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UNRAMIFIED SCALAR EXTENSIONS OF GRADED DIVISION ALGEBRAS

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Abstract. Let E be a graded central division algebra (GCDA) over a grade field R. Let S be an unramified graded field extension of R. We describe the grading on the underlying GCDA E' of $\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S}$ which is analogous to the valuation on a tame division algebra over Henselian valued field.

Let D be a division algebra with a valuation. To this one associates a graded division algebra $GD = \bigoplus_{\gamma \in \Gamma_D} GD_\gamma$, where Γ_D is the value group of D and the summands GD_γ arise from the filtration on D induced by the valuation (see [5] for details). As is illustrated in [5], even though computations in the graded setting are often easier than working directly with D, it seems that not much is lost in passage from D to its corresponding graded division algebra GD. This has provided motivation to systematically study this correspondence, notably by Boulagouaz [1], Hwang, Tignol and Wadsworth [4, 5, 8], and to compare certain functors defined on these objects, notably the Brauer group. Also, the graded method is effectively used to calculate the reduced Whitehead group SK_1 of a division algebra, first on the graded level and then specialise to the non-graded setting by Hazrat, Wadsworth, and Yanchevskiĭ [2, 3, 9].

Let Γ be a torsion-free abelian group. A ring E is a *graded division ring* (with grade group in Γ) if E has additive subgroups E_γ for $\gamma \in \Gamma$ such that $\mathsf{E} = \bigoplus_{\gamma \in \Gamma} \mathsf{E}_\gamma$ and $\mathsf{E}_\gamma \mathsf{E}_\delta \subseteq \mathsf{E}_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$, and each $\mathsf{E}_\gamma \setminus \{0\}$ lies in E^*, the group of units of E. For background on graded division rings and proofs of their properties mentioned here, see [5]. The grade group of E is

$$\Gamma_{\mathsf{E}} = \{\gamma \in \Gamma \mid \mathsf{E}_\gamma \neq \{0\}\},$$

a subgroup of Γ. For $a \in \mathsf{E}_\gamma \setminus \{0\}$ we write $\deg(a) = \gamma$. A significant property is that $\mathsf{E}^* = \bigcup_{\gamma \in \Gamma_{\mathsf{E}}} \mathsf{E}_\gamma \setminus \{0\}$, i.e., every unit of E is actually homogeneous (Γ_{E} torsion-free is needed for this). Thus, E is not a division ring if $|\Gamma_{\mathsf{E}}| > 1$. But, E has no zero divisors (This also depends on having Γ_{E} torsion-free). However,

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E_0 is a division ring, and each E_γ ($\gamma \in \Gamma_{\mathsf{E}}$) is a 1-dimensional left- and right-E_0-vector space.

Let $\mathsf{R} = Z(\mathsf{E})$, the center of E, which is a graded subring of E. Indeed, R is a *graded field*, i.e., a commutative graded division ring. Then E is a left (and right) graded R-vector space, and we write $[\mathsf{E} : \mathsf{R}]$ for $\dim_{\mathsf{R}}(\mathsf{E})$. E is a *graded central division algebra*(GCDA) over R if $[\mathsf{E} : \mathsf{R}] < \infty$. There is a well-defined group homomorphism

$$(1) \qquad \theta_{\mathsf{E}} : \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{R}} \to \mathcal{G}(Z(\mathsf{E}_0)/\mathsf{R}_0) \;\text{ given by }\; \deg(a) + \Gamma_{\mathsf{R}} \mapsto (z \mapsto aza^{-1})$$

for all $a \in \mathsf{E}^*$ and $z \in Z(\mathsf{E}_0)$, where $\mathcal{G}(Z(\mathsf{E}_0)/\mathsf{R}_0)$ denotes the Galois group of $Z(\mathsf{E}_0)$ over R_0. It is known that θ_{E} is surjective (cf. Prop. 2.3 of [5]).

A graded algebra A over a graded field R is said to be a *graded central simple algebra* (GCSA) over R if A is a simple graded ring, $[\mathsf{A} : \mathsf{R}] < \infty$, and $Z(\mathsf{A}) = \mathsf{R}$. There is a theory of GCSA's over a graded field analogous to the theory of central simple algebras (CSA's) over a field (see Sec. 1 of [5]).

