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A SHARP SCHWARZ AND CARATHÉODORY INEQUALITY ON THE BOUNDARY

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Abstract. In this paper, a boundary version of the Schwarz and Carathéodory inequality are investigated. New inequalities of the Carathéodory's inequality and Schwarz lemma at boundary are obtained by taking into account zeros of $f(z)$ function which are different from zero. The sharpness of these inequalities is also proved.

1. Introduction

Let f be a function which is holomorphic on the $D : \{z : |z| < 1\}$ and vanish at $z = 0$, and suppose that $|f| < 1$ for all $z \in D$. Then the inequality

$$|f(z)| \le |z| \tag{1.1}$$

holds for all $z \in D$, and moreover

$$|f'(0)| \le 1. \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z \in D$) or in (1.2) if and only if f is an entire linear function of the form $f(z) = e^{i\alpha} z$, where α is a real number([2], p. 381).

Let the zeros of f be $z_1, z_2, \ldots, z_n$. If we apply inequality (1.1) to the function $f(z) \prod_{k=1}^{n} \left[\frac{1-\overline{z_k}z}{z+z_k} \right]$, we can conclude in the following Schwarz's inequality:

$$|f(z)| \le |z| \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - \overline{z_k}z} \right| \tag{1.3}$$

and

$$|f'(0)| \le \prod_{k=1}^{n} |z_k| .$$

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If $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, $c_p \neq 0$, $p \geq 1$, is a holomorphic function in D and $|f| \leq 1$ for $z \in D$, then at each $z \in D$ we have the inequality

$$|f(z)| \leq |z|^p \prod_{k=1}^{n} \left| \frac{z - z_k}{1 - \overline{z_k} z} \right| \tag{1.4}$$

and

$$|c_p| \leq \prod_{k=1}^{n} |z_k| .$$

If, in addition, the function f can be extended by continuity to a point $z_0 \in \partial D$, $|f(z_0)| = 1$, and the derivative $f'(z_0)$ exists, then (1.1) implies the inequality $|f'(z_0)| \geq 1$, which is known as the Schwarz lemma on the boundary. Previously, R. Osserman, examined sharp Schwarz inequality at the boundary (see [3]).

If the function f has an angular limit $f(z_0)$ at $z_0 \in \partial D$, $|f(z_0)| = 1$, then by Julia-Wolff-Lemma the angular derivative $f'(z_0)$ exists and $1 \leq |f'(z_0)| \leq \infty$ ([4]).

We will obtain more general results at the boundary. In the following Theorems 1.1-1.2, new inequalities of Schwarz inequality at the boundary are obtained and the sharpness of these inequalities is proved.

Introducing the notation

$$\Phi = \sum_{i_1=1}^{n-k+1} \sum_{i_2=i_1+1}^{n-k+2} \cdots \sum_{i_k=i_{k-1}+1}^{n} \left(|z_{i_1}| \, |z_{i_2}| \cdots |z_{i_k}| \right) .$$

Theorem 1.1. *Let f be a holomorphic function in the disc D, $|f| < 1$ for $|z| < 1$, $f(0) = 0$ and $z_1, z_2, \ldots, z_n$ are zeros of the function f in the unit disc that are different from $z = 0$. Further assume that, for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $|f(z_0)| = 1$. Then*

$$|f'(z_0)| \geq \frac{n+1+\sum\limits_{k=1}^{n}(n-2k+1)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)} . \tag{1.5}$$

The inequality (1.5) is sharp, with equality for the function $f(z) = z \prod\limits_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}$, where $z_1, z_2, \ldots, z_n$ are negative real numbers.

