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ON A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS INVOLVING GRUSHIN TYPE OPERATOR

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Abstract. Using variational methods, we prove some results on the non-existence and multiplicity of weak solutions for a class of semilinear elliptic systems of two equations involving Grushin type operators with sign-changing nonlinearities. We also shows that the similar results can be obtained for systems of m equations, where m is arbitrary.

1. Introduction

In recent years, more and more mathematicians have studied the existence of solutions for degenerate elliptic problems. This comes from the fact that they arise in many areas of applied physics, including nuclear physics, field theory, solid waves and problems of false vacuum. These problems are introduced as models for several physical phenomena related to equilibrium of continuous media which somewhere be perfect insulators (see [8, 19]). However, the study have essentially based on the Caffarelli-Kohn-Nirenberg inequalities and their variants, see for example [6, 7, 9, 11, 14, 26] and the references therein. In this paper, we will study the existence of solutions for degenerate elliptic problems involving Grushin type operator $G_s = \Delta_x + |x|^{2s}\Delta_y$ for $s \geq 0$. To our knowledge, the Grushin type operators were firstly introduced in [10], and developed in [13, 15, 17, 22, 23, 24, 25].

Let $\Omega \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ be a bounded domain with smooth boundary $\partial\Omega$, and $0 \in \Omega$. In this paper, we are interested in the semilinear elliptic system with Grushin type operator

$$(1.1) \qquad \begin{cases} L_{\alpha,\beta} w = \lambda \nabla F & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$w = (u, v), \quad L_{\alpha,\beta} = \begin{pmatrix} -G_\alpha & 0 \\ 0 & -G_\beta \end{pmatrix}, \quad G_s = \Delta_x + |x|^{2s}\Delta_y \text{ for } s \geq 0,$$

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$$\Delta_x = \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_y = \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2},$$

and $\nabla F = (F_u, F_v)$ stands for the gradient of a C^1 function $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which may change sign, $\alpha, \beta \geq 0$, λ is a positive parameter. Denoting $N(s) = N_1 + (s+1)N_2$, we assume that $N_1, N_2 \geq 1$ and $N(\alpha) > 2$ and $N(\beta) > 2$.

We point the fact that in [5], N. M. Chuong et al. studied the existence of at least a weak solutions for problem (1.1), using the mountain pass theorem combined with the Ekeland principle. They also observed on the behaviour of the solutions when the parameter $\lambda \to 0$. The goal of this note is to give some sufficient conditions on the nonlinear terms to get the non-existence and multiplicity of weak solutions for (1.1). Thus, the results introduced here extend or complement the obtained results in [5]. Moreover, we do not require in this paper the Ambrosetti-Rabinowitz type condition as in [5]. We also shows that similar arguments can be applied to the systems of m equations, where m is arbitrary. Our paper is inspired by the ideas introduced in [2, 3, 20], in which the authors considered the problem with the p-Laplace operator $-\Delta_p$. Regarding systems of Hamiltonian form involving Grushin type operators, the reader may find in the papers [4, 12].

Throughout this paper for $(t, s) \in \mathbb{R}^2$, we denote $|(t, s)|^2 = |t|^2 + |s|^2$. We assume that $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1-function, satisfying the following conditions:

(**f1**) $F(x, y, 0, 0) = 0$ for a.e. $(x, y) \in \Omega$, $F(x, y, t, s) = F(x, y, 0, s)$ for all $t \leq 0$, $s \in \mathbb{R}$ and a.e. $(x, y) \in \Omega$, $F(x, y, t, s) = F(x, y, t, 0)$ for all $t \in \mathbb{R}$, $s \leq 0$ and a.e. $(x, y) \in \Omega$;

(**f2**) There exists a constant $C > 0$ such that

$$|F_t(x, y, t, s)| + |F_s(x, y, t, s)| \leq C\Big(1 + |t| + |s|\Big)$$

for all $t, s \in \mathbb{R}$ and a.e. $(x, y) \in \Omega$.

We say that a function γ verifies the property (Γ) if and only if

$$(\Gamma) \qquad \gamma(t, s) \leq M(|t|^2 + |s|^2)$$

for all $t, s \in \mathbb{R}$, where $M > 0$ is independent of γ. Let H_i, $i = 1, 2$ be two functions satisfying property Γ. Motivated by an eigenvalue problem considered in [2], we introduce the following assumptions on the behavior of F at the origin and at infinity:

(**f3**) It holds that

$$\limsup_{|(t,s)| \to 0} \frac{F(x, y, t, s)}{H_1(t, s)} \leq 0$$

uniformly in $(x, y) \in \Omega$;

(**f4**) It holds that

$$\limsup_{|(t,s)| \to \infty} \frac{F(x, y, t, s)}{H_2(t, s)} \leq 0$$

uniformly in $(x, y) \in \Omega$.

