Commun. Korean Math. Soc. **29** (2014), No. 1, pp. 27–36
http://dx.doi.org/10.4134/CKMS.2014.29.1.027

ON f-DERIVATIONS FROM SEMILATTICES TO LATTICES

Yong Ho Yon and Kyung Ho Kim

Abstract. In this paper, we introduce the notion of f-derivations from a semilattice S to a lattice L, as a generalization of derivation and f-derivation of lattices. Also, we define the simple f-derivation from S to L, and research the properties of them and the conditions for a lattice L to be distributive. Finally, we prove that a distributive lattice L is isomorphic to the class $SD_f(S,L)$ of all simple f-derivations on S to L for every $\wedge$-homomorphism $f : S \to L$ such that $f(x_0) \vee f(y_0) = 1$ for some $x_0, y_0 \in S$, in particular, $L \cong SD_f(S,L)$ for every $\wedge$-homomorphism $f : S \to L$ such that $f(x_0) = 1$ for some $x_0 \in S$.

1. Introduction

In some of the literature, authors investigated the relationship between the notion of modularity or distributivity and the special operators on lattices such as derivations, multipliers and linear maps.

The notion and some properties of derivations on lattices were introduced in [10, 11]. Szász ([10, 11]) characterized the distributive lattices by multipliers and derivations: a lattice is distributive if and only if the set of all meet-multipliers and of all derivations coincide. In [5] it was shown that every derivation on a lattice is a multiplier and every multiplier is a dual closure. Pataki and Száz ([9]) gave a connection between non-expansive multipliers and quasi-interior operators. Recently, in [12], the linear map on lattices was introduced and modular lattices were characterized by multipliers and linear map. The f-derivation, symmetric bi-derivation, symmetric f bi-derivation and permuting tri-derivation on lattices were introduced and some results about them were proved ([3, 4, 6, 7, 8]).

In this paper, we define an f-derivation from a semilattice S to a lattice L, as a generalization of derivation and f-derivation of lattices, and study some properties of f-derivations from S to L. In Section 4, we define the simple f-derivation from S to L, and research the properties of simple f-derivations and the conditions for a lattice L to be distributive. Also we prove that a

Received March 3, 2013; Revised May 22, 2013.
2010 *Mathematics Subject Classification.* 06A06, 06A11, 06A15.
Key words and phrases. semilattices, lattices, derivation, simple derivation.

distributive lattice L is isomorphic to the class $SD_f(S,L)$ of all simple f-derivations on S to L for every $\wedge$-homomorphism $f : S \to L$ such that $f(x_0) \vee f(y_0) = 1$ for some $x_0, y_0 \in S$, in particular, $L \cong SD_f(S,L)$ for every $\wedge$-homomorphism $f : S \to L$ such that $f(x_0) = 1$ for some $x_0 \in S$.

2. Preliminaries

A *semilattice* is a partially ordered set (shortly, poset) S in which there exists the greatest lower bound $x \wedge y$ for every $x, y \in L$, a $\vee$-*semilattice* is a poset in which there exists the least upper bound $x \vee y$ for every $x, y \in L$, and a *lattice* is a poset L which is semilattice and $\vee$-semilattice.

A lattice L is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every $x, y, z \in L$.

A map f from a semilattice S to a semilattice T is said to be *monotone* (resp. *antitone*) if it satisfies:

$$x \leq y \implies f(x) \leq f(y) \quad (\text{resp. } x \leq y \implies f(x) \geq f(y)),$$

and is called a $\wedge$-*homomorphism* if it satisfies: $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in S$. A map f from a lattice L to a lattice M is called a $\vee$-*homomorphism* if it satisfies: $f(x \vee y) = f(x) \vee f(y)$ for every $x, y \in L$, and is called a *homomorphism* of lattices if f is a $\wedge$-homomorphism and $\vee$-homomorphism. It is well known that every $\wedge$-homomorphism (or $\vee$-homomorphism) is monotone, but the converse is not true.

