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HERMITE-HADAMARD TYPE INEQUALITIES FOR GEOMETRIC-ARITHMETICALLY s-CONVEX FUNCTIONS

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Abstract. In the paper, several properties of geometric-arithmetically s-convex functions are provided, an integral identity in which the integrands are products of a function and a derivative is found, and then some inequalities of Hermite-Hadamard type for integrals whose integrands are products of a derivative and a function whose derivative is of the geometric-arithmetic s-convexity are established.

1. Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

is well known as Hermite-Hadamard inequality for convex functions.

Definition 1.2 ([7])**.** For some $s \in (0, 1]$, a function $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}_0$ is said to be s-convex (in the second sense) if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.3}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

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Definition 1.3 ([13, 14]). A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be geometric-arithmetically convex if

$$f\big(x^t y^{1-t}\big) \le t f(x) + (1-t) f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.4 ([9, Definition 5] and [18, Definition 2.1]). For some $s \in (0,1]$, a function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be geometric-arithmetically s-convex if

$$f\big(x^t y^{1-t}\big) \le t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In recent decades, a lot of inequalities of Hermite-Hadamard type for various kinds of convex functions have been established. Some of them may be reformulated as follows.

Theorem 1.1 ([5, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)\big(|f'(a)|+|f'(b)|\big)}{8}. \tag{1.4}$$

Theorem 1.2 ([12, Theorems 1 and 3]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$.*

(1) *If $|f'(x)|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{s+1/2^s}{(s+1)(s+2)}\right]^{1/q}\big[|f'(a)|^q + |f'(b)|^q\big]^{1/q};\end{aligned} \tag{1.5}$$

(2) *If $|f'(x)|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \frac{b-a}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q} \\ &\quad\times\left\{\left[|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}\right\} \\ &\le \frac{b-a}{2}\left\{\left[|f'(a)|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + |f'(b)|^q\right]^{1/q}\right\}.\end{aligned}$$

Theorem 1.3 ([8, Theorem 2.2 and 2.4]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}_0$ be differentiable on I° and $g : [a, b] \to \mathbb{R}_0$ be continuous and symmetric with respect to $\frac{a+b}{2}$ for $a, b \in I$ with $a < b$.*

(1) *If $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)\,\mathrm{d}x-\int_a^b f(x)g(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}[|f'(a)|+|f'(b)|]\int_0^1\int_{L(t)}^{U(t)} g(x)\,\mathrm{d}x\,\mathrm{d}t, \tag{1.6}$$

where

$$L(t)=\frac{1+t}{2}a+\frac{1-t}{2}b \quad \textit{and} \quad U(t)=\frac{1-t}{2}a+\frac{1+t}{2}b. \tag{1.7}$$

(2) *If $|f'|^q$ is convex on $[a,b]$ for $q\ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)\,\mathrm{d}x-\int_a^b f(x)g(x)\,\mathrm{d}x\right| \le \frac{b-a}{2}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}\int_0^1\int_{L(t)}^{U(t)} g(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{1.8}$$

In recent years, some other kinds of Hermite-Hadamard type inequalities were created in, for example, [1, 2, 3, 4, 10, 11, 16, 17, 19, 20, 21, 22, 23, 25, 26, 27, 24, 28, 29], especially the monographs [6, 15], and related references therein.

In this paper, we will supply several properties of the above defined geometric-arithmetically s-convex functions, find an identity for an integral whose integrand is a product of a function and a derivative, and then establish some integral inequalities of Hermite-Hadamard type for functions whose derivative are of the geometric-arithmetic s-convexity.

2. Properties of geometric-arithmetically s-convex functions

It is clear that when $s=1$ Definition 1.4 becomes Definition 1.3.

