Commun. Korean Math. Soc. **29** (2014), No. 1, pp. 9–22
http://dx.doi.org/10.4134/CKMS.2014.29.1.009

CROSSED MODULES AND STRICT GR-CATEGORIES

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Abstract. In this paper we state some applications of Gr-category theory to the classification problem of crossed modules and to that of group extensions of the type of a crossed module.

1. Introduction

Theory of Gr-categories, or of 2-groups, with its generality has more and more applications. The relationship between this theory and cohomology of groups is stated by Sinh in [13], and it is developed by Cegarra et al. in the theory of graded categorical groups [7] and in that of fibred categorical groups [6]. In another aspect, one can study more deeply the simple case of this theory as considering applications of strict Gr-categories.

A *Gr-category* [13] is a monoidal category in which every morphism is invertible and every object has a weak inverse (Here, a weak inverse of an object x is an object y such that $x \otimes y$ and $y \otimes x$ are both isomorphic to the unit object). A *strict* Gr-category is a strict monoidal category in which every morphism is invertible and every object has a strict inverse (so that $x \otimes y$ and $y \otimes x$ are actually equal to the unit object). This notion is also called *$\mathcal{G}$-groupoid* by Brown and Spencer [4], or *2-group* by Noohi [11], or *strict 2-group* by Baez and Lauda [1], or *strict categorical group* by Joyal and Street [9].

Brown and Spencer showed that the category of crossed modules ([4, Theorem 1]) is equivalent to the category of $\mathcal{G}$-groupoids (morphisms in the category of crossed modules are homomorphisms of crossed modules and those in the category of $\mathcal{G}$-groupoids are functors of groupoids preserving the group structure). The alternative definitions of strict Gr-categories were introduced in [1] by Baez and Lauda, and the equivalent structures to crossed modules were presented in [5] by Brown and Wensley. Many of results on crossed modules were extended for crossed complexes by Brown, Higgins and Sivera [2].

Our aim is to seek new applications of Gr-category theory concerning crossed modules. The above mentioned theorem of Brown and Spencer shows that

Received September 24, 2012; Revised June 12, 2013.
2010 *Mathematics Subject Classification.* 20J05, 20E22, 18D10.
Key words and phrases. crossed module, Gr-category, group extension, group cohomology, obstruction.

strict Gr-categories can be seen as a weakening of crossed modules. Thus, some facts of crossed modules can be found in the more generalized form of Gr-categories with the more natural techniques in proofs.

The present paper consists of two main results. Firstly, we prove that the category **Grstr** of strict Gr-categories and regular Gr-functors is equivalent to the category **Cross** of crossed modules in which a morphism consists of a homomorphism of crossed modules, $(f_1, f_0) : \mathcal{M} \to \mathcal{M}'$, and an element of the group of 2-cocycles $Z^2(\pi_0\mathcal{M}, \pi_1\mathcal{M}')$. This result contains [4, Theorem 1]. Secondly, we use the Gr-category theory to study the group extension problem of the type of a crossed module. For any crossed module $B \stackrel{d}{\to} D$ and any group homomorphism $\psi : Q \to \mathrm{Coker}d$, one can define a strict Gr-category $\mathbb{P}$. Then, for any Gr-functor $F : \mathrm{Dis}\, Q \to \mathbb{P}$ we construct an associated extension of type $B \to D$. Therefore, there is a Schreier bijection (Theorem 7)

$$\mathrm{Hom}_{(\psi,0)}[\mathrm{Dis}\, Q, \mathbb{P}] \leftrightarrow \mathrm{Ext}_{B\to D}(Q, B, \psi),$$

where $\mathrm{Ext}_{B\to D}(Q, B, \psi)$ is the set of equivalence classes of extensions of B by Q of type $B \to D$ inducing ψ. This result implies the classification theorem of Brown and Mucuk ([3, Theorem 5.2]).

We believe that the techniques used here can be effective for the homotopy classification problem of crossed modules, as well as for the dual problem of the group extension problem of the type of a crossed module.

2. Preliminaries

For convenience, we recall here some well-known results on Gr-categories and Gr-functors (see [12]).

We often denote by $\mathbb{G} = (\mathbb{G}, \otimes, I, \mathbf{a}, \mathbf{l}, \mathbf{r})$ a *Gr-category.* If $(F, \widetilde{F}, F_*)$ is a monoidal functor between Gr-categories, it is called a *Gr-functor.* Then the isomorphism $F_* : I' \to FI$ can be deduced from F and $\widetilde{F}$. Hereafter, we refer to $(F, \widetilde{F})$ as a Gr-functor.