Proof. Using the upper bound (1.3) and if $z_0, c \in \partial D$ with $f(z_0) = c$, then we obtain

$$\left| \frac{f(z) - c}{|z| - |z_0|} \right| \geq \frac{1-|f(z)|}{1-|z|} \geq \frac{1-|z| \prod\limits_{k=1}^{n} \left| \frac{z-z_k}{1-\overline{z_k}z} \right|}{1-|z|} \geq \frac{1-|z| \prod\limits_{k=1}^{n} \frac{|z|+|z_k|}{1+|z_k||z|}}{1-|z|}$$

$$= \frac{\prod\limits_{k=1}^{n}(1+|z_k||z|) - |z| \prod\limits_{k=1}^{n}(|z|+|z_k|)}{(1-|z|)\prod\limits_{k=1}^{n}(1+|z_k||z|)}$$

$$= \frac{1-|z|^{n+1}+\sum\limits_{k=1}^{n}\left(|z|^{k}-|z|^{n-k+1}\right)\sum\limits_{i_1=1}^{n-k+1}\sum\limits_{i_2=i_1+1}^{n-k+2}\cdots\sum\limits_{i_k=i_{k-1}+1}^{n}\left(|z_{i_1}||z_{i_2}|\cdots|z_{i_k}|\right)}{(1-|z|)\prod\limits_{k=1}^{n}(1+|z_k||z|)}.$$

Passing to the angular limit in the last inequality yields

$$\frac{n+1+\sum\limits_{k=1}^{n}(n-2k+1)\sum\limits_{i_1=1}^{n-k+1}\sum\limits_{i_2=i_1+1}^{n-k+2}\cdots\sum\limits_{i_k=i_{k-1}+1}^{n}\left(|z_{i_1}||z_{i_2}|\cdots|z_{i_k}|\right)}{\prod\limits_{k=1}^{n}(1+|z_k|)}.$$

The equality in (1.5) is obtained for the function $f(z)=z\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}$, as show simple calculations. □

Theorem 1.2. *Let $f(z)=c_pz^p+c_{p+1}z^{p+1}+\cdots$, $c_p\neq 0$, $p\geq 1$, be a holomorphic function in the disc D, $|f|<1$ for $|z|<1$ and $z_1,z_2,\ldots,z_n$ are zeros of the function f in the unit disc that are different from $z=0$. Further assume that, for some $z_0\in\partial D$, f has an angular limit $f(z_0)$ at z_0, $|f(z_0)|=1$. Then*

$$|f'(z_0)|\geq\frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}. \tag{1.6}$$

The inequality (1.6) *is sharp, with equality for the function*

$$f(z)=z^p\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z},$$

where $z_1,z_2,\ldots,z_n$ are negative real numbers.

Proof. Using the upper bound (1.4) and if $z_0,c\in\partial D$ with $f(z_0)=c$, then we obtain

$$\left|\frac{f(z)-c}{|z|-|z_0|}\right|\geq\frac{1-|f(z)|}{1-|z|}\geq\frac{1-|z|^p\prod\limits_{k=1}^{n}\left|\frac{z-z_k}{1-\overline{z_k}z}\right|}{1-|z|}\geq\frac{1-|z|^p\prod\limits_{k=1}^{n}\frac{|z|+|z_k|}{1+|z_k||z|}}{1-|z|}$$

$$=\frac{\prod\limits_{k=1}^{n}(1+|z_k||z|)-|z|^p\prod\limits_{k=1}^{n}(|z|+|z_k|)}{(1-|z|)\prod\limits_{k=1}^{n}(1+|z_k||z|)}$$

$$=\frac{1-|z|^{n+p}+\sum\limits_{k=1}^{n}\left(|z|^{k}-|z|^{n-k+p}\right)\sum\limits_{i_1=1}^{n-k+1}\sum\limits_{i_2=i_1+1}^{n-k+2}\cdots\sum\limits_{i_k=i_{k-1}+1}^{n}\left(|z_{i_1}||z_{i_2}|\cdots|z_{i_k}|\right)}{(1-|z|)\prod\limits_{k=1}^{n}(1+|z_k||z|)}.$$

Passing to the angular limit in the last inequality yields

$$\frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\sum\limits_{i_1=1}^{n-k+1}\sum\limits_{i_2=i_1+1}^{n-k+2}\cdots\sum\limits_{i_k=i_{k-1}+1}^{n}\left(|z_{i_1}||z_{i_2}|\cdots|z_{i_k}|\right)}{\prod\limits_{k=1}^{n}(1+|z_k|)}.$$

The equality in (1.6) is obtained for the function $f(z) = z^p \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}$, as show simple calculations. □

In the following theorems, we formulated boundary "Carathéodory inequality" (see [1]) as long the Schwarz lemma at the boundary (see [3]). Besides, we have following results, which can be offered as the boundary refinement of the Carathéodory inequality.