Further discussions and symbols for lattice theory can be found in [1, 2].

3. *f*-derivations on semilattices

Throughout this paper, S denotes a semilattice and L a lattice unless otherwise specified.

Let $F(S,L)$ be the class of all maps from S to L. If we define a binary relation $\leq$ on $F(S,L)$ by

$$f \leq g \iff f(x) \leq g(x) \text{ for every } x \in S,$$

then $(F(S,L), \leq)$ is a poset. Moreover, if we define maps $f \vee g, f \wedge g : S \to L$ by

$$(f \vee g)(x) = f(x) \vee g(x) \text{ and } (f \wedge g)(x) = f(x) \wedge g(x),$$

respectively, for every $f, g \in F(S,L)$, then $f \vee g$ is the least upper bound and $f \wedge g$ is the greatest lower bound of f and g. Hence $(F(S,L), \vee, \wedge)$ is a lattice.

Definition 3.1. A map $d : S \to L$ is called an f-*derivation* if there exists a map $f : S \to L$ such that

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y))$$

for every $x, y \in S$.

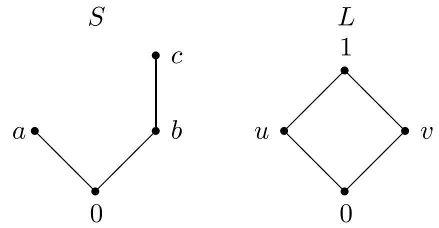

FIGURE 1

Example 1. Let $S = \{0, a, b, c\}$ be a semilattice and $L = \{0, u, v, 1\}$ a lattice with Hasse diagrams of Figure 1. If we define maps f and d from S to L by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ u, & \text{if } x = a, \\ v, & \text{if } x = b, c \end{cases} \quad \text{and} \quad d(x) = \begin{cases} 0, & \text{if } x = 0, c \\ u, & \text{if } x = a, \\ v, & \text{if } x = b, \end{cases}$$

respectively, then d is an f-derivation from S to L.

Theorem 3.2. *If $f : S \to L$ is a $\wedge$-homomorphism, then f is an f-derivation.*

Proof. Let $f : S \to L$ be a $\wedge$-homomorphism. Then for any $x, y \in S$,

$$f(x \wedge y) = f(x) \wedge f(y) = (f(x) \wedge f(y)) \vee (f(x) \wedge f(y)).$$

Hence f is an f-derivation. □

The converse of Theorem 3.2 is not true in general. For example, the f-derivation d of Example 1 is not homomorphism because $d(b \wedge c) = d(b) = v \neq 0 = v \wedge 0 = d(b) \wedge d(c)$.

Lemma 3.3. *Let $f : S \to L$ be a map and $d : S \to L$ an f-derivation. Then the following properties hold.*

(1) *$d(x) \leq f(x)$ for all $x \in S$.*
(2) *$d(x) \wedge d(y) \leq d(x \wedge y) \leq d(x) \vee d(y)$ for all $x, y \in S$.*
(3) *$d(x \wedge y) \leq f(x) \wedge f(y)$ for all $x, y \in S$.*
(4) *If $f(x) = 1$ for every $x \in S$, then every f-derivation is antitone.*
(5) *If f is monotone and there is an element $x_0 \in S$ such that $f(x_0) = d(x_0)$, then $f(x) = d(x)$ for all $x \in S$ with $x \leq x_0$.*

Proof. (1) Let $x \in S$. Then $d(x) = d(x \wedge x) = (d(x) \wedge f(x)) \vee (f(x) \wedge d(x)) = f(x) \wedge d(x)$. Hence $d(x) \leq f(x)$ for all $x \in S$.

(2) Let $x, y \in S$. Then $d(x) \leq f(x)$ and $d(y) \leq f(y)$ by (1) of this lemma. Hence we have

$$\begin{aligned} d(x) \wedge d(y) &= (d(x) \wedge d(y)) \vee (d(x) \wedge d(y)) \\ &\leq (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) = d(x \wedge y). \end{aligned}$$

Also since $d(x) \wedge f(y) \leq d(x)$ and $f(x) \wedge d(y) \leq d(y)$, we have $d(x \wedge y) \leq d(x) \vee d(y)$.