We now supply an example of geometric-arithmetically s-convex functions as follows. Let $s\in(0,1]$ and $f(x)=x^p$ for $x\in\mathbb{R}_+$ and $p>0$. Then

$$f\bigl(x^t y^{1-t}\bigr)\le\begin{cases} tf(x)+(1-t)f(y)\le t^s f(x)+(1-t)^s f(y), & p>0\\ f(tx+(1-t)y)\le tf(x)+(1-t)f(y), & p\ge 1\\ tf(x)+(1-t)f(y)\le f(tx+(1-t)y), & 0<p\le 1\end{cases}$$

for all $x,y\in\mathbb{R}_+$ and $t\in[0,1]$. As a result,

(1) for all $p>0$ and $s\in(0,1]$, the power function x^p is geometric-arithmetically s-convex on $\mathbb{R}_+$;
(2) for $p\ge 1$ the power function x^p is geometric-arithmetically convex on $\mathbb{R}_+$;
(3) and for $0<p\le 1$ the power function x^p is concave on $\mathbb{R}_+$.

It was proved in [18, Proposition 2.3] that, for $s_1,s_2\in(0,1]$ with $s_1<s_2$, if a function $f:[a,b]\subseteq\mathbb{R}_+\to\mathbb{R}_+$ is s_2-geometric-arithmetically convex, then it is also s_1-geometric-arithmetically convex on $[a,b]$.

We now demonstrate several properties of geometric-arithmetically s-convex functions.

Theorem 2.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ and $s \in (0,1]$. Then the following statements are true*:

(1) *The function $f(x)$ is geometric-arithmetically s-convex on I if and only if $f(e^x)$ is s-convex on the interval $\ln I = \{\ln x \mid x \in I\}$, where it is assumed that $\ln 0 = -\infty$.*
(2) *If the function $f(x)$ is geometric-arithmetically convex on I, then it is geometric-arithmetically s-convex on I.*
(3) *If $f(x)$ is decreasing and geometric-arithmetically s-convex on I, then it is s-convex on I.*
(4) *If $f(x)$ is increasing and s-convex on I, then it is also geometric-arithmetically s-convex on I.*

Proof. If a function $f(x)$ is geometric-arithmetically s-convex on I, then we have

$$\begin{aligned} f\bigl(e^{t\ln x+(1-t)\ln y}\bigr) &= f\bigl(x^t y^{1-t}\bigr) \\ &\le t^s f(x) + (1-t)^s f(y) = t^s f\bigl(e^{\ln x}\bigr) + (1-t)^s f\bigl(e^{\ln y}\bigr), \end{aligned}$$

so the function $f(e^x)$ is s-convex on the interval $\ln I$. Conversely, if $f(e^x)$ is s-convex on the interval $\ln I$, we have

$$\begin{aligned} f\bigl(x^t y^{1-t}\bigr) &= f\bigl(e^{t\ln x+(1-t)\ln y}\bigr) \\ &\le t^s f\bigl(e^{\ln x}\bigr) + (1-t)^s f\bigl(e^{\ln y}\bigr) = t^s f(x) + (1-t)^s f(y), \end{aligned}$$

which means that the function $f(x)$ is geometric-arithmetically s-convex on I.

If a function $f(x)$ is geometric-arithmetically convex on I, we have

$$f\bigl(x^t y^{1-t}\bigr) \le t f(x) + (1-t) f(y) \le t^s f(x) + (1-t)^s f(y),$$

so it is also geometric-arithmetically s-convex on I.

If $f(x)$ is decreasing and geometric-arithmetically s-convex on I, we have

$$f(tx+(1-t)y) \le f\bigl(x^t y^{1-t}\bigr) \le t^s f(x) + (1-t)^s f(y),$$

which means that it is s-convex on I.

If $f(x)$ is increasing and s-convex on I, we have

$$f\bigl(x^t y^{1-t}\bigr) \le f(tx+(1-t)y) \le t^s f(x) + (1-t)^s f(y),$$

accordingly, it is geometric-arithmetically s-convex on I. □

3. An integral identity

For establishing new integral inequalities of Hermite-Hadamard type involving the geometric-arithmetically s-convex function, we need the following integral identities.