Theorem 1.3. *Let f be a holomorphic function in the disc D, $f(0) = 0$ and $z_1, z_2, \ldots, z_n$ are zeros of the function f in the unit disc that are different from $z = 0$. Let $\Re f \leqslant A$ for $|z| < 1$. Further assume that, for some $z_0 \in \partial D$, f has an angular limit $f(z_0)$ at z_0, $\Re f(z_0) = A$. Then*

$$|f'(z_0)| \geq \frac{A}{2} \frac{n+1+\sum_{k=1}^{n}(n-2k+1)\Phi}{\prod_{k=1}^{n}(1+|z_k|)}. \tag{1.7}$$

The inequality (1.7) *is sharp, with equality for the function*

$$f(z) = \frac{2Az \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}}{1+z \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}},$$

where $z_1, z_2, \ldots, z_n$ are negative real numbers.

Proof. Consider the function

$$w(z) = \frac{f(z)}{f(z) - 2A}, \quad |z| < 1. \tag{1.8}$$

Since $\Re f(z_0) = A$, $|w(z_0)| = 1$ for $z_0 \in \partial D$ and the function $w(z)$ is defined in (1.8) satisfies the assumptions of Theorem 1.1, we obtain

$$|w'(z_0)| \geq \frac{n+1+\sum_{k=1}^{n}(n-2k+1)\Phi}{\prod_{k=1}^{n}(1+|z_k|)}$$

and

$$\frac{2A\,|f'(z_0)|}{|f(z_0) - 2A|^2} \geq \frac{n+1+\sum_{k=1}^{n}(n-2k+1)\Phi}{\prod_{k=1}^{n}(1+|z_k|)}.$$

Since $|f(z_0) - 2A|^2 \geq [\Re(f(z_0) - 2A)]^2 = A^2$, we obtain inequality (1.7) with an obvious equality case. □

Theorem 1.4. *Let $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, $c_p \neq 0$, $p \geq 1$, be a holomorphic function in the disc D and $z_1, z_2, \ldots, z_n$ are zeros of the function f in the* *unit disc that are different from* $z=0$. *Let* $\Re f \leqslant A$ *for* $|z|<1$. *Further assume that, for some* $z_0 \in \partial D$, f *has an angular limit* $f(z_0)$ *at* z_0, $\Re f(z_0)=A$. *Then*

$$|f'(z_0)| \geq \frac{A}{2}\frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}. \tag{1.9}$$

The inequality (1.9) *is sharp, with equality for the function*

$$f(z)=\frac{2Az^p\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}{1+z^p\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}},$$

where $z_1, z_2, \ldots, z_n$ *are negative real numbers.*

Proof. The function $w(z)$ is defined in (1.8) satisfies the assumptions of Theorem 1.2, we obtain

$$|w'(z_0)| \geq \frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}$$

and

$$\frac{2A\,|f'(z_0)|}{|f(z_0)-2A|^2} \geq \frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}.$$

Since $|f(z_0)-2A|^2 \geq [\Re(f(z_0)-2A)]^2 = A^2$, we obtain inequality (1.9) with an obvious equality case. □

Theorem 1.5. *Let* f *be a holomorphic function in the disc* D *and* $z_1, z_2, \ldots, z_n$ *are zeros of the function* $f(z)-f(0)$ *in the unit disc that are different from* $z=0$. *Let* $\Re f \leqslant A$ *for* $|z|<1$. *Further assume that, for some* $z_0 \in \partial D$, f *has an angular limit* $f(z_0)$ *at* z_0, $\Re f(z_0)=A$. *Then*

$$|f'(z_0)| \geq \left(\frac{A-\Re f(0)}{2}\right)\frac{n+1+\sum\limits_{k=1}^{n}(n-2k+1)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}. \tag{1.10}$$