Let E be any GCDA over the graded field R, and let S be an unramified (or inertial) graded field extension of R (i.e., $\Gamma_{\mathsf{S}} = \Gamma_{\mathsf{R}}$ and S_0 is separable over R_0). Then by Prop. 1.1 of [5], $\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S}$ is a GCSA over S and $\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S} = \mathsf{E} \otimes_{\mathsf{R}_0} \mathsf{S}_0$ as $\mathsf{S} = \mathsf{S}_0 \otimes_{\mathsf{R}_0} \mathsf{R}$. Let $G = \mathcal{G}(Z(\mathsf{E}_0)/\mathsf{R}_0)$ and $T = Z(\mathsf{E}_0) \otimes_{\mathsf{R}_0} \mathsf{S}_0$.

As $Z(\mathsf{E}_0)$ is Galois over R_0 by Prop. 2.3 of [5], $T = \bigoplus_{i=1}^k Te_i$ where e_i's are the primitive idempotents of T and G acts transitively on e_i's (cf. the proof of Prop. 18.18 in [7]). Let $e = e_1$ and $e_{\sigma H} = \sigma(e)$ where σH is the left coset of H in G and $H = \operatorname{stab}_G(e)$ which is the stabilizer group of e in G.

Theorem. *Let E, R, S, G, T and H as above. Let E' be the GCDA over S such that $[\mathsf{E}']_g = [\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S}]_g$ in the graded Bauer group $GBr(\mathsf{S})$ (see Sec. 3 of [5] for the definition of $GBr(\mathsf{S})$). Then,*

(a) $Z(\mathsf{E}'_0) \cong Z(\mathsf{E}_0) \cdot \mathsf{S}_0$;

(b) $\mathsf{E}'_0 \sim \mathsf{E}_0 \otimes_{Z(\mathsf{E}_0)} (Z(\mathsf{E}_0) \cdot \mathsf{S}_0)$ *in* $Br(Z(\mathsf{E}_0) \cdot \mathsf{S}_0)$;

(c) $\Gamma_{\mathsf{E}'}/\Gamma_{\mathsf{R}} = \theta_{\mathsf{E}}^{-1}(\mathcal{G}(Z(\mathsf{E}_0)/(Z(\mathsf{E}_0) \cap \mathsf{S}_0))$*, where* θ_{E} *is the map of* (1)*. So* $\Gamma_{\mathsf{E}'} \subseteq \Gamma_{\mathsf{E}}$ *and* $|\Gamma_{\mathsf{E}} : \Gamma_{\mathsf{E}'}| = [(Z(\mathsf{E}_0) \cap \mathsf{S}_0) : \mathsf{R}_0]$;

(d) *the following diagram is commutative*:

$$\begin{array}{ccc}
\Gamma_{\mathsf{E}'}/\Gamma_{\mathsf{S}} & \xrightarrow{\;i\;} & \Gamma_{\mathsf{E}}/\Gamma_{\mathsf{R}} \\
\theta_{\mathsf{E}'}\big\downarrow & & \theta_{\mathsf{E}}\big\downarrow \\
\mathcal{G}(Z(\mathsf{E}'_0)/\mathsf{S}_0) & \xrightarrow{\;\rho_{\mathsf{S}_0/\mathsf{R}_0}\;} & \mathcal{G}(Z(\mathsf{E}_0)/\mathsf{R}_0)
\end{array}$$

where θ_{E} *is the map of* (1)*,* i *is the inclusion map, and* $\rho_{\mathsf{S}_0/\mathsf{R}_0}$ *is restriction of an automorphism from* $Z(\mathsf{E}'_0) \cong Z(\mathsf{E}_0) \cdot \mathsf{S}_0$ *to* $Z(\mathsf{E}_0)$.

Proof. Let e and $e_{\sigma H}$ be as above. Then $[e(\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S})e]_g = [\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S}]_g = [\mathsf{E}']_g$ in $GBr(\mathsf{S})$. Since

$$(e(\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S})e)_0 = \; e(\mathsf{E}_0 \otimes_{\mathsf{R}_0} \mathsf{S}_0)e = e(\mathsf{E}_0 \otimes_{Z(\mathsf{E}_0)} Z(\mathsf{E}_0) \otimes_{\mathsf{R}_0} \mathsf{S}_0)e$$

$$= \; e(\mathsf{E}_0 \otimes_{Z(\mathsf{E}_0)} (\bigoplus_{i=1}^{k} e_i T))e = \mathsf{E}_0 \otimes_{Z(\mathsf{E}_0)} eT$$
$$\cong \mathsf{E}_0 \otimes_{Z(\mathsf{E}_0)} (Z(\mathsf{E}_0) \cdot \mathsf{S}_0),$$

