text{as } n\to\infty), \tag{18}$$

and

$$\|T_m^n x_n - x_{n-q_m}\| \to 0 \quad (\text{as } n\to\infty). \tag{19}$$

It follows from (2) and (17) that

$$\|x_{n+1} - x_{n-q_1}\| = \beta_n^1\|T_1^n y_n^1 - x_{n-q_1}\| \to 0 \quad (\text{as } n\to\infty). \tag{20}$$

This implies that

$$\|x_{n+1} - x_n\| = 0,\ \|x_n - x_{n-q_i}\| \to 0 \quad (\text{as } n\to\infty),\ i = 1, 2, \ldots, m. \tag{21}$$

Notice that from (2), (17), (18) and (21), for any $i = 1, 2, \ldots, m-2$, we have

$$\begin{aligned}\|x_n - y_n^i\| &= \|x_n - \left((1-\beta_n^{i+1})x_{n-q_{i+1}} + \beta_n^{i+1}T_{i+1}^n y_n^{i+1}\right)\|\\ &\le \|x_n - x_{n-q_{i+1}}\| + \beta_n^{i+1}\|x_{n-q_{i+1}} - T_{i+1}^n y_n^{i+1}\| \to 0 \ (\text{as } n\to\infty),\end{aligned} \tag{22}$$

and it follows from (2), (19) and (21) that for $i = m-1$

$$\begin{aligned}\|x_n - y_n^{m-1}\| &= \|x_n - ((1-\beta_n^m)x_{n-q_m} + \beta_n^m T_m^n x_n)\|\\ &\le \|x_n - x_{n-q_m}\| + \beta_n^m\|x_{n-q_m} - T_m^n x_n\| \to 0 \quad (\text{as } n\to\infty).\end{aligned} \tag{23}$$

It follows from (17), (18), (21) and (22), for $i = 1, 2, \ldots, m-1$

$$\begin{aligned}\|T_i^n x_n - x_n\| &\le \|T_i^n x_n - T_i^n y_n^i\| + \|T_i^n y_n^i - x_{n-q_i}\| + \|x_{n-q_i} - x_n\|\\ &\le (1+v_n)\|x_n - y_n^i\| + \|T_i^n y_n^i - x_{n-q_i}\| + \|x_{n-q_i} - x_n\|\\ &\to 0 \ (\text{as } n\to\infty)\end{aligned} \tag{24}$$

and it follows from (19), (21) and (23)

$$\|T_m^n x_n - x_n\| \le \|T_m^n x_n - x_{n-q_m}\| + \|x_{n-q_m} - x_n\| \to 0 \quad (\text{as } n\to\infty). \tag{25}$$

From (21), (24) and (25), for any $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\|x_n - T_i x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1}x_{n+1}\| + \|T_i^{n+1}x_{n+1} - T_i^{n+1}x_n\|\\ &\quad + \|T_i^{n+1}x_n - T_i x_n\|\\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1}x_{n+1}\| + (1+v_n)\|x_{n+1} - x_n\|\\ &\quad + (1+v_1)\|T_i^n x_n - x_n\|\\ &= (2+v_n)\|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1}x_{n+1}\|\end{aligned}$$

$$+ (1 + v_1)\|T_i^n x_n - x_n\| \to 0 \quad (\text{as } n \to \infty).$$

□

Theorem 3. *Let E be a uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of E, $\{T_i\}_{i=1}^m$ be a family of asymptotically quasi-nonexpansive self-mappings of K with sequences (respectively) $\{u_n^i\}_{i=1}^m \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n^i < +\infty$. If the iteration sequence $\{x_n\}$ is defined by (2) satisfying $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ with $\delta \in (0,1)$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^m$ in K.*

Proof. By using the same proof as in Theorem 2, it can be shown that for any $i = 1, 2, \ldots, m$

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0.$$

So $I - T_i$ is semi-closed at 0.

Since E is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup u$ as $n \to \infty$, without loss of generality. By Lemma 5, we have $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 5, $u, v \in F$. By Lemma 3, $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 6 that $u = v$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^m$. □

Theorem 4. *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E, $\{T_i\}_{i=1}^m$ be a family of asymptotically quasi-nonexpansive self-mappings of K with sequences $\{u_n^i\}_{i=1}^m \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n^i < +\infty$. If the iteration sequence $\{x_n\}$ is defined by (2) satisfying $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ with $\delta \in (0,1)$, and $\{T_i\}_{i=1}^m$ satisfies condition (B) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$ in K.*

Proof. It follows from Theorem 2 that for any $i = 1, 2, \ldots, m$

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0.$$

Since $\{T_i\}_{i=1}^m$ satisfies condition (B) with respect to the sequence $\{x_n\}$, then there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, such that for all $x_n \in K$

$$f(d(x_n, F)) \leq \max_{0 \leq i \leq m} \|x_n - T_i x_n\|.$$

So $\lim_{n\to\infty} f(d(x_n, F)) = 0$, that is $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 1, we obtain $\{x_n\}$ converges strongly to some $p \in F$. □