The inequality (1.10) *is sharp, with equality for the function*

$$f(z)=f(0)+\frac{2(A-\Re f(0))z\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}{1+z\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}},$$

where $z_1, z_2, \ldots, z_n$ *are negative real numbers.*

Proof. Introducing the notation

$$\alpha = A-\Re f(z),\ \beta = A-\Re f(0).$$

If f is not identically constant, then $\alpha>0,\ \beta>0$, $\Re(f(z)-f(0))=\beta-\alpha$ $<\beta$ and $4\beta\Re(f(z)-f(0)) \leq 4\beta^2$. Therefore

$$|f(z)-f(0)-2\beta|^2 = |f(z)-f(0)|^2-4\beta\Re(f(z)-f(0))+4\beta^2 > |f(z)-f(0)|^2.$$

The function

$$\varphi(z)=\frac{f(z)-f(0)}{f(z)-f(0)-2\beta}$$

is a holomorphic function in the disc D, $|\varphi(z)|<1$, $\varphi(0)=0$ and $|\varphi(z_0)|=1$ for $z_0\in\partial D$. The function $\varphi(z)$ satisfies the assumptions of Theorem 1.1, we obtain

$$|\varphi'(z_0)|\geq\frac{n+1+\sum\limits_{k=1}^{n}(n-2k+1)\sum\limits_{i_1=1}^{n-k+1}\sum\limits_{i_2=i_1+1}^{n-k+2}\cdots\sum\limits_{i_k=i_{k-1}+1}^{n}\left(|z_{i_1}||z_{i_2}|\cdots|z_{i_k}|\right)}{\prod\limits_{k=1}^{n}(1+|z_k|)}.$$

Therefore, we take

$$\frac{2\beta\,|f'(z_0)|}{|f(z)-f(0)-2\beta|^2}\geq\frac{n+1+\sum\limits_{k=1}^{n}(n-2k+1)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}.$$

Since $|f(z_0)-f(0)|^2\geq[\Re\left(f(z_0)-f(0)\right)]^2=\beta^2$, we obtain inequality (1.10) with an obvious equality case. □

Theorem 1.6. *Let* $f(z)=f(0)+c_pz^p+c_{p+1}z^{p+1}+\cdots$, $c_p\neq 0$, $p\geq 1$, *be a holomorphic function in the disc* D *and* $z_1,z_2,\ldots,z_n$ *are zeros of the function* $f(z)-f(0)$ *in the unit disc that are different from* $z=0$. *Let* $\Re f\leqslant A$ *for* $|z|<1$. *Further assume that, for some* $z_0\in\partial D$, f *has an angular limit* $f(z_0)$ *at* z_0, $\Re f(z_0)=A$. *Then*

$$|f'(z_0)|\geq\left(\frac{A-\Re f(0)}{2}\right)\frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}. \tag{1.11}$$

The inequality (1.11) *is sharp, with equality for the function*

$$f(z)=f(0)+\frac{2(A-\Re f(0))z^p\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}{1+z^p\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}},$$

where $z_1,z_2,\ldots,z_n$ *are negative real numbers.*

Proof. The function $\varphi(z)$ is defined in Theorem 1.5 satisfies the assumptions of Theorem 1.4, we obtain

$$|\varphi'(z_0)|\geq\frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}.$$

Therefore, we take

$$\frac{2\beta\,|f'(z_0)|}{|f(z)-f(0)-2\beta|^2}\geq\frac{n+p+\sum\limits_{k=1}^{n}(n-2k+p)\Phi}{\prod\limits_{k=1}^{n}(1+|z_k|)}.$$

Since $|f(z_0)-f(0)|^2\geq[\Re\left(f(z_0)-f(0)\right)]^2=\beta^2$, we obtain inequality (1.11) with an obvious equality case. □

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