(3) Let $x, y \in S$. Then $d(x) \leq f(x)$ and $d(y) \leq f(y)$ by (1) of this lemma. Hence we have

$$\begin{aligned} d(x \wedge y) &= (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) \\ &\leq (f(x) \wedge f(y)) \vee (f(x) \wedge f(y)) = f(x) \wedge f(y). \end{aligned}$$

(4) Let $f(x) = 1$ for every $x \in S$ and $x \leq y$. Then $d(x) = d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) = d(x) \vee d(y)$. Hence $d(x) \geq d(y)$.

(5) Let f be monotone and $f(x_0) = d(x_0)$ for some $x_0 \in S$, and let $x \leq x_0$. Then $f(x) \leq f(x_0)$, and $d(x) \wedge d(x_0) \leq f(x) \wedge f(x_0)$ by (1) of this lemma. These imply

$$\begin{aligned} d(x) = d(x \wedge x_0) &= (d(x) \wedge f(x_0)) \vee (f(x) \wedge d(x_0)) \\ &= (d(x) \wedge d(x_0)) \vee (f(x) \wedge f(x_0)) = f(x) \wedge f(x_0) = f(x). \end{aligned}$$

□

Lemma 3.4. *Let $f : S \to L$ be a map and $d : S \to L$ an f-derivation. Then the following are equivalent*:

(1) *d is monotone,*
(2) *d is a $\wedge$-homomorphism.*

Proof. Let $d : S \to L$ be a monotone f-derivation. Then $d(x \wedge y) \leq d(x)$ and $d(x \wedge y) \leq d(y)$ for every $x, y \in S$. This implies $d(x \wedge y) \leq d(x) \wedge d(y)$. Hence $d(x \wedge y) = d(x) \wedge d(y)$ by Lemma 3.3(2).

The converse of this lemma is clear from the properties of $\wedge$-homomorphism. □

Theorem 3.5. *Let $f, d : S \to L$ be maps. Then the following are equivalent*:

(1) *d is a monotone f-derivation,*
(2) *$d(x \wedge y) = f(x) \wedge d(y)$ for every $x, y \in S$.*

Proof. Suppose that $d : S \to L$ is a monotone f-derivation and $x, y \in S$. Then

$$f(x) \wedge d(y) \leq (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)) = d(x \wedge y).$$

Also, since d is a $\wedge$-homomorphism by Lemma 3.4 and $d(x) \leq f(x)$, we have

$$d(x \wedge y) = d(x) \wedge d(y) \leq f(x) \wedge d(y).$$

Hence $d(x \wedge y) = f(x) \wedge d(y)$.

Conversely, suppose that $d(x \wedge y) = f(x) \wedge d(y)$ for every $x, y \in S$. Then $d(x \wedge y) = d(y \wedge x) = f(y) \wedge d(x) = d(x) \wedge f(y)$. This implies

$$d(x \wedge y) = d(x \wedge y) \vee d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)).$$

Hence d is an f-derivation. Also, if $x \leq y$, then $d(x) = d(x \wedge y) = f(x) \wedge d(y) \leq d(y)$. So d is monotone. □

For any f-derivations d_1 and d_2, $d_1 \wedge d_2$ is not f-derivation in general, as the following example shows.