Lemma 3.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $b > a > 0$. Further let $h : [a, b] \to \mathbb{R}_0$ be differentiable. If $f' \in L([a, b])$, then*

(3.1)
$$\begin{aligned} & h(b)f(b) - h(a)f(a) - \int_a^b h'(x)f(x)\,\mathrm{d}\,x \\ = & \frac{\ln b - \ln a}{2}\biggl\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}h\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)f'\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\,\mathrm{d}\,t \\ & + \int_0^1 a^{(1-t)/2}b^{(1+t)/2}h\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)f'\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\,\mathrm{d}\,t\biggr\}. \end{aligned}$$

Proof. Since

$$\int_a^b h'(x)f(x)\,\mathrm{d}\,x = \int_a^{\sqrt{ab}} h'(x)f(x)\,\mathrm{d}\,x + \int_{\sqrt{ab}}^b h'(x)f(x)\,\mathrm{d}\,x.$$

Letting $x = a^{(1+t)/2}b^{(1-t)/2}$ for $t \in [0, 1]$ results in

$$\begin{aligned} & \int_a^{\sqrt{ab}} h'(x)f(x)\,\mathrm{d}\,x \\ = & -\int_0^1 f\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\,\mathrm{d}\,h\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr) \\ = & -h\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)f\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\big|_0^1 \\ & - \frac{\ln b - \ln a}{2}\int_0^1 a^{(1+t)/2}b^{(1-t)/2}h\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)f'\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\,\mathrm{d}\,t \\ = & \, h\bigl(\sqrt{ab}\bigr)f\bigl(\sqrt{ab}\bigr) - h(a)f(a) \\ & - \frac{\ln b - \ln a}{2}\int_0^1 a^{(1+t)/2}b^{(1-t)/2}h\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)f'\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\,\mathrm{d}\,t. \end{aligned}$$

Putting $x = a^{(1-t)/2}b^{(1+t)/2}$ for $t \in [0, 1]$ brings out

$$\begin{aligned} & \int_{\sqrt{ab}}^b h'(x)f(x)\,\mathrm{d}\,x \\ = & \int_0^1 f\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\,\mathrm{d}\,h\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr) \\ = & \, h\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)f\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\big|_0^1 \\ & - \frac{\ln b - \ln a}{2}\int_0^1 a^{(1-t)/2}b^{(1+t)/2}h\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)f'\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\,\mathrm{d}\,t \\ = & \, h(b)f(b) - h\bigl(\sqrt{ab}\bigr)f\bigl(\sqrt{ab}\bigr) \\ & - \frac{\ln b - \ln a}{2}\int_0^1 a^{(1-t)/2}b^{(1+t)/2}h\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)f'\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\,\mathrm{d}\,t. \end{aligned}$$

Lemma 3.1 is thus proved. □

4. Some integral inequalities of Hermite-Hadamard type

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type involving the geometric-arithmetically s-convex function.

Theorem 4.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $b > a > 0$. Further let $h : [a, b] \to \mathbb{R}_0$ be a differentiable function. If $f' \in L([a, b])$ and $|f'|^q$ is a geometric-arithmetically s-convex function on $[a, b]$ for $s \in (0, 1]$ and $q > 1$, then*

$$\left|h(b)f(b) - h(a)f(a) - \int_a^b h'(x)f(x)\,\mathrm{d}\,x\right| \le \frac{(\ln b - \ln a)\|h\|_\infty}{2}\left[\frac{1}{2^s(s+1)}\right]^{1/q}$$
$$\times \Big\{\big[a^{q/[2(q-1)]}L\big(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\big)\big]^{1-1/q}\big[(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q\big]^{1/q}$$
$$+\big[b^{q/[2(q-1)]}L\big(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\big)\big]^{1-1/q}\big[|f'(a)|^q + \big(2^{s+1}-1\big)|f'(b)|^q\big]^{1/q}\Big\},$$

where $\|h\|_\infty = \sup_{x\in[a,b]} h(x)$ and $L(u, v)$ is the logarithmic mean defined by

$$L(u,v) = \begin{cases} \dfrac{u-v}{\ln u - \ln v}, & u \ne v, \\ u, & u = v, \end{cases} \quad u, v > 0. \tag{4.1}$$