We denote by $S^{p,s}(\Omega)$, $1 \leq p < \infty$ the set of element $u \in L^p(\Omega)$ such that $\frac{\partial u}{\partial x_i}$, $|x|^s \frac{\partial u}{\partial y_j} \in L^p(\Omega)$ for all $i = 1, 2, \ldots, N_1$, and $j = 1, 2, \ldots, N_2$. Then $S^{p,s}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{p,s} = \left[\int_\Omega \Big(|u|^p + |\nabla_x u|^p + |x|^{ps}|\nabla_y u|^p\Big)\, dxdy\right]^{\frac{1}{p}},$$

where

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{N_1}}\right), \quad \nabla_y = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_{N_2}}\right).$$

The space $S_0^{p,s}(\Omega)$ is the closure of $C_0^1(\Omega)$ in the space $S^{p,s}(\Omega)$. In the special case when $p = 2$, we conclude the Hilbert spaces $S^{2,s}(\Omega)$ and $S_0^{2,s}(\Omega)$. In [22], the authors proved that for all $u \in S_0^{2,s}(\Omega)$,

$$\left(\int_\Omega |u|^q\, dxdy\right)^{\frac{1}{q}} \leq C\left[\int_\Omega (|\nabla_x u|^2 + |x|^{2s}|\nabla_y u|^2)\, dxdy\right]^{\frac{1}{2}}, \tag{1.2}$$

where $q = \frac{2N(s)}{N(s)-2} - \tau$, $C > 0$, $s \geq 0$ and $N(s) = N_1 + (s+1)N_2$, provided that $\tau > 0$ is small enough. Furthermore, the embedding $S_0^{2,s}(\Omega)$ into $L^q(\Omega)$ is compact for $2 \leq q < 2_s^* = \frac{2N(s)}{N(s)-2}$. Therefore, in the space $S_0^{2,s}(\Omega)$, we can use the following equivalent norm

$$\|u\|_{2,s} = \left[\int_\Omega (|\nabla_x u|^2 + |x|^{2s}|\nabla_y u|^2)\, dxdy\right]^{\frac{1}{2}}.$$

Next, for $\alpha, \beta \geq 0$, we define the space $H = S_0^{2,\alpha}(\Omega) \times S_0^{2,\beta}(\Omega)$. Then H is a Hilbert space under the norm

$$\|w\| = \|u\|_{2,\alpha} + \|v\|_{2,\beta}$$

and the inner product is

$$\langle u, v\rangle_H = \int_\Omega \Big(\nabla_x u_1 \cdot \nabla_x v_1 + \nabla_x u_2 \cdot \nabla_x v_2 + |x|^{2\alpha}\nabla_y u_1 \cdot \nabla_y v_1 + |x|^{2\beta}\nabla_y u_2 \cdot \nabla_y v_2\Big) dxdy$$

for all $u = (u_1, v_1)$, $v = (v_1, v_2) \in H$.

Definition 1.1. We say that a function $w = (u, v) \in H$ is a weak solution of system (1.1) if and only if

$$\int_\Omega \Big(\nabla_x u \cdot \nabla_x \varphi_1 + |x|^{2\alpha}\nabla_y u \cdot \nabla_y \varphi_1\Big)\, dxdy = \lambda \int_\Omega F_u(x, y, u, v)\varphi_1\, dxdy,$$
$$\int_\Omega \Big(\nabla_x v \cdot \nabla_x \varphi_2 + |x|^{2\beta}\nabla_y v \cdot \nabla_y \varphi_2\Big)\, dxdy = \lambda \int_\Omega F_v(x, y, u, v)\varphi_2\, dxdy$$

for all $\varphi = (\varphi_1, \varphi_2) \in H$.

The main results of this paper can be described in the following theorems.

Theorem 1.2. *Assume $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, satisfies*

$$|F_t(x, y, t, s)| + |F_s(x, y, t, s)| \le C(|t| + |s|), \quad C > 0 \tag{1.3}$$

for all $(t, s) \in \mathbb{R}^2$ and a.e. $(x, y) \in \Omega$. Then there is a constant $\lambda_* > 0$ such that system (1.1) has no nontrivial weak solution for any $\lambda < \lambda_*$.

Theorem 1.3. *Assume that the conditions* (**f1**)-(**f4**) *are satisfied. Moreover, if in addition we assume that there exist a ball $B \subset \Omega$ and $t_0, s_0 > 0$ such that $F(x, y, t_0, s_0) > 0$ for all $(x, y) \in B$, then there exists a constant $\lambda^* > 0$ such that system (1.1) has at least two nontrivial, nonnegative weak solutions for any $\lambda \ge \lambda^*$.*

Our paper is organized as follows. In Section 2, we prove Theorems 1.2 and 1.3 using variational arguments. In Section 3, we make some comments regarding extensions of system (1.1).