$(e(\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S})e)_0$ is simple. So, by Prop. 1.4(b) of [5], $\mathsf{E}'_0 \sim \mathsf{E}_0 \otimes_{Z(\mathsf{E}_0)} (Z(\mathsf{E}_0) \cdot \mathsf{S}_0)$ in $Br(Z(\mathsf{E}_0) \cdot \mathsf{S}_0)$, and $Z(\mathsf{E}'_0) \cong Z(\mathsf{E}_0) \cdot \mathsf{S}_0$, and $\Gamma_{\mathsf{E}'} = \Gamma_{e(\mathsf{E}\otimes_{\mathsf{R}}\mathsf{S})e} \subseteq \Gamma_{\mathsf{E}}$. Since $e(\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S})e = e(\mathsf{E} \otimes_{\mathsf{R}_0} \mathsf{S}_0)e = \bigoplus_{\gamma\in\Gamma_{\mathsf{E}}} e(\mathsf{E}_\gamma \otimes_{\mathsf{R}_0} \mathsf{S}_0)e$, and $H = \mathrm{stab}_G(e) = \rho_{\mathsf{S}_0/\mathsf{R}_0}(\mathcal{G}((Z(\mathsf{E}_0) \cdot \mathsf{S}_0)/\mathsf{S}_0)) = \mathcal{G}(Z(\mathsf{E}_0)/(Z(\mathsf{E}_0) \cap \mathsf{S}_0))$, it suffices to show that $e(\mathsf{E}_\gamma \otimes_{\mathsf{R}_0} \mathsf{S}_0)e \neq 0$ if and only if $\theta_{\mathsf{E}}(\gamma + \Gamma_{\mathsf{R}}) \in H$ (So, $\Gamma_{\mathsf{E}'}/\Gamma_{\mathsf{R}} = \Gamma_{e(\mathsf{E}\otimes_{\mathsf{R}}\mathsf{S})e}/\Gamma_{\mathsf{R}} = \theta_{\mathsf{E}}^{-1}(H)$). To show this, observe that $T = \bigoplus_{\sigma H} Te_{\sigma H}$, where σH are the left cosets of H in G and $e_{\sigma H} = \sigma(e)$, as G acts transitively on e_i's. Let $a \in \mathsf{E}_\gamma - (0)$, and let $\tau = \theta_{\mathsf{E}}(\gamma + \Gamma_{\mathsf{R}}) \in G = \mathcal{G}(Z(\mathsf{E}_0)/\mathsf{R}_0) = \mathcal{G}(T/\mathsf{S}_0)$. Then τ is the conjugation by $a \otimes 1$ on $T = Z(\mathsf{E}_0) \otimes_{\mathsf{R}_0} \mathsf{S}_0$. So, $(a\otimes 1)e(a\otimes 1)^{-1} = \tau(e) = e_{\tau H}$.

(1) When $\tau = \theta_{\mathsf{E}}(\gamma + \Gamma_{\mathsf{R}}) \in H$, i.e., $\gamma + \Gamma_{\mathsf{R}} \in \theta_{\mathsf{E}}^{-1}(H)$, $e(a \otimes 1)e(a \otimes 1)^{-1} = ee_{\tau H} = e$. So, $e(\mathsf{E}_\gamma \otimes_{\mathsf{R}_0} \mathsf{S}_0)e \neq 0$.

(2) When $\gamma + \Gamma_{\mathsf{R}} \notin \theta_{\mathsf{E}}^{-1}(H)$, i.e., $\tau \notin H$, Then for $b = \sum_{i=1}^{n} a_i \otimes s_i \in \mathsf{E}_\gamma \otimes_{\mathsf{R}_0} \mathsf{S}_0$, where $a_i \in \mathsf{E}_\gamma$ and $s_i \in \mathsf{S}_0$, $ebe = e(\sum_{i=1}^{n}(a_i \otimes 1)e(1 \otimes s_i)) = e(\sum_{i=1}^{n} e_{\tau H}(a_i \otimes s_i)) = ee_{\tau H}b = 0$. So, $e(\mathsf{E}_\gamma \otimes_{\mathsf{R}_0} \mathsf{S}_0)e = 0$.