Proof. Using Lemma 3.1 and by Hölder integral inequality, we obtain

$$\left|h(b)f(b) - h(a)f(a) - \int_a^b h'(x)f(x)\,\mathrm{d}\,x\right|$$
$$\le \frac{\ln b - \ln a}{2}\bigg\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}h\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|\,\mathrm{d}\,t$$
$$+\int_0^1 a^{(1-t)/2}b^{(1+t)/2}h\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|\,\mathrm{d}\,t\bigg\}$$
$$\le \frac{(\ln b - \ln a)\|h\|_\infty}{2}\bigg\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|\,\mathrm{d}\,t$$
$$+\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|\,\mathrm{d}\,t\bigg\}$$
$$\le \frac{(\ln b - \ln a)\|h\|_\infty}{2}$$
$$\times\left\{\left[\int_0^1 \big(a^{(1+t)/2}b^{(1-t)/2}\big)^{q/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 \big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|^q\,\mathrm{d}\,t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1 \big(a^{(1-t)/2}b^{(1+t)/2}\big)^{q/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 \big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|^q\,\mathrm{d}\,t\right]^{1/q}\right\}.$$

Since

$$\int_0^1 \left(a^{(1+t)/2}b^{(1-t)/2}\right)^{q/(q-1)} \,\mathrm{d}t = a^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right)$$

and

$$\int_0^1 \left(a^{(1-t)/2}b^{(1+t)/2}\right)^{q/(q-1)} \,\mathrm{d}t = b^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right),$$

by the geometric-arithmetic s-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}\int_0^1 \left|f'\left(a^{(1+t)/2}b^{(1-t)/2}\right)\right|^q \,\mathrm{d}t &\le \int_0^1 \left[\left(\frac{1+t}{2}\right)^s |f'(a)|^q + \left(\frac{1-t}{2}\right)^s |f'(b)|^q\right] \mathrm{d}t \\ &= \frac{\left(2^{s+1}-1\right)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)}.\end{aligned}$$

Similarly, we have

$$\int_0^1 \left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|^q \,\mathrm{d}t \le \frac{|f'(a)|^q + \left(2^{s+1}-1\right)|f'(b)|^q}{2^s(s+1)}.$$

A combination of the above equalities and inequalities immediately gives Theorem 4.1. □

Theorem 4.2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $b > a > 0$, and let $h : [a,b] \to \mathbb{R}_0$ be differentiable. If $f' \in L([a,b])$ and $|f'|^q$ is a geometric-arithmetically s-convex function on $[a,b]$ for $s \in (0,1]$ and $q > 1$, then*

$$\begin{aligned}&\left|h(b)f(b) - h(a)f(a) - \int_a^b h'(x)f(x)\,\mathrm{d}x\right| \le \frac{(\ln b - \ln a)\|h\|_\infty}{2^{s+1}}\left(\frac{1}{sq+1}\right)^{1/q} \\ &\times \left\{\left[a^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left[\left(2^{sq+1}-1\right)^{1/q}|f'(a)| + |f'(b)|\right]\right. \\ &+ \left.\left[b^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left[|f'(a)| + \left(2^{sq+1}-1\right)^{1/q}|f'(b)|\right]\right\}.\end{aligned}$$

Proof. From Lemma 3.1 and by the geometric-arithmetically s-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}&\left|h(b)f(b) - h(a)f(a) - \int_a^b h'(x)f(x)\,\mathrm{d}x\right| \\ &\le \frac{\ln b - \ln a}{2}\left\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}h\left(a^{(1+t)/2}b^{(1-t)/2}\right)\left|f'\left(a^{(1+t)/2}b^{(1-t)/2}\right)\right|\mathrm{d}t\right. \\ &\quad\left.+ \int_0^1 a^{(1-t)/2}b^{(1+t)/2}h\left(a^{(1-t)/2}b^{(1+t)/2}\right)\left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|\mathrm{d}t\right\} \\ &\le \frac{(\ln b - \ln a)\|h\|_\infty}{2}\left\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\left|f'\left(a^{(1+t)/2}b^{(1-t)/2}\right)\right|\mathrm{d}t\right. \\ &\quad\left.+ \int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left|f'\left(a^{(1-t)/2}b^{(1+t)/2}\right)\right|\mathrm{d}t\right\}\end{aligned}$$

$$\leq \frac{(\ln b - \ln a)\|h\|_\infty}{2}\biggl\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\biggl[\biggl(\frac{1+t}{2}\biggr)^s|f'(a)| + \biggl(\frac{1-t}{2}\biggr)^s|f'(b)|\biggr]\,\mathrm{d}\,t + \int_0^1 a^{(1-t)/2}b^{(1+t)/2}\biggl[\biggl(\frac{1-t}{2}\biggr)^s|f'(a)| + \biggl(\frac{1+t}{2}\biggr)^s|f'(b)|\biggr]\,\mathrm{d}\,t\biggr\}.$$