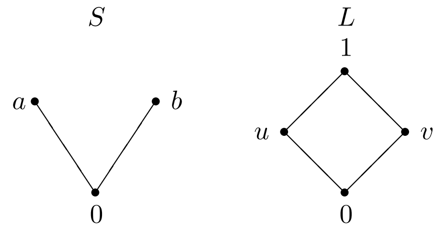

FIGURE 2

Example 2. Let $S = \{0, a, b\}$ be a semilattice and $L = \{0, u, v, 1\}$ a lattice with Hasse diagrams given by Figure 2. Let $f : S \to L$ be a map given by $f(x) = 1$ for all $x \in S$. If we define maps $d_1, d_2 : S \to L$ by

$$d_1(x) = \begin{cases} 0, & \text{if } x = a \\ u, & \text{if } x = 0, b \end{cases} \quad \text{and} \quad d_2(x) = \begin{cases} 0, & \text{if } x = b \\ u, & \text{if } x = 0, a, \end{cases}$$

respectively, then d_1 and d_2 are f-derivations, but $d_1 \wedge d_2$ is not an f-derivation, because $(d_1 \wedge d_2)(a \wedge b) = u \neq 0 = ((d_1 \wedge d_2)(a) \wedge f(b)) \vee (f(a) \wedge (d_1 \wedge d_2)(b))$.

Theorem 3.6. *Let $f : S \to L$ be a map. Then the class $MD_f(S, L)$ of all monotone f-derivations from S to L is a subsemilattice of $F(S, L)$.*

Proof. Let d_1 and d_2 be monotone f-derivations from S to L. Then by Theorem 3.5, we have

$$\begin{aligned}(d_1 \wedge d_2)(x \wedge y) = d_1(x \wedge y) \wedge d_2(x \wedge y) &= (f(x) \wedge d_1(y)) \wedge (f(x) \wedge d_2(y)) \\ &= f(x) \wedge (d_1(y) \wedge d_2(y)) = f(x) \wedge (d_1 \wedge d_2)(y)\end{aligned}$$

for every $x, y \in S$. This implies that $d_1 \wedge d_2$ is a monotone f-derivation by Theorem 3.5. Hence $MD_f(S, L)$ is a subsemilattice of $F(S, L)$. □

The subsemilattice $MD_f(S, L)$ of $F(S, L)$ is not a $\vee$-subsemilattice in general, as the following example show.

Example 3. Let $S = \{0, a, b\}$ be a semilattice and $L = \{0, u, v, w, 1\}$ a lattice with Hasse diagrams given by Figure 3. If we define maps $f, d_1, d_2 : S \to L$ by

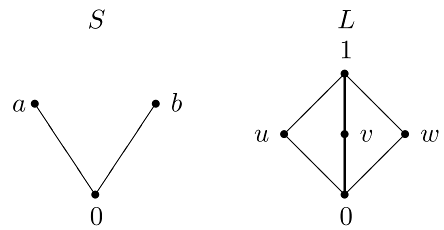

FIGURE 3

$$f(x) = \begin{cases} w, & \text{if } x = 0, a \\ 1, & \text{if } x = b, \end{cases} \quad d_1(x) = \begin{cases} 0, & \text{if } x = 0, a \\ u, & \text{if } x = b, \end{cases} \quad d_2(x) = \begin{cases} 0, & \text{if } x = 0, a \\ v, & \text{if } x = b, \end{cases}$$

respectively, then d_1 and d_2 are monotone f-derivations, but $d_1 \vee d_2$ is not f-derivation, because $(d_1 \vee d_2)(a \wedge b) = 0 \neq w = ((d_1 \vee d_2)(a) \wedge f(b)) \vee (f(a) \wedge (d_1 \vee d_2)(b))$.

Theorem 3.7. *Let $f : S \to L$ be a map. If L is distributive, then $MD_f(S, L)$ is a sublattice of $F(S, L)$.*

Proof. Suppose that L is distributive and $d_1, d_2 \in MD_f(S, L)$. Then for any $x, y \in S$, we have

$$\begin{aligned}(d_1 \vee d_2)(x \wedge y) &= d_1(x \wedge y) \vee d_2(x \wedge y)\\ &= (f(x) \wedge d_1(y)) \vee (f(x) \wedge d_2(y)) \quad \text{(by Theorem 3.5)}\\ &= f(x) \wedge (d_1(y) \vee d_2(y)) \quad \text{(by distributivity of } L)\\ &= f(x) \wedge (d_1 \vee d_2)(y).\end{aligned}$$

