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NEW RESULTS FOR THE SERIES ${}_2F_1(x)$ WITH AN APPLICATION

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Abstract. The well known quadratic transformation formula due to Gauss:

$$(1-x)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b\,; \end{matrix}\; -\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix} a,\, a-b+\frac{1}{2}\,; \\ b+\frac{1}{2}\,; \end{matrix}\; x^2\right]$$

plays an important role in the theory of (generalized) hypergeometric series. In 2001, Rathie and Kim have obtained two results closely related to the above quadratic transformation for ${}_2F_1$. Our main objective of this paper is to deduce some interesting known or new results for the series ${}_2F_1(x)$ by using the above Gauss's quadratic transformation and its contiguous relations and then apply our results to provide a list of a large number of integrals involving confluent hypergeometric functions, some of which are (presumably) new. The results established here are (potentially) useful in mathematics, physics, statistics, engineering, and so on.

1. Introduction and preliminaries

The *generalized hypergeometric series* ${}_pF_q$ is defined by (see [16, p. 73]):

$$\begin{aligned}{}_pF_q\left[\begin{matrix} \alpha_1,\, \ldots,\, \alpha_p\,; \\ \beta_1,\, \ldots,\, \beta_q\,; \end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\, \ldots,\, \alpha_p;\, \beta_1,\, \ldots,\, \beta_q;\, z),\end{aligned} \tag{1.1}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [19, p. 2 and p. 6] and [20, p. 2 and pp. 4–6]):

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N} := \{1,\, 2,\, 3,\, \ldots\}) \end{cases} \tag{1.2}$$

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$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers, $\mathbb{C}$ the set of complex numbers, and $\Gamma(\lambda)$ is the familiar Gamma function. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1.1), that is, that

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q). \tag{1.3}$$

For the detailed conditions for the convergence of the series in (1.1), see, for example, [19, Section 1.4] and [20, Section 1.5]. It is only noted that if one of the numerator parameters, say, a_j is a negative integer, then the series in (1.1) reduces to a polynomial in z of degree $-a_j$.

It should also be remarked here that whenever hypergeometric function ${}_2F_1$ and generalized hypergeometric functions ${}_pF_q$ are expressed in terms of the Gamma function, the results are usually important, in particular, from the application point of view. Therefore, the well known summation theorems such as those of Gauss, Gauss's second, Bailey and Kummer for the series ${}_2F_1$ and Watson, Dixon and Whipple for the series ${}_3F_2$ and their extensions and generalizations (see [8], [9], [10], [11] and [13]) play an important role in the theory of generalized hypergeometric series. For applications of the above-mentioned classical summation theorems, we refer to [1], [2], [5], [6], [11], [13], [15] and [16].

By employing the above-mentioned classical summation theorems, Bailey [1] obtained a large number of known or new results involving certain products of hypergeometric series, quadratic transformation formulas and other results. In our present investigation we are interested in the following quadratic transformation formula due to Gauss [4]:

$$(1-x)^{-2a}\,{}_2F_1\left[\begin{array}{c} a,\, b\,; \\ 2b\,; \end{array}\ -\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{array}{r} a,\, a-b+\dfrac{1}{2}\,; \\ b+\dfrac{1}{2}\,; \end{array}\ x^2\right]. \tag{1.4}$$

Bailey [1] rederived this result by using the following classical Watson's summation theorem (see, for example, [19, p. 251] and [18]):

$$\begin{aligned} {}_3F_2&\left[\begin{array}{r} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c\,; \end{array}\ 1\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b+c\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}b\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a+c\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}b+c\right)}, \end{aligned} \tag{1.5}$$

provided $\Re(2c-a-b) > -1$. By making use of the following results closely related to the classical Watson's theorem (1.5) obtained earlier by Lavoie *et al.*

[8]:

$$
\tag{1.6} {}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c+1\,; \end{matrix}\; 1\right] \quad (\Re(2c-a-b) > -3)
$$

$$
= \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(a\right)\,\Gamma\left(b\right)}
$$

$$
\cdot\left\{\frac{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+1\right)\,\Gamma\left(c-\frac{1}{2}b+1\right)}\right\}
$$

and

$$
\tag{1.7} {}_3F_2\left[\begin{matrix} a,\, b,\, c\,; \\ \frac{1}{2}(a+b+1),\, 2c-1\,; \end{matrix}\; 1\right] \quad (\Re(2c-a-b) > 1)
$$

$$
= \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(a\right)\,\Gamma\left(b\right)}
$$

$$
\cdot\left\{\frac{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} + \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a\right)\,\Gamma\left(c-\frac{1}{2}b\right)}\right\}.
$$