Using Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\biggl[\biggl(\frac{1+t}{2}\biggr)^s|f'(a)| + \biggl(\frac{1-t}{2}\biggr)^s|f'(b)|\biggr]\,\mathrm{d}\,t\\
&\leq \biggl[\int_0^1 \bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)^{q/(q-1)}\,\mathrm{d}\,t\biggr]^{1-1/q}\\
&\quad\times\biggl\{\biggl[\int_0^1\biggl(\frac{1+t}{2}\biggr)^{sq}\,\mathrm{d}\,t\biggr]^{1/q}|f'(a)| + \biggl[\int_0^1\biggl(\frac{1-t}{2}\biggr)^{sq}\,\mathrm{d}\,t\biggr]^{1/q}|f'(b)|\biggr\}\\
&= \bigl[a^{q/[2(q-1)]}L\bigl(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\bigr)\bigr]^{1-1/q}\biggl[\frac{1}{2^{sq}(sq+1)}\biggr]^{1/q}\\
&\quad\times\bigl[\bigl(2^{sq+1}-1\bigr)^{1/q}|f'(a)| + |f'(b)|\bigr].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\biggl[\biggl(\frac{1-t}{2}\biggr)^s|f'(a)| + \biggl(\frac{1+t}{2}\biggr)^s|f'(b)|\biggr]\,\mathrm{d}\,t\\
&\leq \bigl[b^{q/[2(q-1)]}L\bigl(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\bigr)\bigr]^{1-1/q}\biggl[\frac{1}{2^{sq}(sq+1)}\biggr]^{1/q}\\
&\quad\times\bigl[|f'(a)| + \bigl(2^{sq+1}-1\bigr)^{1/q}|f'(b)|\bigr].
\end{aligned}$$

Combining the above equalities and inequalities results in Theorem 4.2. □

Theorem 4.3. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I$ with $b > a > 0$, and let $h : [a, b] \to \mathbb{R}_0$ be differentiable. If $f' \in L([a, b])$ and $|f'|^q$ is a geometric-arithmetically s-convex function on $[a, b]$ for $s \in (0, 1]$ and $q \geq 1$, then*

$$\begin{aligned}
&\biggl|h(b)f(b) - h(a)f(a) - \int_a^b h'(x)f(x)\,\mathrm{d}\,x\biggr| \\
&\leq \frac{(\ln b - \ln a)\|h\|_\infty}{2^{1+s/q}}\bigl\{a^{1/2}L\bigl(a^{1/2}, b^{1/2}\bigr)\bigl[2^s|f'(a)|^q + |f'(b)|^q\bigr]^{1/q}\\
&\qquad + b^{1/2}L\bigl(a^{1/2}, b^{1/2}\bigr)\bigl[|f'(a)|^q + 2^s|f'(b)|^q\bigr]^{1/q}\bigr\}.
\end{aligned}\tag{4.2}$$

Proof. By Lemma 3.1 and Hölder integral inequality, we have

$$\biggl|h(b)f(b) - h(a)f(a) - \int_a^b h'(x)f(x)\,\mathrm{d}\,x\biggr|$$

$$\leq \frac{\ln b - \ln a}{2}\bigg\{ \int_0^1 a^{(1+t)/2}b^{(1-t)/2}h\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|\,\mathrm{d}t + \int_0^1 a^{(1-t)/2}b^{(1+t)/2}h\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|\,\mathrm{d}t\bigg\}$$

$$\leq \frac{(\ln b - \ln a)\|h\|_\infty}{2}\bigg\{ \int_0^1 a^{(1+t)/2}b^{(1-t)/2}\big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|\,\mathrm{d}t + \int_0^1 a^{(1-t)/2}b^{(1+t)/2}\big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|\,\mathrm{d}t\bigg\}$$

$$\leq \frac{(\ln b - \ln a)\|h\|_\infty}{2}\bigg\{ \bigg[\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\,\mathrm{d}t\bigg]^{1-1/q} \times\bigg[\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|^q\,\mathrm{d}t\bigg]^{1/q} + \bigg[\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\,\mathrm{d}t\bigg]^{1-1/q} \times\bigg[\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|^q\,\mathrm{d}t\bigg]^{1/q}\bigg\}.$$