Two Gr-functors $(F, \widetilde{F})$ and $(F', \widetilde{F'})$ from $\mathbb{G}$ to $\mathbb{G}'$ are *homotopic* if there is a *monoidal natural equivalence*, or a *homotopy* $\alpha : (F, \widetilde{F}, F_*) \to (F', \widetilde{F}', F'_*)$ which is a natural isomorphism such that

$$F'_* = \alpha_I \circ F_*.$$

Each Gr-category $\mathbb{G}$ determines three invariants:

i) the set $\Pi = \pi_0\mathbb{G}$ of isomorphism classes of the objects in $\mathbb{G}$ is a group where the multiplication is induced by the tensor product in $\mathbb{G}$,

ii) the set $A = \pi_1\mathbb{G}$ of automorphisms of the unit object I is an abelian group where the operation, denoted by $+$, is the composition. Further, A is a left Π-module,

iii) an element $\overline{k} \in H^3(\Pi, A)$ is induced by the associativity constraint of $\mathbb{G}$.

Based on these data, one can construct a Gr-category, denoted by $S_{\mathbb{G}}$, which is equivalent to $\mathbb{G}$, as follows. $S_{\mathbb{G}}$ is a category whose objects are the elements of

Π and whose morphisms are automorphisms $(x,a): x \to x$, where $x \in \Pi,\ a \in A$. The composition of two morphisms is induced by the addition in A

$$(x,a) \circ (x,b) = (x, a+b).$$

The operations on $S_{\mathbb{G}}$ are given by

$$x \otimes y = x.y, \quad x, y \in \Pi,$$
$$(x,a) \otimes (y,b) = (xy, a + xb), \quad a, b \in A.$$

The unit constraints of the Gr-category $S_{\mathbb{G}}$ are strict, and its associativity constraint is

$$\mathbf{a}_{x,y,z} = (xyz, k(x,y,z)).$$

The Gr-category $S_{\mathbb{G}}$ is called a *reduction* of the Gr-category $\mathbb{G}$. We say that $S_{\mathbb{G}}$ is of *type* (Π, A, k), or simply *type* (Π, A), if $\pi_0\mathbb{G}$ and $\pi_1\mathbb{G}$ are replaced with the group Π and the Π-module A, respectively.

Let $\mathbb{S} = (\Pi, A, k),\ \mathbb{S}' = (\Pi', A', k')$ be Gr-categories. A functor $F: \mathbb{S} \to \mathbb{S}'$ is of *type* (φ, f) if

$$F(x) = \varphi(x),\ F(x,a) = (\varphi(x), f(a)),$$

where $\varphi: \Pi \to \Pi',\ f: A \to A'$ are group homomorphisms satisfying $f(xa) = \varphi(x)f(a)$ for $x \in \Pi, a \in A$. Note that if Π'-module A' is considered as a Π-module under the action $xa' = \varphi(x).a'$, then $f: A \to A'$ is a homomorphism of Π-modules. In this case, we call (φ, f) a *pair of homomorphisms*, and

$$\xi = \varphi^* k' - f_* k \in Z^3(\Pi, A'), \tag{1}$$

where

$$(f_*k)(x,y,z) = f(k(x,y,z)),$$
$$(\varphi^*k')(x,y,z) = k'(\varphi x, \varphi y, \varphi z),$$

is an *obstruction* of the functor F.

The results on monoidal functors of type (φ, f) presented in [12] can be summarized in the following proposition.

Proposition 1. *Let $\mathbb{G}, \mathbb{G}'$ be two Gr-categories and $\mathbb{S}, \mathbb{S}'$ be their reduced Gr-categories, respectively.*

i) *Every Gr-functor $(F, \widetilde{F}): \mathbb{G} \to \mathbb{G}'$ induces a Gr-functor $S_F: \mathbb{S} \to \mathbb{S}'$ of type (φ, f).*

ii) *Every Gr-functor $(F, \widetilde{F}): \mathbb{S} \to \mathbb{S}'$ is a functor of type (φ, f).*

iii) *The functor $F: \mathbb{S} \to \mathbb{S}'$ of type (φ, f) is realizable, i.e., there exist isomorphisms $\widetilde{F}_{x,y}$ so that $(F, \widetilde{F})$ is a Gr-functor, if and only if the obstruction $\overline{\xi}$ vanishes in $H^3(\Pi, A')$. Then, there is a bijection*

$$\mathrm{Hom}_{(\varphi,f)}[\mathbb{S}, \mathbb{S}'] \leftrightarrow H^2(\Pi, A'),$$

where $\mathrm{Hom}_{(\varphi,f)}[\mathbb{S}, \mathbb{S}']$ is the set of homotopy classes of the Gr-functors of type (φ, f) from $\mathbb{S}$ to $\mathbb{S}'$.