Rathie and Kim [17] established the following two results contiguous to the quadratic transformation formula (1.4):

$$
\tag{1.8} (1-x)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b+1\,; \end{matrix}\; -\frac{4x}{(1-x)^2}\right]
$$

$$
= {}_2F_1\left[\begin{matrix} a,\, a-b+\dfrac{1}{2}\,; \\ b+\dfrac{1}{2}\,; \end{matrix}\; x^2\right] + \frac{2ax}{2b+1}\,{}_2F_1\left[\begin{matrix} a+1,\, a-b+\dfrac{1}{2}\,; \\ b+\dfrac{3}{2}\,; \end{matrix}\; x^2\right]
$$

and

$$
\tag{1.9} (1-x)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\, b\,; \\ 2b-1\,; \end{matrix}\; -\frac{4x}{(1-x)^2}\right]
$$

$$
= {}_2F_1\left[\begin{matrix} a,\, a-b+\dfrac{3}{2}\,; \\ b-\dfrac{1}{2}\,; \end{matrix}\; x^2\right] - \frac{2ax}{2b-1}\,{}_2F_1\left[\begin{matrix} a+1,\, a-b+\dfrac{3}{2}\,; \\ b+\dfrac{1}{2}\,; \end{matrix}\; x^2\right].
$$

Our work here is organized as follows: In Section 2, we establish some new or known results for the series ${}_2F_1\,[a,\, b\,;\, c;\, x]$ by employing the Gauss's quadratic transformation formula (1.4) and its contiguous results (1.8) and (1.9). In Section 3, we provide a list of a large number of integrals involving confluent hypergeometric functions.

2. Some results for the series ${}_2F_1(x)$

The results to be established in this section are as follows:

$$(2.1)\qquad {}_2F_1\left[\begin{matrix}\gamma,\ \gamma+\dfrac{1}{2}\,; \\ 2\,\gamma\,;\end{matrix}\ z\right] = (1-z)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1}.$$

$$(2.2)\qquad {}_2F_1\left[\begin{matrix}\gamma,\ \gamma-\dfrac{1}{2}\,; \\ 2\,\gamma\,;\end{matrix}\ z\right] = \left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma-1}.$$

$$(2.3)\qquad {}_2F_1\left[\begin{matrix}\gamma,\ \gamma-\dfrac{1}{2}\,; \\ 2\,\gamma+1\,;\end{matrix}\ z\right] = \frac{2\gamma\sqrt{1-z}+1}{2\gamma+1}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma}.$$

$$(2.4)\qquad {}_2F_1\left[\begin{matrix}\gamma+1,\ \gamma-\dfrac{1}{2}\,; \\ 2\,\gamma+1\,;\end{matrix}\ z\right] = \frac{2\gamma+\sqrt{1-z}}{2\gamma+1}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2\gamma}.$$

$$(2.5)\qquad {}_2F_1\left[\begin{matrix}2\,\alpha,\ \beta+1\,; \\ \beta\,;\end{matrix}\ z\right] = \left[1+\frac{z}{\beta}\,(2\alpha-\beta)\right]\,(1-z)^{-2\,\alpha-1}.$$

Proof. The derivation of the above results are quite straight forward. For example, when we establish the result (2.1), first replace x by $-x$ in (1.4) and set $z=\frac{4x}{1+x^2}$ with $a=\gamma+\frac{1}{2}$ and $b=\gamma$, and make use of the well-known binomial theorem:

$$(2.6)\qquad (1-z)^{-a} = {}_1F_0\left[\begin{matrix}a\,; \\ -\,;\end{matrix}\ z\right]\qquad (|z|<1),$$

we get (2.1). In a similar manner as in getting (2.1), first replacing x by $-x$ and setting $z=\frac{4x}{(1+x)^2}$, and taking

(i) $a=\gamma-\frac{1}{2}$ and $b=\gamma$;
(ii) $a=\gamma-\frac{1}{2}$ and $b=\gamma$;
(iii) $a=\gamma-\frac{1}{2}$ and $b=\gamma+1$

in (1.4), (1.8) and (1.9), respectively, we obtain (2.2), (2.3) and (2.4), respectively.