Since $|f'|^q$ is a geometric-arithmetically s-convex function on $[a,b]$ and $\frac{1+t}{2} \leq 1$ and $\frac{1-t}{2} \leq \frac{1}{2}$ for $0 \leq t \leq 1$, then

$$\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|^q\,\mathrm{d}t \leq \int_0^1 a^{(1+t)/2}b^{(1-t)/2}\bigg[\bigg(\frac{1+t}{2}\bigg)^s|f'(a)|^q + \bigg(\frac{1-t}{2}\bigg)^s|f'(b)|^q\bigg]\,\mathrm{d}t \leq \int_0^1 a^{(1+t)/2}b^{(1-t)/2}\bigg[|f'(a)|^q + \bigg(\frac{1}{2}\bigg)^s|f'(b)|^q\bigg]\,\mathrm{d}t = \bigg(\frac{1}{2}\bigg)^s\big\{a^{1/2}L\big(a^{1/2},b^{1/2}\big)\big[2^s|f'(a)|^q + |f'(b)|^q\big]\big\}.$$

Similarly, we also have

$$\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|^q\,\mathrm{d}t \leq \bigg(\frac{1}{2}\bigg)^s\big\{b^{1/2}L\big(a^{1/2},b^{1/2}\big)\big[|f'(a)|^q + 2^s|f'(b)|^q\big]\big\}.$$

Substituting the last two inequalities into the first one reveals (4.2). Theorem 4.3 is thus proved. □

Theorem 4.4. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $a,b \in I$ with $b > a > 0$, and let $h : [a,b] \to \mathbb{R}_0$ be differentiable. If $f' \in L([a,b])$*

and $|f'|^q$ *is a geometric-arithmetically* 1*-convex function on* $[a,b]$ *for* $q \ge 1$*, then*

$$\begin{aligned}
(4.3)\qquad &\left|h(b)f(b)-h(a)f(a)-\int_a^b h'(x)f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{(\ln b-\ln a)\|h\|_\infty}{2}\biggl\{\bigl[a^{1/2}L\bigl(a^{1/2},b^{1/2}\bigr)\bigr]^{1-1/q} \\
&\quad\times\biggl(\frac{a^{1/2}}{2}L\bigl(a^{1/2},b^{1/2}\bigr)[|f'(a)|^q+|f'(b)|^q] \\
&\quad+\frac{1}{\ln b-\ln a}\bigl[a^{1/2}L\bigl(a^{1/2},b^{1/2}\bigr)-a\bigr][|f'(a)|^q-|f'(b)|^q]\biggr)^{1/q} \\
&\quad+\bigl[b^{1/2}L\bigl(a^{1/2},b^{1/2}\bigr)\bigr]^{1-1/q}\biggl(\frac{b^{1/2}}{2}L\bigl(a^{1/2},b^{1/2}\bigr)[|f'(a)|^q+|f'(b)|^q] \\
&\quad+\frac{1}{\ln b-\ln a}\bigl[b-b^{1/2}L\bigl(a^{1/2},b^{1/2}\bigr)\bigr][|f'(b)|^q-|f'(a)|^q]\biggr)^{1/q}\biggr\}.
\end{aligned}$$