3. Classification of crossed modules by strict Gr-categories

Definition. A *crossed module* is a quadruple $\mathcal{M} = (B, D, d, \theta)$ where $d : B \to D$, $\theta : D \to \mathrm{Aut}B$ are group homomorphisms satisfying the following relations
C_1. $\theta d = \mu$,
C_2. $d(\theta_x(b)) = \mu_x(d(b))$, $x \in D, b \in B$,
where μ_x is an inner automorphism given by conjugation with x.

In this paper, the crossed module (B, D, d, θ) is sometimes denoted by $B \stackrel{d}{\to} D$, or by $B \to D$. For convenience, we denote by the addition for the operation in B and by the multiplication for that in D.

The following properties follow from the definition of a crossed module.

Proposition 2. *Let $\mathcal{M} = (B, D, d, \theta)$ be a crossed module.*
i) $\mathrm{Ker}d \subset Z(B)$*, where $Z(B)$ is the center of B.*
ii) $\mathrm{Im}d$ *is a normal subgroup in D.*
iii) *The homomorphism θ induces a homomorphism $\varphi : D \to \mathrm{Aut}(\mathrm{Ker}d)$ by*

$$\varphi_x = \theta_x|_{\mathrm{Ker}d}.$$

iv) $\mathrm{Ker}d$ *is a left* $\mathrm{Coker}d$*-module under the action*

$$sa = \varphi_x(a),\ a \in \mathrm{Ker}d,\ x \textit{ is a representative of } s \in \mathrm{Coker}d.$$

The groups $\mathrm{Coker}\, d$, $\mathrm{Ker}\, d$ are also denoted by $\pi_0\mathcal{M}$, $\pi_1\mathcal{M}$, respectively.

It is known that a strict Gr-category can be seen as a crossed module (see [4], [9, Remark 3.1]). In order to motivate the readers, we state this in detail, whence the classification theorem of crossed modules is obtained (Theorem 5).

For any crossed module (B, D, d, θ) we can construct a strict Gr-category $\mathbb{P}_{B\to D} = \mathbb{P}$, called the Gr-category *associated* to the crossed module $B \to D$, as follows.

$$\mathrm{Ob}\mathbb{P} = D,\ \ \mathrm{Hom}(x, y) = \{b \in B/x = d(b)y\},$$

where x, y are objects of $\mathbb{P}$. The composition of two morphisms is given by

$$(x \stackrel{b}{\to} y \stackrel{c}{\to} z) = (x \stackrel{b+c}{\to} z).$$

The tensor product on objects is given by the multiplication in the group D, and for two morphisms $(x \stackrel{b}{\to} y)$, $(x' \stackrel{b'}{\to} y')$, one defines

$$(x \stackrel{b}{\to} y) \otimes (x' \stackrel{b'}{\to} y') = (xx' \stackrel{b+\theta_y b'}{\longrightarrow} yy'). \tag{2}$$

The associativity and unit constraints are identities. Then, by definition of crossed module, we can easily check that $\mathbb{P}$ is a strict Gr-category.

Conversely, for any strict Gr-category $(\mathbb{P}, \otimes)$ we define an *associated* crossed module $\mathcal{M}_{\mathbb{P}} = (B, D, d, \theta)$ as follows. Set

$$D = \mathrm{Ob}\mathbb{P},\ B = \{x \stackrel{b}{\to} 1/x \in D\}.$$

The operations in D and in B are given by

$$xy = x \otimes y, \quad b + c = b \otimes c,$$

respectively. Then D becomes a group in which the unit is 1, the inverse of x is x^{-1} ($x \otimes x^{-1} = 1$). B is a group in which the unit is the morphism $(1 \xrightarrow{id_1} 1)$ and the inverse of $(x \xrightarrow{b} 1)$ is the morphism $(x^{-1} \xrightarrow{\overline{b}} 1)$ $(b \otimes \overline{b} = id_1)$.

The homomorphisms $d : B \to D$ and $\theta : D \to \mathrm{Aut}B$ are given by

$$d(x \xrightarrow{b} 1) = x,$$
$$\theta_y(x \xrightarrow{b} 1) = (yxy^{-1} \xrightarrow{id_y + b + id_{y^{-1}}} 1),$$

respectively. It is easy to see that (B, D, d, θ) is a crossed module.

Definition. A *homomorphism* $(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$ of crossed modules consists of group homomorphisms $f_1 : B \to B'$, $f_0 : D \to D'$ satisfying

H_1. $f_0 d = d' f_1$,

H_2. $f_1(\theta_x b) = \theta'_{f_0(x)} f_1(b)$ for all $x \in D$, $b \in B$,

We need two following lemmas to prove the classification theorem.