2. Proofs of the main results

Proof of Theorem 1.2. Let $\lambda_{1,\alpha}$, $\lambda_{1,\beta}$ be the first eigenvalue of the operators $-G_\alpha$ and $-G_\beta$ with Dirichlet boundary, i.e.,

$$\lambda_{1,\alpha} = \inf_{u \in S_0^{2,\alpha}(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla_x u|^2 + |x|^{2\alpha}|\nabla_y u|^2)\, dxdy}{\int_\Omega |u|^2\, dxdy} > 0,$$
$$\lambda_{1,\beta} = \inf_{v \in S_0^{2,\beta}(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla_x v|^2 + |x|^{2\beta}|\nabla_y v|^2)\, dxdy}{\int_\Omega |v|^2\, dxdy} > 0.$$

Then we have

$$\begin{aligned} 0 < \lambda_{\alpha,\beta} &= \min\{\lambda_{1,\alpha}, \lambda_{1,\beta}\} \\ &\le \frac{\int_\Omega \left[|\nabla_x u|^2 + |\nabla_x v|^2 + |x|^{2\alpha}|\nabla_y u|^2 + |x|^{2\beta}|\nabla_y v|^2\right] dxdy}{\int_\Omega \left[|u|^2 + |v|^2\right] dxdy} \end{aligned} \tag{2.4}$$

for all $w = (u, v) \in H\setminus\{0\}$ (see [22]).

If $w = (u, v) \in H$ is a non-trivial weak solution of system (1.1), it follows by multiplying the first equation of (1.1) by u and the second by v, and integrating by parts that

$$\int_\Omega (|\nabla_x u|^2 + |x|^{2\alpha}|\nabla_y u|^2)\, dx = \lambda \int_\Omega F_u(x, y, u, v) u\, dxdy,$$
$$\int_\Omega (|\nabla_x v|^2 + |x|^{2\beta}|\nabla_y v|^2)\, dx = \lambda \int_\Omega F_v(x, y, u, v) v\, dxdy,$$

which implies by using (1.3) that

$$
\begin{aligned}
&\int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2 + |x|^{2\alpha}|\nabla_y u|^2 + |x|^{2\beta}|\nabla_y v|^2)\,dxdy \\
&= \lambda \int_\Omega \Big(F_u(x,y,u,v)u + F_v(x,y,u,v)v\Big)\,dxdy \qquad (2.5)\\
&\le \lambda C \int_\Omega (|u|^2 + |v|^2)\,dxdy,
\end{aligned}
$$

where C is a positive constant. Thus, by taking $\lambda_* = \frac{\lambda_{\alpha,\beta}}{C} > 0$, where $\lambda_{\alpha,\beta}$ is given by (2.4), we conclude the proof of Theorem 1.2. □

In order to prove Theorem 1.3, we shall use critical point theory. For all $\mu, \lambda \in \mathbb{R}$, we consider the functional $T_\lambda : H \to \mathbb{R}$ given by

$$
\begin{aligned}
T_\lambda(w) &= \frac{1}{2}\int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2 + |x|^{2\alpha}|\nabla_y u|^2 + |x|^{2\beta}|\nabla_y v|^2)\,dxdy \\
&\quad - \lambda \int_\Omega F(x,y,u,v)\,dxdy \qquad (2.6)\\
&= J(w) - \lambda I(w), \quad w = (u,v) \in H,
\end{aligned}
$$

where

$$
\begin{aligned}
J(w) &= \frac{1}{2}\int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2 + |x|^{2\alpha}|\nabla_y u|^2 + |x|^{2\beta}|\nabla_y v|^2)\,dxdy, \qquad (2.7)\\
I(w) &= \int_\Omega F(x,y,u,v)\,dxdy, \quad w = (u,v) \in H.
\end{aligned}
$$

A simple computation implies that T_λ is well-defined and of C^1 class in H. Thus, weak solutions of system (1.1) correspond to the critical points of T_λ. Moreover, we first have the following result.

Lemma 2.1. *The functional T_λ given by* (2.6) *is weakly semi-continuous in H.*

Proof. Let $\{w_m\} = \{(u_m, v_m)\}$ be a sequence that converges weakly to $w = (u,v)$ in H. By the semi-continuity of norm, it is sufficient to show that the functional I is weakly continuous in H, i.e.,

$$
\lim_{m\to\infty} \int_\Omega F(x,y,u_m,v_m)\,dxdy = \int_\Omega F(x,y,u,v)\,dxdy. \qquad (2.8)
$$