In order to derive (2.5), let $\mathcal{L}$ denote the left-hand side of (2.5). Then, by definition, we have

$$\mathcal{L} = \sum_{n=0}^{\infty}\frac{(2\alpha)_n}{n!}\,z^n\,\frac{(\beta+1)_n}{(\beta)_n}.$$

Noting $(\beta+1)_n/(\beta)_n = 1+\frac{n}{\beta}$, we have

$$\mathcal{L} = \sum_{n=0}^{\infty}\frac{(2\alpha)_n}{n!}\,z^n + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{(2\alpha)_n}{(n-1)!}\,z^n.$$

Setting $n-1=n'$ in the second series and dropping the prime on n and noting $(\alpha)_{n+1}=\alpha\,(\alpha+1)_n$, we have

$$\mathcal{L}=\sum_{n=0}^{\infty}\frac{(2\alpha)_n}{n!}\,z^n+\frac{2\alpha\,z}{\beta}\sum_{n=0}^{\infty}\frac{(2\alpha+1)_n}{n!}\,z^n.$$

By applying the binomial theorem (2.6) to the two last series, we get

$$\begin{aligned}\mathcal{L}&={}_1F_0\left[\begin{matrix}2\,\alpha\,;\\ -\,;\end{matrix}\;z\right]+\frac{2\alpha\,z}{\beta}\,{}_1F_0\left[\begin{matrix}2\,\alpha+1\,;\\ -\,;\end{matrix}\;z\right]\\ &=(1-z)^{-2\alpha}+\frac{2\alpha\,z}{\beta}\,(1-z)^{-2\alpha-1}\\ &=\left[1+\frac{z}{\beta}\,(2\alpha-\beta)\right]\,(1-z)^{-2\,\alpha-1},\end{aligned}$$

which is the right-hand side of (2.5). □

It is remarked in passing that (2.1) and (2.2) are known results (see [12]) while the results (2.3) and (2.4) seem to be new.

3. Integrals involving confluent hypergeometric functions

In this section, by using the results established in Section 2, we provide a list of a large number of certain interesting integrals involving confluent hypergeometric functions. To do this, we recall a well-known integral involving confluent hypergeometric function (see, for example, [12, p. 278]):

$$\int_0^{\infty} e^{-h\,t}\,t^{d-1}\,{}_1F_1\left[\begin{matrix}a\,;\\ b\,;\end{matrix}\;kt\right]dt=\frac{\Gamma(d)}{h^d}\,{}_2F_1\left[\begin{matrix}d,\,a\,;\\ b\,;\end{matrix}\;\frac{k}{h}\right],\tag{3.1}$$

provided $\Re(d)>0$, $\Re(h)>0$ and $|k|<|h|$. It is noted that the integral formula (3.1) is a special case of a known general result recorded in [14, p. 546, Entry 3.38.1].

Many interesting integrals involving confluent hypergeometric functions can easily be obtained from (3.1) by using the results in Section 2. Since these integrals in this section are easily derivable, we give them here without their proofs:

1. $\Re(h)>0$, $\Re(a)>0$ and $|k|<|h|$

$$\begin{aligned}&\int_0^{\infty} e^{-h\,t}\,t^{a-1}\,{}_1F_1\left[\begin{matrix}a+\dfrac{1}{2}\,;\\ 2a\,;\end{matrix}\;kt\right]dt\\ &=\frac{\Gamma(a)}{h^a}\,{}_2F_1\left[\begin{matrix}a+\dfrac{1}{2},\,a\,;\\ 2a\,;\end{matrix}\;\frac{k}{h}\right]=\frac{\Gamma(a)}{h^a}\left(1-\frac{k}{h}\right)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}.\end{aligned}\tag{3.2}$$

2. $\Re(h) > 0$, $\Re(a) > -\frac{1}{2}$ and $|k| < |h|$

$$\begin{aligned}
&\int_0^\infty e^{-h\,t}\, t^{a-\frac{1}{2}}\, {}_1F_1\left[\begin{matrix} a\,; \\ 2a\,; \end{matrix}\; kt\right] dt \\
&= \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\, {}_2F_1\left[\begin{matrix} a+\dfrac{1}{2},\ a\,; \\ 2a\,; \end{matrix}\; \frac{k}{h}\right] \\
&= \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}} \left(1-\frac{k}{h}\right)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}.
\end{aligned} \tag{3.3}$$

3. $\Re(h) > 0$, $\Re(a) > -1$ and $|k| < |h|$

$$\begin{aligned}
&\int_0^\infty e^{-h\,t}\, t^{a}\, {}_1F_1\left[\begin{matrix} a+\dfrac{1}{2}\,; \\ 2a+1\,; \end{matrix}\; kt\right] dt \\
&= \frac{\Gamma\left(a+1\right)}{h^{a+1}}\, {}_2F_1\left[\begin{matrix} a+\dfrac{1}{2},\ a+1\,; \\ 2a+1\,; \end{matrix}\; \frac{k}{h}\right] \\
&= \frac{\Gamma\left(a+1\right)}{h^{a+1}} \left(1-\frac{k}{h}\right)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}.
\end{aligned} \tag{3.4}$$