Corollary 5. *Let E be a uniformly convex Banach space, K is a nonempty closed convex subset of E, $\{T_i\}_{i=1}^m$ be a family of asymptotically quasi-nonexpansive self-mappings of K with sequences $\{u_n^i\}_{i=1}^m \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} u_n^i <$*

$+\infty$. *If the iteration sequence* $\{x_n\}$ *is defined as follows*:

$$\begin{cases} x_j \in E \ (j = 1, 2, \ldots, r), \\ x_{n+1} = (1-\beta_n^1)x_{n-1} + \beta_n^1 T_1^n y_n^1, \\ y_n^i = (1-\beta_n^{i+1})x_{n-(i+1)} + \beta_n^{i+1} T_{i+1}^n y_n^{i+1}, \ i = 1, 2, \ldots, m-2, \\ y_n^{m-1} = (1-\beta_n^m)x_{n-m} + \beta_n^m T_m^n x_n, \ 2 \le m \le r \le n, \end{cases}$$

where $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ *with* $\delta \in (0,1)$.

(1) *If* E *satisfying Opial's condition, then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^m$ *in* K.

(2) *If* T *satisfies condition* (B) *with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a common fixed point of* T.

Proof. By taking $\{q_i\}_{i=1}^m = i$ for all $n \ge 1$ in (2), from Theorem 3 and Theorem 4, the conclusion of the corollary follows. □

Corollary 6. *Let* E *be a uniformly convex Banach space,* K *is a nonempty closed convex subset of* E, $\{T_i\}_{i=1}^m$ *be a family of asymptotically quasi-nonexpansive self-mappings of* K *with sequences* $\{u_n^i\}_{i=1}^m \subset [0,\infty)$ *such that* $\sum_{n=1}^\infty u_n^i < +\infty$. *If the iteration sequence* $\{x_n\}$ *is defined as follows*:

$$\begin{cases} x_1 \in E, \\ x_{n+1} = (1-\beta_n^1)x_n + \beta_n^1 T_1^n y_n^1, \\ y_n^1 = (1-\beta_n^2)x_{n-1} + \beta_n^2 T_2^n y_n^2, \\ \quad \vdots \\ y_n^{m-2} = (1-\beta_n^{m-1})x_{n-(m-2)} + \beta_n^{m-1} T_{m-1}^n y_n^{m-1}, \\ y_n^{m-1} = (1-\beta_n^m)x_{n-(m-1)} + \beta_n^m T_m^n x_n, \ m \ge 1, \end{cases}$$

where $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ *with* $\delta \in (0,1)$.

(1) *If* E *satisfying Opial's condition, then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^m$ *in* K.

(2) *If* T *satisfies condition* (B) *with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a common fixed point of* T.

Proof. By taking $r = 1, q_1 = 0$ and $\{q_i\}_{i=2}^m = i-1$ for all $n \ge 1$ in (2), from Theorem 3 and Theorem 4, the conclusion of the corollary follows. This completes the proof. □

Corollary 7. *Let* E *be a uniformly convex Banach space,* K *is a nonempty closed convex subset of* E, $\{T_i\}_{i=1}^m$ *be a family of nonexpansive self-mappings of* K *with sequences* $\{u_n^i\}_{i=1}^m \subset [0,\infty)$ *such that* $\sum_{n=1}^\infty u_n^i < +\infty$ *and* $F = \bigcap_{i=1}^m F(T_i) \ne \emptyset$. *If the iteration sequence* $\{x_n\}$ *is defined by* (1) *satisfying* $\{\beta_n^i\}_{i=1}^m \subset [\delta, 1-\delta]$ *with* $\delta \in (0,1)$.

(1) *If* E *satisfying Opial's condition, then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_i\}_{i=1}^m$ *in* K.

(2) *If T satisfies condition (B) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a common fixed point of T.*

Proof. By taking $r = 1$, $\{T_i^n\}_{i=1}^m = T$ and $\{q_i\}_{i=1}^m = 0$ for all $n \geq 1$ in (2), we get (1), which is Rhoades and Soltuz introduced in [13]. From Theorem 3 and Theorem 4, the conclusion of the corollary follows. □

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Juan Xiao
School of Mathematics and Statistics
Southwest University
Chongqing 400715, P. R. China
E-mail address: juanxiao@swu.edu.cn

Lei Deng
School of Mathematics and Statistics
Southwest University
Chongqing 400715, P. R. China
E-mail address: denglei@swu.edu.cn

Ming-ge Yang
College of Mathematics Science
Luoyang Normal University
Luoyang, Henan 471022, P. R. China
E-mail address: yang_mingge@sina.com