This implies that $d_1 \vee d_2$ is a monotone f-derivation by Theorem 3.5. Hence $MD_f(S, L)$ is $\vee$-subsemilattice of $F(S, L)$, and $MD_f(S, L)$ is a sublattice of $F(S, L)$ by Theorem 3.6. □

4. Simple f-derivation

Lemma 4.1. *Let $f : S \to L$ be a $\wedge$-homomorphism and $u \in L$. If we define a map $f_u : S \to L$ by*

$$f_u(x) = f(x) \wedge u$$

for each $x \in S$, then f_u is an f-derivation from S to L.

Proof. Suppose that $f : S \to L$ is a $\wedge$-homomorphism and $x, y \in S$. Then

$$\begin{aligned}f_u(x \wedge y) &= f(x \wedge y) \wedge u = (f(x) \wedge f(y)) \wedge u\\ &= ((f(x) \wedge f(y)) \wedge u) \vee ((f(x) \wedge f(y)) \wedge u)\\ &= ((f(x) \wedge u) \wedge f(y)) \vee (f(x) \wedge (f(y) \wedge u))\\ &= (f_u(x) \wedge f(y)) \vee (f(x) \wedge f_u(y)).\end{aligned}$$

Hence f_u is an f-derivation. □

For each $u \in L$, the f-derivation f_u in Lemma 4.1 is called a *simple f-derivation* from S to L.

Proposition 4.2. *Let $f : S \to L$ be a $\wedge$-homomorphism. Then the following properties hold.*

(1) *The $\wedge$-homomorphism f is the greatest element in $MD_f(S, L)$.*
(2) *Every simple f-derivation is monotone.*
(3) *If S has the greatest element 1, then every monotone f-derivation is a simple f-derivation.*

Proof. (1) Let $f : S \to L$ be a $\wedge$-homomorphism. Then f is monotone, and it is an f-derivation by Theorem 3.2. Also, $d \leq f$ for every $d \in MD_f(S, L)$ by Lemma 3.3(1). Hence f is the greatest element in $MD_f(S, L)$.

(2) Let $x \leq y$. Since f is a $\wedge$-homomorphism, f is monotone. This implies $f(x) \leq f(y)$, and

$$f_u(x) = f(x) \wedge u \leq f(y) \wedge u = f_u(y).$$

Hence f_u is monotone.

(3) Suppose that S has the greatest element 1. If d is a monotone f-derivation, then by Theorem 3.5, $d(x) = d(x \wedge 1) = f(x) \wedge d(1) = f_{d(1)}(x)$ for every $x \in S$. Hence d is a simple f-derivation $f_{d(1)}$. □

Let $SD_f(S, L)$ be the class of all simple f-derivations with respect to a $\wedge$-homomorphism $f : S \to L$. Then $SD_f(S, L) \subseteq MD_f(S, L) \subseteq F(S, L)$. In particular, if S has the greatest element 1, then $SD_f(S, L) = MD_f(S, L)$ by Proposition 4.2.

Theorem 4.3. *Let $f : S \to L$ be a $\wedge$-homomorphism. Then $SD_f(S, L)$ is a subsemilattice of $F(S, L)$ with $f_u \wedge f_v = f_{u \wedge v}$.*

Proof. Let $f_u, f_v \in SD_f(S, L)$. Then we have

$$\begin{aligned}(f_u \wedge f_v)(x) &= f_u(x) \wedge f_v(x) = (f(x) \wedge u) \wedge (f(x) \wedge v)\\ &= f(x) \wedge (u \wedge v) = f_{u \wedge v}(x).\end{aligned}$$

This implies $f_u \wedge f_v = f_{u \wedge v} \in SD_f(S, L)$. Hence $SD_f(S, L)$ is a subsemilattice of $F(S, L)$. □

The subsemilattice $SD_f(S, L)$ of $F(S, L)$ is not $\vee$-subsemilattice in general. In Example 3, two derivation d_1 and d_2 are simple f-derivations with $d_1 = f_u$ and $d_2 = f_v$, respectively, but $d_1 \vee d_2$ is not f-derivation.