4. $\Re(h) > 0$, $\Re(a) > -\frac{1}{2}$ and $|k| < |h|$

$$\begin{aligned}
&\int_0^\infty e^{-h\,t}\, t^{a-\frac{1}{2}}\, {}_1F_1\left[\begin{matrix} a+1\,; \\ 2a+1\,; \end{matrix}\; kt\right] dt \\
&= \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\, {}_2F_1\left[\begin{matrix} a+\dfrac{1}{2},\ a+1\,; \\ 2a+1\,; \end{matrix}\; \frac{k}{h}\right] \\
&= \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}} \left(1-\frac{k}{h}\right)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}.
\end{aligned} \tag{3.5}$$

5. $\Re(h) > 0$, $\Re(a) > 0$ and $|k| < |h|$

$$\begin{aligned}
&\int_0^\infty e^{-h\,t}\, t^{a-1}\, {}_1F_1\left[\begin{matrix} a-\dfrac{1}{2}\,; \\ 2a\,; \end{matrix}\; kt\right] dt \\
&= \frac{\Gamma\left(a\right)}{h^{a}}\, {}_2F_1\left[\begin{matrix} a-\dfrac{1}{2},\ a\,; \\ 2a\,; \end{matrix}\; \frac{k}{h}\right] = \frac{\Gamma\left(a\right)}{h^{a}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}.
\end{aligned} \tag{3.6}$$

6. $\Re(h) > 0$, $\Re(a) > \frac{1}{2}$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{a-\frac{3}{2}}\, {}_1F_1\left[\begin{matrix} a\,; \\ 2a\,; \end{matrix}\, kt\right] dt \\ &= \frac{\Gamma\left(a-\frac{1}{2}\right)}{h^{a-\frac{1}{2}}}\, {}_2F_1\left[\begin{matrix} a-\dfrac{1}{2},\, a\,; \\ 2a\,; \end{matrix}\, \frac{k}{h}\right] = \frac{\Gamma\left(a-\frac{1}{2}\right)}{h^{a-\frac{1}{2}}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a-1}. \end{aligned} \tag{3.7}$$

7. $\Re(h) > 0$, $\Re(a) > 0$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{a-1}\, {}_1F_1\left[\begin{matrix} a+\dfrac{1}{2}\,; \\ 2a+1\,; \end{matrix}\, kt\right] dt \\ &= \frac{\Gamma\left(a\right)}{h^{a}}\, {}_2F_1\left[\begin{matrix} a+\dfrac{1}{2},\, a\,; \\ 2a+1\,; \end{matrix}\, \frac{k}{h}\right] = \frac{\Gamma\left(a\right)}{h^{a}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}. \end{aligned} \tag{3.8}$$

8. $\Re(h) > 0$, $\Re(a) > -\frac{1}{2}$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{a-\frac{1}{2}}\, {}_1F_1\left[\begin{matrix} a\,; \\ 2a+1\,; \end{matrix}\, kt\right] dt \\ &= \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\, {}_2F_1\left[\begin{matrix} a+\dfrac{1}{2},\, a\,; \\ 2a+1\,; \end{matrix}\, \frac{k}{h}\right] = \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}. \end{aligned} \tag{3.9}$$

9. $\Re(h) > 0$, $\Re(a) > 0$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{a-1}\, {}_1F_1\left[\begin{matrix} a-\dfrac{1}{2}\,; \\ 2a+1\,; \end{matrix}\, kt\right] dt \\ &= \frac{\Gamma\left(a\right)}{h^{a}}\, {}_2F_1\left[\begin{matrix} a-\dfrac{1}{2},\, a\,; \\ 2a+1\,; \end{matrix}\, \frac{k}{h}\right] = \frac{\Gamma\left(a\right)}{h^{a}}\, \frac{2a\sqrt{1-\frac{k}{h}}+1}{2a+1} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}. \end{aligned} \tag{3.10}$$

10. $\Re(h) > 0$, $\Re(a) > \frac{1}{2}$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{a-\frac{3}{2}}\, {}_1F_1\left[\begin{matrix} a\,; \\ 2a+1\,; \end{matrix}\, kt\right] dt \\ &= \frac{\Gamma\left(a-\frac{1}{2}\right)}{h^{a-\frac{1}{2}}}\, {}_2F_1\left[\begin{matrix} a-\dfrac{1}{2},\, a\,; \\ 2a+1\,; \end{matrix}\, \frac{k}{h}\right] \\ &= \frac{\Gamma\left(a-\frac{1}{2}\right)}{h^{a-\frac{1}{2}}}\, \frac{2a\sqrt{1-\frac{k}{h}}+1}{2a+1} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a}. \end{aligned} \tag{3.11}$$