Lemma 3. *Let $(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$ be a homomorphism of crossed modules. Let $\mathbb{P}$ and $\mathbb{P}'$ be the two Gr-categories associated to the crossed modules (B, D, d, θ) and (B', D', d', θ'), respectively.*

i) *There exists a functor $F : \mathbb{P} \to \mathbb{P}'$ defined by $F(x) = f_0(x)$, $F(b) = f_1(b)$, for $x \in \mathrm{Ob}\mathbb{P}$, $b \in \mathrm{Mor}\mathbb{P}$.*

ii) *Natural isomorphisms $\widetilde{F}_{x,y} : F(x)F(y) \to F(xy)$ together with F is a Gr-functor if and only if $\widetilde{F}_{x,y} = \varphi(\overline{x}, \overline{y})$, where $\varphi \in Z^2(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$.*

Proof. i) By the construction of the Gr-category associated to a crossed module and by the fact that f_1 is a group homomorphism, one can easily check that F is a functor.

ii) The group homomorphisms f_1, f_0 satisfying H_2 is equivalent to the equation

$$F(b \otimes c) = F(b) \otimes F(c)$$

for any two morphisms $(x \xrightarrow{b} x')$, $(y \xrightarrow{c} y')$ in $\mathbb{P}$.

Besides, since f_0 is a homomorphism and $F(x) = f_0(x)$, $\widetilde{F}_{x,y} : F(x)F(y) \to F(xy)$ is a morphism in $\mathbb{P}$ if and only if $d'(\widetilde{F}_{x,y}) = 1'$, i.e.,

$$\widetilde{F}_{x,y} \in \mathrm{Ker}d' \subset Z(B').$$

Then, the naturality of $(F,\widetilde{F})$, or the commutativity of the diagram

$$\begin{array}{ccc} F(x)F(y) & \xrightarrow{\widetilde{F}_{x,y}} & F(xy) \\ \Big\downarrow{\scriptstyle F(b)\otimes F(c)} & & \Big\downarrow{\scriptstyle F(b\otimes c)} \\ F(x')F(y') & \xrightarrow[\widetilde{F}_{x',y'}]{} & F(x'y'), \end{array}$$

is equivalent to the equation $\widetilde{F}_{x,y}=\widetilde{F}_{x',y'}$ for $x=(db)x', y=(dc)y'$. This defines a function $\varphi : \operatorname{Coker} d\times \operatorname{Coker} d\to \operatorname{Ker} d'$ by

$$\varphi(\overline{x},\overline{y})=\widetilde{F}_{x,y}.$$

Since $F(1)=1'$, the compatibility of $(F,\widetilde{F})$ with the unit constraints is equivalent to the normality of φ. The compatibility of $(F,\widetilde{F})$ with the associativity constraints is equivalent to

$$\theta'_{F(x)}(\widetilde{F}_{y,z})+\widetilde{F}_{x,yz}=\widetilde{F}_{x,y}+\widetilde{F}_{xy,z},$$

or

$$\overline{x}\varphi(\overline{y},\overline{z})+\varphi(\overline{x},\overline{y}\,\overline{z})=\varphi(\overline{x},\overline{y})+\varphi(\overline{x}\,\overline{y},\overline{z}),$$

where the action of $\operatorname{Coker} d$ on $\operatorname{Ker} d'$ is canonically induced by that of $\operatorname{Coker} d'$ on $\operatorname{Ker} d'$ via f_0, $\overline{x}b'=\overline{f_0(x)}b'$. Thus, $\varphi\in Z^2(\operatorname{Coker} d,\operatorname{Ker} d')$. □

Definition. A Gr-functor $(F,\widetilde{F}):\mathbb{P}\to\mathbb{P}'$ is called *regular* if the functor F preserves the operation $\otimes$, that means

S_1. $F(x)\otimes F(y)=F(x\otimes y)$,

S_2. $F(b)\otimes F(c)=F(b\otimes c)$

for $x,y\in \mathrm{Ob}\mathbb{P},\ b,c\in\mathrm{Mor}\mathbb{P}$.

Thanks to Lemma 3, one can define the category

Cross

whose objects are crossed modules and whose morphisms are triples (f_1,f_0,φ), where $(f_1,f_0):(B\to D)\to(B'\to D')$ is a homomorphism of crossed modules and $\varphi\in Z^2(\operatorname{Coker} d,\operatorname{Ker} d')$.