Theorem 4.4. *Let $f : S \to L$ be a $\wedge$-homomorphism. If L is distributive, then $SD_f(S, L)$ is a sublattice of $F(S, L)$ with $f_u \vee f_v = f_{u \vee v}$ and $f_u \wedge f_v = f_{u \wedge v}$ for every $u, v \in L$.*

Proof. Suppose that L be distributive and $f_u, f_v \in SD_f(S, L)$. Then for any $x \in S$, we have

$$\begin{aligned}(f_u \vee f_v)(x) &= f_u(x) \vee f_v(x) = (f(x) \wedge u) \vee (f(x) \wedge v)\\ &= f(x) \wedge (u \vee v) = f_{u \vee v}(x).\end{aligned}$$

This implies $f_u \vee f_v = f_{u \vee v} \in SD_f(S, L)$. Hence $SD_f(S, L)$ is a $\vee$-subsemilattice and it is sublattice of $F(S, L)$ by Theorem 4.3 □

Let $f : S \to L$ be a $\wedge$-homomorphism. We define a map $\phi : L \to F(S, L)$ by

$$\phi(u) = f_u$$

for each $u \in L$. Also, we define a subset $D(L)$ of a lattice L as following.

$$D(L) = \{u \in L \mid u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w) \text{ for all } v, w \in L\}.$$

If $D(L)$ is a sublattice of L, then $D(L)$ is distributive. In particular, L is distributive if and only if $D(L) = L$.

Lemma 4.5. *If $f : S \to L$ be a $\wedge$-homomorphism, then the map $\phi : L \to F(S, L)$ is a $\wedge$-homomorphism.*

Proof. Let $u, v \in L$. Then for every $x \in S$,

$$\begin{aligned} f_{u\wedge v}(x) &= f(x) \wedge (u \wedge v) = (f(x) \wedge u) \wedge (f(x) \wedge v) \\ &= f_u(x) \wedge f_v(x) = (f_u \wedge f_v)(x). \end{aligned}$$

Hence $\phi(u\wedge v) = f_{u\wedge v} = f_u \wedge f_v = \phi(u)\wedge\phi(v)$, and ϕ is a $\wedge$-homomorphism. □

Theorem 4.6. *Let $f : S \to L$ be a $\wedge$-homomorphism. Then the following are equivalent*:

(1) *$\phi : L \to F(S, L)$ is a $\vee$-homomorphism.*
(2) *$SD_f(S, L)$ is a $\vee$-subsemilattice of $F(S, L)$ with $f_u \vee f_v = f_{u\vee v}$ for every $u, v \in L$.*
(3) $\text{Im}f \subseteq D(L)$.

Proof. (1)⇒(2) Suppose that $\phi : L \to F(S, L)$ is a $\vee$-homomorphism. Then it is clear that $SD_f(S, L) = \text{Im}\phi$ is a $\vee$-subsemilattice of $F(S, L)$, and for every $u, v \in L$,

$$f_u \vee f_v = \phi(u) \vee \phi(v) = \phi(u \vee v) = f_{u\vee v}.$$

(2)⇒(3) Suppose that $SD_f(S, L)$ is a $\vee$-subsemilattice of $F(S, L)$ with $f_u \vee f_v = f_{u\vee v}$ for every $u, v \in L$. If $f(x) \in \text{Im}f$, then

$$\begin{aligned} f(x) \wedge (v \vee w) &= f_{v\vee w}(x) = (f_v \vee f_w)(x) \\ &= f_v(x) \vee f_w(x) = (f(x) \wedge v) \vee (f(x) \wedge w) \end{aligned}$$

for every $v, w \in L$. This implies $f(x) \in D(L)$. Hence $\text{Im}f \subseteq D(L)$.