Indeed, we have

$$
\begin{aligned}
&\int_\Omega \Big[F(x,y,u_m,v_m) - F(x,y,u,v)\Big]\,dxdy \\
&= \int_\Omega \nabla F(x,y,w + \theta_m(w_m - w))\cdot(w_m - w)\,dxdy \\
&= \int_\Omega F_u(x,y,u + \theta_{1,m}(u_m - u), v + \theta_{2,m}(v_m - v))(u_m - u)\,dxdy
\end{aligned}
$$

$$+\int_\Omega F_v(x,y,u+\theta_{1,m}(u_m-u),v+\theta_{2,m}(v_m-v))(v_m-v)\,dxdy,$$

where $\theta_m=(\theta_{1,m},\theta_{2,m})$ and $0\le\theta_{1,m}(x),\theta_{2,m}(x)\le 1$ for all $x\in\Omega$. Now, using (**f2**) and Hölder's inequality we conclude that

$$\begin{aligned}
&\left|\int_\Omega [F(x,y,u_m,v_m)-F(x,y,u,v)]\,dxdy\right|\\
&\le \int_\Omega |F_u(x,y,u+\theta_{1,m}(u_m-u),v+\theta_{2,m}(v_m-v))||u_m-u|\,dxdy\\
&\quad+\int_\Omega |F_v(x,y,u+\theta_{1,m}(u_m-u),v+\theta_{2,m}(v_m-v))||v_m-v|\,dxdy\\
&\le C\int_\Omega \Big(1+|u+\theta_{1,m}(u_m-u)|+|v+\theta_{2,m}(v_m-v))|\Big)\\
&\quad(|u_m-u|+|v_m-v|)\,dxdy\\
&\le C\Big(|\Omega|^{\frac12}+\|u+\theta_{1,m}(u_m-u)\|_{L^2}+\|v+\theta_{1,m}(v_m-v)\|_{L^2}\Big)\\
&\quad\Big(\|u_m-u\|_{L^2}+\|v_m-v\|_{L^2}\Big).
\end{aligned}$$

On the other hand, since $H\hookrightarrow L^2(\Omega)\times L^2(\Omega)$ is compact, the sequence $\{w_m\}$ converges strongly to $w=(u,v)$ in the space $L^2(\Omega)\times L^2(\Omega)$, i.e., $\{u_m\}$ converges strongly to u in $L^2(\Omega)$ and $\{v_m\}$ converges strongly to v in $L^2(\Omega)$. Hence, it is easy to see that the sequences $\{\|u+\theta_{1,m}(u_m-u)\|_{L^2}\}$ and $\{\|v+\theta_{2,m}(v_m-v)\|_{L^2}\}$ are bounded. Thus, it follows that (2.8) holds true. □

Lemma 2.2. *For any $\lambda\in\mathbb{R}$, the functional T_λ is coercive and bounded from below on H.*

Proof. For any $\lambda\in\mathbb{R}$, by conditions (**f2**) and (**f4**), there exists a constant $C_\lambda=C(\lambda)>0$ such that

$$\lambda F(x,y,t,s)\le \frac{\lambda_{\alpha,\beta}}{4M}H_2(t,s)+C_\lambda \tag{2.9}$$

for all $t,s\in\mathbb{R}$ and a.e. $x\in\Omega$. Hence, we have for all $w=(u,v)\in H$ that

$$\begin{aligned}
T_\lambda(w)&=\frac12\int_\Omega(|\nabla_x u|^2+|\nabla_x v|^2+|x|^{2\alpha}|\nabla_y u|^2+|x|^{2\beta}|\nabla_y v|^2)\,dxdy\\
&\quad-\lambda\int_\Omega F(x,y,u,v)\,dxdy\\
&\ge \frac12\|u\|_{2,\alpha}^2+\frac12\|v\|_{2,\beta}^2-\int_\Omega\left(\frac{\lambda_{\alpha,\beta}}{4M}H_2(u,v)+C_\lambda\right)dxdy\\
&\ge \frac14\|u\|_{2,\alpha}^2+\frac14\|v\|_{2,\beta}^2-C_\lambda|\Omega|,
\end{aligned}$$

which helps us to show that T_λ is coercive and bounded from below in H. □

Lemma 2.3. *If $w = (u, v) \in H$ is a weak solution of system* (1.1)*, then $u \geq 0$ and $v \geq 0$ in Ω.*

Proof. Indeed, if $w = (u, v) \in H$ is a weak solution of system (1.1), then we have

$$\begin{aligned} 0 &= T'_\lambda(w)(w^-) \\ &= \int_\Omega \Big(\nabla u_x \cdot \nabla_x u^- + \nabla v_x \cdot \nabla_x v^- + |x|^{2\alpha} \nabla u_y \cdot \nabla_y u^- + |x|^{2\alpha} \nabla v_y \cdot \nabla_y v^- \Big) dxdy \\ &\qquad - \lambda \int_\Omega \Big(F_u(x,y,u,v) u^- + F_v(x,y,u,v) v^- \Big)\, dxdy \\ &= \int_\Omega \Big(|\nabla_x u^-|^2 + |\nabla_x v^-|^2 + |x|^{2\alpha} |\nabla_y u^-|^2 + |x|^{2\alpha} |\nabla_y v^-|^2 \Big)\, dxdy, \end{aligned}$$

where $w^- = (u^-, v^-)$ with $u^- = \min\{0, u(x)\}$ is the negative part of u and $v^- = \min\{0, v(x)\}$ is the negative part of v. Moreover, since

$$0 = \|u^-\|^2_{2,\alpha} + \|v^-\|^2_{2,\beta} \geq \lambda_{\alpha,\beta} \int_\Omega \Big(|u^-|^2 + |v^-|^2 \Big)\, dxdy,$$

it follows that $u(x) \geq 0$ and $v(x) \geq 0$ for a.e. $x \in \Omega$, i.e., $w \geq 0$ for a.e. $x \in \Omega$. □