Proof. Using Lemma 3.1 and by Hölder integral inequality, we obtain

$$\begin{aligned}
&\left|h(b)f(b)-h(a)f(a)-\int_a^b h'(x)f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{\ln b-\ln a}{2}\biggl\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}h\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\bigl|f'\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\bigr|\,\mathrm{d}\,t \\
&\quad+\int_0^1 a^{(1-t)/2}b^{(1+t)/2}h\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\bigl|f'\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\bigr|\,\mathrm{d}\,t\biggr\} \\
&\le \frac{(\ln b-\ln a)\|h\|_\infty}{2}\biggl\{\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\bigl|f'\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\bigr|\,\mathrm{d}\,t \\
&\quad+\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\bigl|f'\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\bigr|\,\mathrm{d}\,t\biggr\} \\
&\le \frac{(\ln b-\ln a)\|h\|_\infty}{2}\biggl\{\biggl[\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\,\mathrm{d}\,t\biggr]^{1-1/q} \\
&\quad\times\biggl[\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\bigl|f'\bigl(a^{(1+t)/2}b^{(1-t)/2}\bigr)\bigr|^q\,\mathrm{d}\,t\biggr]^{1/q} \\
&\quad+\biggl[\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\,\mathrm{d}\,t\biggr]^{1-1/q} \\
&\quad\times\biggl[\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\bigl|f'\bigl(a^{(1-t)/2}b^{(1+t)/2}\bigr)\bigr|^q\,\mathrm{d}\,t\biggr]^{1/q}\biggr\}.
\end{aligned}$$

Since

$$\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\,\mathrm{d}\,t=a^{1/2}L\bigl(a^{1/2},b^{1/2}\bigr),$$

$$\int_0^1 t a^{(1+t)/2} b^{(1-t)/2}\,\mathrm{d}\,t = \frac{2\big[a^{1/2}L\big(a^{1/2},b^{1/2}\big)-a\big]}{\ln b-\ln a},$$
$$\int_0^1 a^{(1-t)/2} b^{(1+t)/2}\,\mathrm{d}\,t = b^{1/2}L\big(a^{1/2},b^{1/2}\big),$$
$$\int_0^1 t a^{(1-t)/2} b^{(1+t)/2}\,\mathrm{d}\,t = \frac{2\big[b-b^{1/2}L\big(a^{1/2},b^{1/2}\big)\big]}{\ln b-\ln a}.$$

By the geometric-arithmetic 1-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}&\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\big|f'\big(a^{(1+t)/2}b^{(1-t)/2}\big)\big|^q\,\mathrm{d}\,t\\ &\le \int_0^1 a^{(1+t)/2}b^{(1-t)/2}\bigg(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\bigg)\,\mathrm{d}\,t\\ &=\frac12 a^{1/2}L\big(a^{1/2},b^{1/2}\big)[|f'(a)|^q+|f'(b)|^q]\\ &\quad+\frac{1}{\ln b-\ln a}\big[a^{1/2}L\big(a^{1/2},b^{1/2}\big)-a\big][|f'(a)|^q-|f'(b)|^q]\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\big|f'\big(a^{(1-t)/2}b^{(1+t)/2}\big)\big|^q\,\mathrm{d}\,t\\ &\le \frac12 b^{1/2}L(b^{1/2},b^{1/2})[|f'(a)|^q+|f'(b)|^q]\\ &\quad+\frac{1}{\ln b-\ln a}\big[b-b^{1/2}L\big(a^{1/2},b^{1/2}\big)\big][|f'(b)|^q-|f'(a)|^q].\end{aligned}$$

Combinatining the above equalities and inequalities leads to the inequality (4.3). Theorem 4.4 is proved. ☐

Corollary 4.4.1. *Under the conditions of Theorem 4.4, if $q=1$, then*

$$\begin{aligned}&\bigg|h(b)f(b)-h(a)f(a)-\int_a^b h'(x)f(x)\,\mathrm{d}\,x\bigg|\\ &\le\frac{(\ln b-\ln a)\|h\|_\infty}{2}\bigg\{\frac12\big[\big(a^{1/2}+b^{1/2}\big)L\big(a^{1/2},b^{1/2}\big)(|f'(a)|+|f'(b)|)\big]\\ &\quad+\frac{1}{\ln b-\ln a}\big[(a+b)-\big(a^{1/2}+b^{1/2}\big)L\big(a^{1/2},b^{1/2}\big)\big](|f'(b)|-|f'(a)|)\bigg\}.\end{aligned}\tag{4.4}$$

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