Lemma 4. *Let $\mathbb{P}$, $\mathbb{P}'$ be corresponding Gr-categories associated to the crossed modules (B,D,d,θ), (B',D',d',θ'), and $(F,\widetilde{F}):\mathbb{P}\to\mathbb{P}'$ be a regular Gr-functor. Then, the triple (f_1,f_0,φ), where*

$$f_1(b)=F(b),\ f_0(x)=F(x),\ \varphi(\overline{x},\overline{y})=\widetilde{F}_{x,y},$$

for $b\in B, x,y\in D,\overline{x},\overline{y}\in\operatorname{Coker} d$, is a morphism in the category **Cross**.

Proof. By the condition S_1, f_0 is a group homomorphism. Since F preserves the composition of morphisms, f_1 is a group homomorphism.

Since any $b\in B$ can be seen as a morphism $(db\overset{b}{\to}1)$ in $\mathbb{P}$, $(F(db)\overset{F(b)}{\to}1')$ is a morphism in $\mathbb{P}'$, whence H_1 holds: $f_0(d(b))=d'(f_1(b))$ for all $b\in B$.

According to the proof of Theorem 3, H_2 holds thanks to the fact that the homomorphism f_1 satisfies S_2. Therefore, the pair (f_1, f_0) is a homomorphism of crossed modules.

Now, also by Lemma 3, $\widetilde{F}_{x,y} \in \operatorname{Ker} d' \subset Z(B')$, and it defines a function $\varphi \in Z^2(\operatorname{Coker} d, \operatorname{Ker} d')$ by $\varphi(\overline{x}, \overline{y}) = \widetilde{F}_{x,y}$. □

We write **Grstr** for the category of strict Gr-categories and regular Gr-functors.

Theorem 5 (Classification Theorem)**.** *There exists an equivalence*

$$\begin{array}{rccc} \Phi : & \mathbf{Cross} & \to & \mathbf{Grstr} \\ & (B \to D) & \mapsto & \mathbb{P}_{B\to D} \\ & (f_1, f_0, \varphi) & \mapsto & (F, \widetilde{F}) \end{array}$$

where $F(x) = f_0(x), F(b) = f_1(b), \widetilde{F}_{x,y} = \varphi(\overline{x}, \overline{y})$ *for* $x, y \in D$*,* $b \in B$*.*

Proof. Let $\mathbb{P}$ and $\mathbb{P}'$ be the Gr-categories associated to crossed modules $B \to D$ and $B' \to D'$, respectively. By Lemma 3, the correspondence $(f_1, f_0, \varphi) \mapsto (F, \widetilde{F})$ defines an injection on the homsets,

$$\Phi : \operatorname{Hom}_{\mathbf{Cross}}(B \to D, B' \to D') \to \operatorname{Hom}_{\mathbf{Grstr}}(\mathbb{P}_{B\to D}, \mathbb{P}_{B'\to D'}).$$

By Lemma 4, Φ is surjective.

If $\mathbb{P}$ is a strict Gr-category and $\mathcal{M}_{\mathbb{P}}$ is its associated crossed module, then $\Phi(\mathcal{M}_{\mathbb{P}}) = \mathbb{P}$ (rather than an isomorphism). Thus, Φ is an equivalence. □

Remark 1*.* The category 2-**Gp** of $\mathcal{G}$-groupoids is a subcategory of the category **Grstr** whose morphisms are monoidal functors $(F, \widetilde{F})$ in which $\widetilde{F} = id$. The category **CrossMd** of crossed modules is a subcategory of the category **Cross** whose morphisms are (f_1, f_0, φ) in which $\varphi = 0$. Therefore, Theorem 5 contains [4, Theorem 1].

4. Classification of group extensions of the type of a crossed module

We now recall the notion of an *extension of groups of the type of a crossed module* due to Dedeker [8] (see also [2, 3]).

Note that if B is a normal subgroup in D, then the quadruple (B, D, d, θ) is a crossed module in which $d : B \to D$ is an inclusion, $\theta : D \to \operatorname{Aut}B$ is given by conjugation.

Definition. Let $\mathcal{M} = (B \xrightarrow{d} D)$ be a crossed module. An *extension* of B by Q *of type* $\mathcal{M}$ is a diagram of homomorphisms of groups

$$\begin{array}{lccccccccc} \mathcal{E}: & 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow & 1, \\ & & & \| & & \downarrow{\scriptstyle \varepsilon} & & & & \\ & & & B & \xrightarrow{d} & D & & & & \end{array}$$

where the top row is exact, the quadruple (B, E, j, θ') is a crossed module in which θ' is given by conjugation, and (id_B, ε) is a homomorphism of crossed modules.

Two extensions of B by Q of type $B \xrightarrow{d} D$ are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow & 1, \\ & & \| & & \downarrow{\scriptstyle \alpha} & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{j'} & E' & \xrightarrow{p'} & Q & \longrightarrow & 1, \end{array} \qquad \begin{array}{c} E \xrightarrow{\varepsilon} D \\ \\ E' \xrightarrow{\varepsilon'} D \end{array}$$

such that $\varepsilon'\alpha = \varepsilon$. Obviously, α is an isomorphism.