Lemma 2.4. *There exists $\lambda^* > 0$ such that $\inf_H T_\lambda < 0$ for any $\lambda \geq \lambda^*$.*

Proof. Let us consider a sufficiently large compact subset B' of B, where B is a ball such that $F(x, y, t_0, s_0) > 0$ for all $(x, y) \in B$ and some $t_0, s_0 > 0$. Consider u_0 and v_0, smooth functions with compact support in B, such that $u_0(x, y) = t_0$, $v_0(x, y) = s_0$ in B', $0 \leq u_0(x, y) \leq t_0$ and $0 \leq v_0(x, y) \leq s_0$ for all $(x, y) \in B \backslash B'$. Then we get

$$\begin{aligned} \int_\Omega F(x,y,u_0,v_0)\, dxdy &= \int_{B'} F(x,y,u_0,v_0)\, dxdy + \int_{B\backslash B'} F(x,y,u_0,v_0)\, dxdy \\ &\geq \int_{B'} F(x,y,t_0,s_0)\, dxdy - C \int_{B\backslash B'} (|u_0|^2 + |v_0|^2)\, dxdy \\ &\geq \int_{B'} F(x,y,t_0,s_0)\, dxdy - C(1 + |t_0|^2 + |s_0|^2)|B\backslash B'| \\ &> 0, \end{aligned}$$

provided that $|B\backslash B'|$ is small enough. Hence, denoting $w_0 = (u_0, v_0) \in H$, we have

$$\begin{aligned} T_\lambda(w_0) &= \frac{1}{2} \int_\Omega (|\nabla_x u_0|^2 + |\nabla_x v_0|^2 + |x|^{2\alpha} |\nabla_y u_0|^2 + |x|^{2\beta} |\nabla_y v_0|^2)\, dxdy \\ &\qquad - \lambda \int_\Omega F(x,y,u_0,v_0)\, dxdy \\ &\leq \frac{1}{2} \|u_0\|^2_{2,\alpha} + \frac{1}{2} \|v_0\|^2_{2,\beta} - \lambda \int_\Omega F(x,y,u_0,v_0)\, dxdy < 0 \end{aligned}$$

for λ large enough. So there is a positive constant λ^* such that $\inf_H T_\lambda < 0$ for all $\lambda \geq \lambda^*$. □

Now, we fix $\lambda \geq \lambda^*$. Then by Lemmas 2.1-2.4 the functional T_λ has a global minimum $w_1 = (u_1, v_1)$ such that $T_\lambda(w_1) < 0$ and thus system (1.1) has a non-negative nontrivial weak solution $w_1 = (u_1, v_1) \in H$. In order to obtain the second one, we use the mountain pass theorem [1]. To this purpose, we first show that T_λ has the mountain pass geometry.

Lemma 2.5. *For any $\lambda \in \mathbb{R}$, there exist $\rho \in (0, \|w_1\|_H)$ and a constant $r > 0$ such that $T_\lambda(w) \geq r$ for all $w = (u, v) \in H$ with $\|w\|_H = \rho$.*

Proof. For any $\lambda \in \mathbb{R}$, it follows from (**f2**) and (**f3**) that there is $C_\lambda > 0$, which depends on λ, such that

$$\lambda F(x, y, t, s) \leq \frac{\lambda_{\alpha,\beta}}{4M} H_1(t, s) + C_\lambda(|t|^p + |s|^q) \tag{2.10}$$

for all $(x, y, t, s) \in \Omega\times\mathbb{R}\times\mathbb{R}$, where $2 < p < 2^*_\alpha = \frac{2N(\alpha)}{N(\alpha)-2}$, $2 < q < 2^*_\beta = \frac{2N(\beta)}{N(\beta)-2}$. Hence, by the Sobolev embeddings, it follows that

$$\begin{aligned} T_\lambda(w) &= \frac{1}{2}\int_\Omega (|\nabla_x u|^2 + |\nabla_x v|^2 + |x|^{2\alpha}|\nabla_y u|^2 + |x|^{2\beta}|\nabla_y v|^2)\, dxdy \\ &\quad - \lambda \int_\Omega F(x, y, u, v)\, dxdy \\ &\geq \frac{1}{2}\|u\|^2_{2,\alpha} + \frac{1}{2}\|v\|^2_{2,\beta} - \frac{\lambda_{\alpha,\beta}}{4M}\int_\Omega H_1(u, v)\, dxdy - C_\lambda \int_\Omega (|u|^p + |v|^q)\, dxdy \\ &\geq \left(\frac{1}{4} - \overline{C}_\lambda \|u\|^{p-2}_{2,\alpha}\right)\|u\|^2_{2,\alpha} + \left(\frac{1}{4} - \overline{C}_\lambda \|v\|^{q-2}_{2,\beta}\right)\|v\|^2_{2,\beta}, \end{aligned}$$

where $\overline{C}_\lambda$ is a positive constant. Since $2 < p < 2^*_\alpha = \frac{2N(\alpha)}{N(\alpha)-2}$, $2 < q < 2^*_\beta = \frac{2N(\beta)}{N(\beta)-2}$, there are two constants $r > 0$ and $0 < \rho < \|w_1\|_H$ such that $T_\lambda(w) \geq r$ for all $w \in H$ with $\|w\|_H = \rho$. □