Let $\mathsf{B} = e(\mathsf{E} \otimes_{\mathsf{R}} \mathsf{S})e$. Then, as B_0 is simple, by Cor. 2.3 of [8], $\theta_{\mathsf{E}'} = \theta_{\mathsf{B}}$ where θ_{B} is the map defined in (2.5) of [8]. So, the commutativity of the above diagram is followed. □

Now, we will use this theorem to prove the analogous theorem for tame division algebras over a Henselian valued field.

Corollary (Theorem 3.1 of [6])**.** *Let (F, v) be a Henselian valued field, and let D be any tame CDA (central division algebra) over F. Let L be any inertial extension field of F (i.e., $[\overline{L} : \overline{F}] = [L : F]$ and $\overline{L}$ is separable over $\overline{F}$). Let D_L be the CDA over L with $D_L \sim D \otimes_F L$ in $Br(L)$. Then,*

(a) $Z(\overline{D_L}) \cong Z(\overline{D}) \cdot \overline{L}$;

(b) $\overline{D_L} \cong (\overline{D})_{Z(\overline{D})\cdot\overline{L}}$;

(c) $\Gamma_{D_L}/\Gamma_F = \theta_D^{-1}(\mathcal{G}(Z(\overline{D})/(Z(\overline{D}) \cap \overline{L}))$, *where θ_D is a well-defined group homomorphism*

(2) $\quad \theta_D : \Gamma_D/\Gamma_F \to \mathcal{G}(Z(\overline{D})/\overline{F})$ *given by* $v(d) + \Gamma_F \mapsto (\overline{z} \mapsto \overline{dzd^{-1}})$

for all $d \in D^*$ and $\overline{z} \in Z(\overline{D})$. So $\Gamma_{D_L} \subseteq \Gamma_D$ and $|\Gamma_D : \Gamma_{D_L}| = [(Z(\overline{D})\cap\overline{L}) : \overline{F}]$;

(d) *the following diagram is commutative*:

$$\begin{array}{ccc} \Gamma_{D_L}/\Gamma_L & \xrightarrow{\ i\ } & \Gamma_D/\Gamma_F \\ {\scriptstyle\theta_{D_L}}\downarrow & & {\scriptstyle\theta_D}\downarrow \\ \mathcal{G}(Z(\overline{D_L})/\overline{L}) & \xrightarrow{\rho_{L/F}} & \mathcal{G}(Z(\overline{D})/\overline{F}) \end{array}$$

where θ_D is the map of (2), i is the inclusion map, and ρ_{L0/F_0} is restriction of an automorphism from $Z(\overline{D_L}) \cong Z(\overline{D}) \cdot \overline{L}$ to $Z(\overline{D})$.

Proof. Let GD and GF be the graded division algebra and the graded field derived from D and F, respectively, as defined in Sec. 4 of [5]. Then GD is a $GCDA$ over GF by Prop. 4.3 of [5], and GL is unramified over GF as $\Gamma_{GL} = \Gamma_L = \Gamma_F = \Gamma_{GF}$ and $(GL)_0(=\overline{L})$ is separable over $(GF)_0(=\overline{F})$ as L is inertial over F. Also, GD_L is the $GCDA$ over GL such that $[GD_L]_g = [GD \otimes_{GF} GL]_g$ in $GBr(GL)$ by Cor. 5.7 of [5]. So, by applying the above theorem with $E = GD$, $R = GF$ and $S = GL$,

(a) $Z(\overline{D_L}) = Z((GD_L)_0) \cong Z((GD)_0) \cdot (GL)_0 = Z(\overline{D}) \cdot \overline{L}$.

(b) $\overline{D_L} = (GD_L)_0 \sim (GD)_0 \otimes_{Z((GD)_0)} (Z((GD)_0) \cdot (GL)_0) = \overline{D} \otimes_{Z(\overline{D})} (Z(\overline{D}) \cdot \overline{L})$. So, $\overline{D_L} \cong (\overline{D})_{Z((\overline{D}) \cdot \overline{L}}$.

(c) Let θ_{GD} be the map of (1) with $E = GD$. Then, as $\theta_{GD} = \theta_D$, $\Gamma_{D_L}/\Gamma_F = \Gamma_{GD_L}/\Gamma_{GF} = \theta_{GD}^{-1}(\mathcal{G}(Z(G((GD)_0))/(Z(GD)_0 \cap (GL)_0)) = \theta_D^{-1}(\mathcal{G}(Z(\overline{D})/(Z(\overline{D}) \cap \overline{L}))$.

(d) It is clear that the commutative diagram of the theorem becomes the above diagram. □

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