(3)⇒(1) Suppose that $\text{Im}f \subseteq D(L)$ and $u, v \in L$. Since $f(x) \in D(L)$ for every $x \in S$, we have

$$\begin{aligned} f_{u\vee v}(x) &= f(x) \wedge (u \vee v) = (f(x) \wedge u) \vee (f(x) \wedge v) \\ &= f_u(x) \vee f_v(x) = (f_u \vee f_v)(x) \end{aligned}$$

for every $x \in S$. Hence $\phi(u \vee v) = f_{u\vee v} = f_u \vee f_v = \phi(u) \vee \phi(v)$, and ϕ is a $\vee$-homomorphism. □

Corollary 4.7. *Let $f : S \to L$ be a surjective $\wedge$-homomorphism. Then the following are equivalent*:

(1) *$\phi : L \to F(S, L)$ is a $\vee$-homomorphism.*
(2) *$SD_f(S, L)$ is a $\vee$-subsemilattice of $F(S, L)$ with $f_u \vee f_v = f_{u\vee v}$ for every $u, v \in L$.*
(3) *L is distributive.*

Proof. Since $f : S \to L$ is surjective, $L = \text{Im}f \subseteq D(L)$ in Theorem 4.6(3), i.e., $L = D(L)$, and L is distributive. □

In Theorem 4.6 and Corollary 4.7, the map $\phi : L \to F(S, L)$ is a homomorphism of lattices and $SD_f(S, L)$ is a sublattice of $F(S, L)$ by Lemma 4.5 and Theorem 4.3.

Theorem 4.8. *Let L be a lattice with the greatest element 1 and $f : S \to L$ a $\wedge$-homomorphism. If $\mathrm{Im} f \subseteq D(L)$ and $f(x_0) = 1$ for some $x_0 \in S$, then $SD_f(S, L)$ is isomorphic to L.*

Proof. Suppose that $\mathrm{Im} f \subseteq D(L)$ and $f(x_0) = 1$ for some $x_0 \in S$. Then $\phi : L \to F(S, L)$ is a homomorphism of lattices by Lemma 4.5 and Theorem 4.6. If $\phi(u) = f_u = f_v = \phi(v)$, then

$$u = 1 \wedge u = f(x_0) \wedge u = f_u(x_0) = f_v(x_0) = f(x_0) \wedge v = 1 \wedge v = v.$$

Hence ϕ is one-to-one, and L is isomorphic to $\mathrm{Im}\phi = SD_f(S, L)$. □

Theorem 4.9. *Let L be a distributive lattice with the greatest element 1 and $f : S \to L$ a $\wedge$-homomorphism. If $f(x_0) \vee f(y_0) = 1$ for some $x_0, y_0 \in S$, then $SD_f(S, L)$ is isomorphic to L.*

Proof. Suppose that L is distributive and $f(x_0) \vee f(y_0) = 1$ for some $x_0, y_0 \in S$. Then $\phi : L \to F(S, L)$ is a homomorphism of lattices by Lemma 4.5 and Corollary 4.7. If $\phi(u) = f_u = f_v = \phi(v)$, then we have

$$\begin{aligned} u &= 1 \wedge u = (f(x_0) \vee f(y_0)) \wedge u = (f(x_0) \wedge u) \vee (f(y_0) \wedge u) \\ &= f_u(x_0) \vee f_u(y_0) = f_v(x_0) \vee f_v(y_0) = (f(x_0) \wedge v) \vee (f(y_0) \wedge v) \\ &= (f(x_0) \vee f(y_0)) \wedge v = 1 \wedge v = v. \end{aligned}$$

Hence ϕ is one-to-one, and L is isomorphic to $\mathrm{Im}\phi = SD_f(S, L)$. □

Corollary 4.10. *Let L be a distributive lattice with the greatest element 1 and $f : S \to L$ a $\wedge$-homomorphism. If $f(x_0) = 1$ for some $x_0 \in S$, then $SD_f(S, L)$ is isomorphic to L.*