Lemma 2.6. *The functional T_λ satisfies the Palais-Smale condition in H.*

Proof. Let $\{w_m\} = \{(u_m, v_m)\}$ be a Palais-Smale sequence for the functional T_λ in H, i.e.,

$$|T_\lambda(w_m)| \leq \overline{c} \text{ for all } m, \quad T'_\lambda(w_m) \to 0 \text{ in } H^{-1} \text{ as } m \to \infty, \tag{2.11}$$

where H^{-1} is the dual space of H.

Since T_λ is coercive on H, the sequence $\{w_m\}$ is bounded in H. Since H is a Hilbert space, there exists $w = (u, v) \in H$ such that, passing to a subsequence, still denoted by $\{w_m\} = \{(u_m, v_m)\}$, it converges weakly to $w = (u, v)$ in H and hence $\{w_m\}$ converges strongly to w in $L^2(\Omega, \mathbb{R}^2)$, i.e., $\{u_m\}$ converges strongly to u in $L^2(\Omega)$ and $\{v_m\}$ converges strongly to v in $L^2(\Omega)$. By the definition of T_λ we have

(2.12)
$$\begin{aligned}
&T'_\lambda(w_m)(w_m - w)\\
&= \int_\Omega \Big(\nabla_x u_m \cdot (\nabla_x u_m - \nabla_x u) + \nabla_x v_m \cdot (\nabla_x v_m - \nabla_x v)\Big)\,dxdy\\
&\quad + \int_\Omega \Big(|x|^{2\alpha}\nabla_y u_m \cdot (\nabla_y u_m - \nabla_y u) + |x|^{2\beta}\nabla_y v_m \cdot (\nabla_y v_m - \nabla_y v)\Big)\,dxdy\\
&\quad - \lambda \int_\Omega \Big(F_u(x,y,u_m,v_m)(u_m - u) + F_v(x,y,u_m,v_m)(v_m - v)\Big)\,dxdy
\end{aligned}$$

and

(2.13)
$$\begin{aligned}
&T'_\lambda(w)(w - w_m)\\
&= \int_\Omega \Big(\nabla_x u \cdot (\nabla_x u - \nabla_x u_m) + \nabla_x v \cdot (\nabla_x v - \nabla_x v_m)\Big)\,dxdy\\
&\quad + \int_\Omega \Big(|x|^{2\alpha}\nabla_y u \cdot (\nabla_y u - \nabla_y u_m) + |x|^{2\beta}\nabla_y v \cdot (\nabla_y v - \nabla_y v_m)\Big)\,dxdy\\
&\quad - \lambda \int_\Omega \Big(F_u(x,y,u,v)(u - u_m) + F_v(x,y,u,v)(v - v_m)\Big)\,dxdy.
\end{aligned}$$

By (2.12) and (2.13) we get

$$\begin{aligned}
&T'_\lambda(w_m)(w_m - w) + T'_\lambda(w)(w - w_m)\\
&= \int_\Omega (|\nabla_x u_m - \nabla_x u|^2 + |\nabla_x v_m - \nabla_x v|^2)\,dxdy\\
&\quad + \int_\Omega (|x|^{2\alpha}|\nabla_y u_m - \nabla_y u|^2 + |x|^{2\beta}|\nabla_y v_m - \nabla_y v|^2)\,dxdy\\
&\quad - \lambda \int_\Omega \Big(F_u(x,y,u_m,v_m) - F_u(x,y,u,v)\Big)(u_m - u)\,dxdy\\
&\quad - \lambda \int_\Omega \Big(F_v(x,y,u_m,v_m) - F_v(x,y,u,v)\Big)(v_m - v)\,dxdy.
\end{aligned} \tag{2.14}$$

By (2.11), we get

$$\lim_{m\to\infty} T'_{\mu,\lambda}(w_m)(w_m - w) + T'_{\mu,\lambda}(w)(w - w_m) = 0. \tag{2.15}$$