11. $\Re(h) > 0$, $\Re(a) > 0$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{a-1}\, {}_1F_1\left[\begin{matrix} a+\frac{1}{2}\,; \\ 2a+2\,; \end{matrix}\; kt\right] dt \\ &= \frac{\Gamma(a)}{h^a}\, {}_2F_1\left[\begin{matrix} a+\frac{1}{2},\, a\,; \\ 2a+2\,; \end{matrix}\; \frac{k}{h}\right] \\ &= \frac{\Gamma(a)}{h^a}\, \frac{(2a+1)\sqrt{1-\frac{k}{h}}+1}{2a+2} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a+1}. \end{aligned} \tag{3.12}$$

12. $\Re(h) > 0$, $\Re(a) > -\frac{1}{2}$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{a-\frac{1}{2}}\, {}_1F_1\left[\begin{matrix} a\,; \\ 2a+2\,; \end{matrix}\; kt\right] dt \\ &= \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\, {}_2F_1\left[\begin{matrix} a+\frac{1}{2},\, a\,; \\ 2a+2\,; \end{matrix}\; \frac{k}{h}\right] \\ &= \frac{\Gamma\left(a+\frac{1}{2}\right)}{h^{a+\frac{1}{2}}}\, \frac{(2a+1)\sqrt{1-\frac{k}{h}}+1}{2a+2} \left(\frac{2}{1+\sqrt{1-\frac{k}{h}}}\right)^{2a+1}. \end{aligned} \tag{3.13}$$

13. $\Re(h) > 0$, $\Re(a) > 0$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{2a-1}\, {}_1F_1\left[\begin{matrix} b+1\,; \\ b\,; \end{matrix}\; kt\right] dt \\ &= \frac{\Gamma(2a)}{h^{2a}}\, {}_2F_1\left[\begin{matrix} 2a,\, b+1\,; \\ b\,; \end{matrix}\; \frac{k}{h}\right] = \frac{\Gamma(2a)}{h^{2a}} \left[1+\frac{k(2a-b)}{bh}\right] \left(1-\frac{k}{h}\right)^{-2a-1}. \end{aligned} \tag{3.14}$$

14. $\Re(h) > 0$, $\Re(b) > -1$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{b}\, {}_1F_1\left[\begin{matrix} 2a\,; \\ b\,; \end{matrix}\; kt\right] dt \\ &= \frac{\Gamma(b+1)}{h^{b+1}}\, {}_2F_1\left[\begin{matrix} 2a,\, b+1\,; \\ b\,; \end{matrix}\; \frac{k}{h}\right] = \frac{\Gamma(b+1)}{h^{b+1}} \left[1+\frac{k(2a-b)}{bh}\right] \left(1-\frac{k}{h}\right)^{-2a-1}. \end{aligned} \tag{3.15}$$

Remark. If we set $b = a$ in (3.14) and (3.15), we, respectively, get

15. $\Re(h) > 0$, $\Re(a) > 0$ and $|k| < |h|$

$$\begin{aligned} &\int_0^\infty e^{-h\,t}\, t^{2a-1}\, {}_1F_1\left[\begin{matrix} a+1\,; \\ a\,; \end{matrix}\; kt\right] dt \\ &= \frac{\Gamma(2a)}{h^{2a}}\, {}_2F_1\left[\begin{matrix} 2a,\, a+1\,; \\ a\,; \end{matrix}\; \frac{k}{h}\right] = \Gamma(2a)\,(h+k)\,(h-k)^{-2a-1}; \end{aligned} \tag{3.16}$$

16. $\Re(h) > 0$, $\Re(a) > -1$ and $|k| < |h|$

(3.17)

$$\int_0^\infty e^{-h\,t}\, t^a\, {}_1F_1\left[\begin{matrix} 2a\,; \\ a\,; \end{matrix}\; kt\right] dt$$

$$= \frac{\Gamma(a+1)}{h^{a+1}}\, {}_2F_1\left[\begin{matrix} 2a,\; a+1\,; \\ a\,; \end{matrix}\; \frac{k}{h}\right] = \frac{\Gamma(a+1)}{h^{a+1}} \left(1+\frac{k}{h}\right)\left(1-\frac{k}{h}\right)^{-2a-1},$$

which are found to be special cases of an integral formula in [12, p. 278]. The results (3.10) and (3.15) are believed to be new.

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