In the diagram

$$(3) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & Q & \longrightarrow & 1, \\ & & \| & & \downarrow{\scriptstyle \varepsilon} & & \vdots{\scriptstyle \psi} & & \\ & & B & \xrightarrow{d} & D & \xrightarrow{q} & \mathrm{Coker}d & & \end{array}$$

where q is the canonical homomorphism, since the top row is exact and $q \circ \varepsilon \circ j = q \circ d = 0$, there is a homomorphism $\psi : Q \to \mathrm{Coker}d$ such that the right hand side square commutes. Moreover, ψ only depends on the equivalence class of the extension $\mathcal{E}$, and we say that the extension $\mathcal{E}$ *induces* the homomorphism ψ.

Our objective is to study the set

$$\mathrm{Ext}_{B \to D}(Q, B, \psi)$$

of equivalence classes of extensions of B by Q of type $B \to D$ inducing $\psi : Q \to \mathrm{Coker}d$. The classification theorem for such group extensions has been proved in some ways (see Brown and Mucuk [3], Theorem 5.2).

In the present paper, we use the obstruction theory of Gr-functors to prove Theorem 5.2 in [3]. Further, the second assertion of this theorem can be seen as a consequence of Schreier Theory (Theorem 7) by means of Gr-functors between strict Gr-categories $\mathbb{P}_{B \to D}$ and Dis$Q$, where Dis$Q$ is the Gr-category of type $(Q, 0, 0)$ (and it is just the Gr-category associated to the crossed module $(0, Q, 0, 0)$). Each such Gr-functor induces an extension in $\mathrm{Ext}_{B \to D}(Q, B, \psi)$ as in the following lemma.

Lemma 6. *Let $B \to D$ be a crossed module and $\psi : Q \to \mathrm{Coker}d$ be a group homomorphism. Then, for each Gr-functor $(F, \widetilde{F}) : \mathrm{Dis}Q \to \mathbb{P}$ which satisfies $F(1) = 1$ and induces the pair $(\psi, 0) : (Q, 0) \to (\mathrm{Coker}\, d, \mathrm{Ker}\, d)$, there exists an extension $\mathcal{E}_F$ of B by Q of type $B \to D$ inducing ψ.*

The extension $\mathcal{E}_F$ is called a *crossed extension associated* to the Gr-functor F.

Proof. Let $(F,\widetilde{F}):\ \mathrm{Dis}\,Q\to\mathbb{P}$ be a Gr-functor which satisfies $F(1)=1$ and induces the pair $(\psi,0)$. One defines the function $f:Q\times Q\to B$ by

$$f(x,y)=\widetilde{F}_{x,y}.$$

Since $\widetilde{F}_{x,y}$ is a morphism in $\mathbb{P}$,

$$F(x).F(y)=df(x,y).F(xy).$$

As in the proof of Lemma 3, f is a normalized function satisfying

$$\theta_{F(x)}f(y,z)+f(x,yz)=f(x,y)+f(xy,z). \tag{4}$$

One can define the crossed product $E_0=[B,f,Q]$, i.e., $E_0=B\times Q$ with the operation

$$(b,x)+(c,y)=(b+\theta_{F(x)}c+f(x,y),xy).$$

The set E_0 is a group due to the normality of f and the equation (4). The zero is $(0,1)$ and $-(b,x)=(b',x^{-1})$, where $\theta_{F(x)}(b')=-b-f(x,x^{-1})$. Then, we have an exact sequence of groups

$$\mathcal{E}_F:0\to B\overset{j_0}{\to}E_0\overset{p_0}{\to}Q\to 1,$$

where $j_0(b)=(b,1)$; $p_0(b,x)=x$, $b\in B$, $x\in Q$. Since $j_0(B)$ is a normal subgroup in E_0, $j_0:B\to E_0$ is a crossed module where the action $\theta^0:E_0\to \mathrm{Aut}B$ is given by conjugation.