Proof. If L is distributive, then $L = D(L)$. This implies $\mathrm{Im} f \subseteq L = D(L)$. Hence $SD_f(S, L)$ is isomorphic to L by Theorem 4.8. □

Example 4. In Example 3, there are three $\wedge$-homomorphisms from the semilattice $S = \{0, a, b\}$ to the lattice $L = \{0, u, v, w, 1\}$ such that $\mathrm{Im} f \subseteq D(L) = \{0, 1\}$ and $f(x_0) = 1$ for some $x_0 \in S$;

$$f^1(x) = \begin{cases} 0, & \text{if } x = 0, a \\ 1, & \text{if } x = b, \end{cases} \quad f^2(x) = \begin{cases} 0, & \text{if } x = 0, b \\ 1, & \text{if } x = a, \end{cases} \quad f^3(x) = 1 \text{ for every } x \in S.$$

By Theorem 4.8, $SD_{f^i}(S, L) \cong L$ for every $i = 1, 2, 3$. The $\wedge$-homomorphism $f : S \to L$ of Example 3 does not satisfy the property $\mathrm{Im} f \subseteq D(L)$, and $\phi : L \to SD_f(S, L)$ is not homomorphism of lattice, because $\phi(u \vee v) = \phi(1) = f_1 \neq f_u \vee f_v = \phi(u) \vee \phi(v)$.

In Example 1, the lattice $L = \{0, u, v, 1\}$ is distributive and $f(a) \vee f(b) = u \vee v = 1$. Hence $SD_f(S, L) \cong L$ by Theorem 4.9.

In Example 2, the lattice $L = \{0, u, v, 1\}$ is distributive and $f(x) = 1$ for every $x \in S$. Hence $SD_f(S, L) \cong L$ by Corollary 4.10. In this example, there are seven $\wedge$-homomorphisms from S to L such that $f(x_0) = 1$ for some $x_0 \in S$. For each f of them, $SD_f(S, L) \cong L$.

References

[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, United States, 1974.
[2] G. Birkhoff, *Lattice Theory*, American Mathematical Society, New York, 1940.
[3] Y. Çeven, *Symmetric bi-derivations of lattices*, Quaest. Math. **32** (2009), no. 2, 241–245.
[4] Y. Çeven and M. A. Öztürk, *On f-derivations of lattices*, Bull. Korean Math. Soc. **45** (2008), no. 4, 701–707.
[5] L. Ferrari, *On derivations of lattices*, Pure Math. Appl. **12** (2001), no. 4, 365–382.
[6] S. Harmaitree and U. Leerawat, *On f-derivations in lattices*, Far East J. Math. Sci. **51** (2011), no. 1, 27–40.
[7] S. A. Ozbal and A. Firat, *Symmetric f bi-derivations of lattices*, Ars Combin. **97** (2010), 471–477.
[8] M. A. Öztürk, H. Yazarlı, and K. H. Kim, *Permuting tri-derivations in lattices*, Quaest. Math. **32** (2009), no. 3, 415–425.
[9] G. Pataki and Á. Száz, *Characterizations of nonexpansive multipliers on partially ordered sets*, Math. Slovaca **51** (2001), no. 4, 371–382.
[10] Ǵ. Szász, *Translationen der Verbände*, Acta Fac. Rer. Nat. Univ. Comenianae **5** (1961), 449–453.
[11] ———, *Derivations of lattices*, Acta Sci. Math. (Szeged) **37** (1975), 149–154.
[12] Y. H. Yon and K. H. Kim, *On expansive linear map and $\vee$-multipliers of lattices*, Quaest. Math. **33** (2010), no. 4, 417–427.

Yong Ho Yon
Innovation Center for Engineering Education
Mokwon University
Daejeon 302-729, Korea
E-mail address: yhyon@mokwon.ac.kr

Kyung Ho Kim
Department of Mathematics
Korea National University of Transpotation
Chungju 380-702, Korea
E-mail address: ghkim@ut.ac.kr