On the other hand, by using the condition (**f2**), Hölder's inequality and the compact embeddings we deduce that

$$\begin{aligned}
&\left|\int_\Omega \Big(F_u(x,y,u_m,v_m) - F_u(x,y,u,v)\Big)(u_m - u)\,dxdy\right|\\
&\le C\int_\Omega (2 + |u_m| + |v_m| + |u| + |v|)|u_m - u|\,dxdy
\end{aligned} \tag{2.16}$$

$$\leq C\Big(2|\Omega|^{\frac{1}{2}} + \|u_m\|_{L^2} + \|v_m\|_{L^2}^2 + \|u\|_{L^2} + \|v\|_{L^2}^2\Big)\|u_m - u\|_{L^2}^2,$$

which tends 0 as $m \to \infty$ and

$$\begin{aligned}
&\left|\int_\Omega \Big(F_v(x,y,u_m,v_m) - F_v(x,y,u,v)\Big)(v_m - v)\,dxdy\right| \\
&\leq C\int_\Omega (2 + |u_m| + |v_m| + |u| + |v|)|v_m - v|\,dxdy \qquad (2.17)\\
&\leq C\Big(2|\Omega|^{\frac{1}{2}} + \|u_m\|_{L^2} + \|v_m\|_{L^2}^2 + \|u\|_{L^2} + \|v\|_{L^2}^2\Big)\|v_m - v\|_{L^2}^2,
\end{aligned}$$

which tends 0 as $m \to \infty$.

By relations (2.12)-(2.17) we conclude that $\{w_m\}$ converges strongly to w in H. Thus, the functional T_λ satisfies the Palais-Smale condition in H. □

Proof of Theorem 1.3. By Lemmas 2.1-2.4, for all $\lambda \geq \lambda^*$, system (1.1) admits a non-negative, non-trivial weak solution w_1 as the global minimizer of T_λ. Setting

$$\overline{c} := \inf_{\gamma\in\mathbf{\Gamma}} \max_{w\in\gamma([0,1])} T_\lambda(w), \qquad (2.18)$$

where

$$\mathbf{\Gamma} := \{\gamma \in C([0,1], H) : \gamma(0) = 0, \gamma(1) = w_1\}.$$

Lemmas 2.5-2.6 show that all assumptions of the mountain pass theorem in [1] are satisfied, $T_\lambda(w_1) < 0$ and $\|w_1\| > \rho$. Then, $\overline{c}$ is a critical value of T_λ, i.e., there exists $w_2 \in H$ such that $T_\lambda'(w_2)(\varphi) = 0$ for all $\varphi \in H$ or w_2 is a weak solution of (1.1). Moreover, w_2 is a not trivial solution and $w_2 \not\equiv w_1$ since $T_\lambda(w_2) = \overline{c} > 0 > T_\lambda(w_1)$. Theorem 1.3 is completely proved. □

3. Final comments

In this section, we make some comments regarding extensions of system (1.1). While uniform elliptic problems (equations and systems) are intensively studied in the last decades, the degenerate elliptic problems still contain some unknown things, especially problems involving Grushin-type operators.

Firstly, the operator $G_s = \Delta_x + |x|^{2s}\Delta_y$ can be naturally extended to the more complicated form

$$\Delta_{x_0} + \sum_{i=1}^m |x_0|^{2s_i}\Delta_{x_i}$$

in the domain $\Omega = \{(x_0, x_1, \ldots, x_m) : \ x_i \in \mathbb{R}^{N_i}, i = 0, 1, \ldots, m\}$. Then system (1.1) becomes

(3.19)

$$\begin{cases}
\Delta_{x_0}u + \sum_{i=1}^m |x_0|^{2s_i}\Delta_{x_i}u = \lambda F_u(x_0, x_1, \ldots, x_m, u, v) & \text{in } \Omega,\\
\Delta_{x_0}v + \sum_{i=1}^m |x_0|^{2s_i}\Delta_{x_i}v = \lambda F_v(x_0, x_1, \ldots, x_m, u, v) & \text{in } \Omega,\\
u = v = 0 & \text{on } \partial\Omega,
\end{cases}$$

where $\Omega \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, and λ is a positive parameter. Following [5, 22], the critical exponent for this case is

$$\frac{N_0 + \sum_{i=1}^m (s_i+1)N_i + 2}{N_0 + \sum_{i=1}^m (s_i+1)N_i - 2}.$$

In [5], the authors studied the existence of a nontrivial solution for (3.19) in the superlinear case. More precisely, the nonlinear term was assumed to satisfy the Ambrosetti-Rabinowitz type condition in [1], see the condition (H2) of [5].