The map $\varepsilon:E_0\to D$ given by

$$\varepsilon(b,x)=dbF(x),\ (b,x)\in E_0, \tag{5}$$

is a group homomorphism. Then, the pair (id,ε) is a homomorphism of crossed modules. In fact, $\varepsilon\circ j_0=d$. Further, for all $(b,x)\in E_0, c\in B$,

$$\theta^0_{(b,x)}(c)=j_0^{-1}[\mu_{(b,x)}(c,1)]=\mu_b(\theta_{F(x)}c),$$

$$\theta_{\varepsilon(b,x)}(c)=\theta_{dbF(x)}(c)=\mu_b(\theta_{F(x)}c).$$

Hence, $\theta^0_{(b,x)}(c)=\theta_{\varepsilon(b,x)}(c)$. Therefore, one obtains an extension of type $B\to D$,

$$\begin{array}{rcccccccl}
\mathcal{E}_F: & 0 & \longrightarrow & B & \xrightarrow{j_0} & E_0 & \xrightarrow{p_0} & Q & \longrightarrow 1.\\
 & & & \Vert & & \downarrow\varepsilon & & \vdots\,\psi & \\
 & & & B & \xrightarrow{d} & D & \xrightarrow{q} & \mathrm{Coker}d &
\end{array}$$

For all $x\in Q$,

$$q\varepsilon(b,x)=q(db.F(x))=qF(x)=\psi(x),$$

i.e., the extension $\mathcal{E}_F$ induces $\psi:Q\to\mathrm{Coker}d$. □

Under the hypothesis of Lemma 6, we state the following result.

Theorem 7 (Schreier Theory for group extensions of the type of a crossed module)**.** *There is a bijection*

$$\Omega : \mathrm{Hom}_{(\psi,0)}[\mathrm{Dis}Q, \mathbb{P}_{B\to D}] \to \mathrm{Ext}_{B\to D}(Q, B, \psi).$$

Proof. Step 1 : *Gr-functors* $(F, \widetilde{F})$ *and* $(F', \widetilde{F}')$ *are homotopic if and only if the corresponding associated extensions* $\mathcal{E}_F, \mathcal{E}_{F'}$ *are equivalent.*

First, let us recall that every graded monoidal functor $(F, \widetilde{F})$ is homotopic to one $(G, \widetilde{G})$ in which $G(1) = 1$. Hence, we can restrict our attention to this kind of graded monoidal functors.

Let $F, F' : \mathrm{Dis}Q \to \mathbb{P}$ be homotopic by a homotopy $\alpha : F \to F'$. By Lemma 6, there exist the extensions $\mathcal{E}_F$ and $\mathcal{E}_{F'}$ associated to F and F', respectively. By definition of a homotopy, $\alpha_1 = 0$ and

$$F(x) = d(\alpha_x).F'(x). \tag{6}$$

The naturality of α leads to

$$\widetilde{F}_{x,y} + \alpha_{xy} = \alpha_x \otimes \alpha_y + \widetilde{F}'_{x,y}.$$

By the relation (2),

$$f(x, y) + \alpha_{xy} = \alpha_x + \theta_{F'(x)}(\alpha_y) + f'(x, y), \tag{7}$$

where $f(x, y) = \widetilde{F}_{x,y}, f'(x, y) = \widetilde{F}'_{x,y}$. Now we set

$$\begin{aligned} \alpha^* : E_0 &\to E'_0 \\ (b, x) &\mapsto (b + \alpha_x, x). \end{aligned}$$

For all $(b, x), (c, y) \in E_0$, one has

$$\begin{aligned} \alpha^*[(b, x) + (c, y)] &= \alpha^*[(b + \theta_{F(x)}c + f(x, y), xy)] \\ &= (b + \theta_{F(x)}c + f(x, y) + \alpha_{xy}, xy), \\ \alpha^*(b, x) + \alpha^*(c, y) &= (b + \alpha_x, x) + (c + \alpha_y, y) \\ &= (b + \alpha_x + \theta_{F'(x)}(c + \alpha_y) + f'(x, y), xy). \end{aligned}$$

The condition α^* being a group homomorphism is equivalent to

$$b + \theta_{F(x)}c + f(x, y) + \alpha_{xy} = b + \alpha_x + \theta_{F'(x)}(c + \alpha_y) + f'(x, y).$$

This follows from

$$\begin{aligned} \theta_{F(x)}(c) + f(x, y) + \alpha_{xy} &\overset{(6)}{=} \theta_{d\alpha_x F'(x)}(c) + f(x, y) + \alpha_{xy} \\ &\overset{(C_1)}{=} \mu_{\alpha_x}(\theta_{F'(x)}c) + f(x, y) + \alpha_{xy} \\ &\overset{(7)}{=} \alpha_x + \theta_{F'(x)}(c + \alpha_y) + f'(x, y). \end{aligned}$$

Further, it is easy to see that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E_F & \xrightarrow{p} & Q & \longrightarrow & 1, \qquad E_F \xrightarrow{\varepsilon} D \\ & & \| & & \downarrow{\alpha^*} & & \| & & \\ 0 & \longrightarrow & B & \xrightarrow{j'} & E_{F'} & \xrightarrow{p'} & Q & \longrightarrow & 1, \qquad E_{F'} \xrightarrow{\varepsilon'} D \end{array}$$

We next show that $\varepsilon'\alpha^* = \varepsilon$. It follows from the equations (5), (6) that

$$\begin{aligned}\varepsilon'\alpha^*(b,x) &= \varepsilon'(b+\alpha_x, x) = d(b+\alpha_x)\cdot F'(x)\\ &= d(b)\cdot d(\alpha_x)\cdot F'(x) = d(b)\cdot F(x) = \varepsilon(b,x).\end{aligned}$$

Thus, two extensions $\mathcal{E}_F$ and $\mathcal{E}_{F'}$ are equivalent.