In this work, we do not use this condition. Putting $x = (x_0, x_1, \ldots, x_m)$, we can preserve the hypotheses (**f1**)-(**f4**) and proceed with the functional energy

$$\begin{aligned} T_\lambda(w) &= \frac{1}{2}\int_\Omega \Big(|\nabla_{x_0} u|^2 + \sum_{i=1}^m |x_0|^{2s_i} |\nabla_{x_i} u|^2 \Big)\, dx \\ &\quad + \frac{1}{2}\int_\Omega \Big(|\nabla_{x_0} v|^2 + \sum_{i=1}^m |x_0|^{2s_i} |\nabla_{x_i} v|^2 \Big)\, dx - \lambda \int_\Omega F(x,u,v)\, dx. \end{aligned}$$

Secondly, it should be noticed that using the same argument used above, we can deal with the system of m unknowns

$$\left\{ \begin{array}{rcll} L_{\alpha_1,\alpha_2,\ldots,\alpha_m} w &=& \lambda \nabla F & \text{in } \Omega, \\ w &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{3.20}$$

where $w = (u_1, u_2, \ldots, u_m)$

$$L_{\alpha_1,\alpha_2,\ldots,\alpha_m} = \begin{pmatrix} -G_{\alpha_1} & 0 & & 0 \\ 0 & -G_{\alpha_2} & & 0 \\ & & \cdots & \\ 0 & 0 & & -G_{\alpha_m} \end{pmatrix}, \quad G_s = \Delta_x + |x|^{2s}\Delta_y \text{ for } s \geq 0,$$

and $\nabla F = (F_{u_1}, F_{u_2}, \ldots, F_{u_m})$ stands for the gradient of a C^1 function $F : \Omega \times \mathbb{R}^m \to \mathbb{R}$ which may change sign, $\Delta_x = \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2}$, and $\Delta_y = \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2}$. More precisely, we assume that $F : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a C^1-function, satisfying the following conditions:

(**f1**)$'$ $F(x, y, 0, 0, \ldots, 0) = 0$ for a.e. $(x, y) \in \Omega$, and $F(x, y, t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_m) = F(x, y, t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_m)$ for all $t_i \leq 0$, $t_j \in \mathbb{R}$ for all $j \neq i$ and a.e. $(x, y) \in \Omega$, $i, j = 1, 2, \ldots, m$;

(**f2**)$'$ There exists a constant $C > 0$ such that

$$\sum_{i=1}^m |F_{t_i}(x, y, t_1, t_2, \ldots, t_m)| \leq C\Big(1 + \sum_{i=1}^m |t_i|\Big)$$

for all $t = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m$ and a.e. $(x, y) \in \Omega$.

We say that a function γ verifies the property $(\Gamma)'$ if and only if

$$\gamma(t_1, t_2, \ldots, t_m) \leq M \sum_{i=1}^m |t_i|^2 \tag{Γ'}$$

for all $t = (t_1, t_2, \dots, t_m) \in \mathbb{R}^m$, where $M > 0$ is independent of γ.

Let H_i, $i = 1, 2$ be two functions satisfying property Γ' and denote

$$|t|_{\mathbb{R}^m} = \Big(\sum_{i=1}^{m} |t_i|^2\Big)^{\frac{1}{2}}, \quad t = (t_1, t_2, \dots, t_m) \in \mathbb{R}^m.$$

We introduce the following assumptions on the behavior of F at origin and at infinity:

(f3)$'$ It holds that

$$\limsup_{|t|_{\mathbb{R}^m} \to 0} \frac{F(x, y, t_1, t_2, \dots, t_m)}{H_1(t_1, t_2, \dots, t_m)} \leq 0$$

uniformly in $(x, y) \in \Omega$;

(f4)$'$ It holds that

$$\limsup_{|t|_{\mathbb{R}^m} \to \infty} \frac{F(x, y, t_1, t_2, \dots, t_m)}{H_2(t_1, t_2, \dots, t_m)} \leq 0$$

uniformly in $(x, y) \in \Omega$.

Then, using the similar arguments as in the proofs of Theorems 1.2 and 1.3, we can obtain the following results.

Theorem 3.1. *Assume $F : \Omega \times \mathbb{R}^m \to \mathbb{R}$, satisfies*

$$\sum_{i=1}^{m} |F_{t_i}(x, y, t_1, t_2, \dots, t_m)| \leq C \sum_{i=1}^{m} |t_i|, \quad C > 0 \tag{3.21}$$

for all $t = (t_1, t_2, \dots, t_m) \in \mathbb{R}^m$ and a.e. $(x, y) \in \Omega$. Then there is a constant $\lambda_* > 0$ such that system (3.20) *has no nontrivial weak solution for any $\lambda < \lambda_*$.*

Theorem 3.2. *Assume that the conditions* **(f1)**$'$-**(f4)**$'$ *are satisfied. Moreover, if in addition we assume that there exist a ball $B \subset \Omega$ and $t_i^0 > 0$, $i = 1, 2, \dots, m$ such that $F(x, y, t_1^0, t_2^0, \dots, t_m^0) > 0$ for all $(x, y) \in B$, then there exists a constant $\lambda^* > 0$ such that system* (3.20) *has at least two non-trivial, non-negative weak solutions for any $\lambda \geq \lambda^*$.*

It is clear that our arguments in this paper are applicable to elliptic problems involving the Caffarelli-Kohn-Nirenberg inequalities in order to obtain better results in [7, 16, 18]. Finally, the study of existence of solutions for problem (1.1) with critical exponents is an interesting topic, see [21, 25].

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