Conversely, if $\alpha^* : E_F \to E_{F'}$ is an isomorphism such that (id_B, α^*, id_Q) is an equivalence of two extensions, then one can easily check that

$$\alpha^*(b,x) = (b+\alpha_x, x),$$

where $\alpha : Q \to B$ is a function satisfying $\alpha_1 = 0$. Thus, α is a homotopy between F and F' as we see by retracing our steps.

Step 2 : Ω *is surjective.*

Let $\mathcal{E}$ be an extension E of B by Q of type $B \to D$ inducing $\psi : Q \to \operatorname{Coker} d$ as in the commutative diagram (3). We prove that $\mathcal{E}$ is equivalent to a crossed extension $\mathcal{E}_F$ which is associated to some Gr-functor $(F, \widetilde{F}) : \mathrm{Dis}Q \to \mathbb{P}$.

For each $x \in Q$, choose a representative $e_x \in E$ such that $p(e_x) = x,\ e_1 = 0$. Each element of E is written uniquely as $b + e_x$, where $b \in B, x \in Q$. The system of representatives $\{e_x\}$ induces a normalized function $f : Q \times Q \to B$ by

$$e_x + e_y = f(x,y) + e_{xy}, \tag{8}$$

and automorphisms φ_x of B by

$$\varphi_x = \mu_{e_x} : b \mapsto e_x + b - e_x.$$

It follows from the condition H_2 of the homomorphism $(id, \varepsilon) : (B \to E) \to (B \to D)$ of crossed modules that

$$\theta_{\varepsilon e_x} = \varphi_x.$$

Then, the group structure of E can be described by

$$(b+e_x) + (c+e_y) = b + \varphi_x(c) + f(x,y) + e_{xy},$$

Now, we defines a Gr-functor $(F, \widetilde{F}) :\ \mathrm{Dis}\,Q \to \mathbb{P}$ as follows. Since $\psi(x) = \psi p(e_x) = q\varepsilon(e_x)$, $\varepsilon(e_x)$ is a representative of $\psi(x)$ in D. So, one sets

$$F(x) = \varepsilon(e_x),\ \widetilde{F}_{x,y} = f(x,y).$$

Since $\overline{F(x)} = q\varepsilon(e_x) = \psi(x)$, then F induces $(\psi, 0)$. The relation (8) shows that $\widetilde{F}_{x,y}$ are actually morphisms in $\mathbb{P}$. Clearly, $F(1) = 1$. This together with the normality of the function $f(x,y)$ implies the compatibility of $(F, \widetilde{F})$ with the unit constraints. The associativity law of the operation in E leads to the relation (4). This proves that $(F,\widetilde{F})$ is compatible with the associativity constraints. The naturality of $\widetilde{F}_{x,y}$ and the condition of F preserving the composition are obvious.

Finally, one can easily check that the crossed extension $\mathcal{E}_F$ associated to $(F,\widetilde{F})$ is equivalent to the extension $\mathcal{E}$ by the isomorphism $\beta:(b,x)\mapsto b+e_x$. □

Let $\mathbb{P}=\mathbb{P}_{B\to D}$ be the Gr-category associated to the crossed module $B\overset{d}{\to}D$. Since $\pi_0\mathbb{P}=\mathrm{Coker}d,\ \pi_1\mathbb{P}=\mathrm{Ker}d$, its reduced category is

$$S_{\mathbb{P}}=(\mathrm{Coker}d,\mathrm{Ker}d,k),\ \overline{k}\in H^3(\mathrm{Coker}d,\mathrm{Ker}d).$$

Then, according to (1), the homomorphism $\psi:Q\to\mathrm{Coker}d$ induces an *obstruction*

$$\psi^*k\in Z^3(Q,\mathrm{Ker}d).$$

By this notion of obstruction one can state a new proof of [3, Theorem 5.2] whose techniques differ from those used in [3].

Theorem 8. *Let (B,D,d,θ) be a crossed module and $\psi:Q\to\mathrm{Coker}d$ be a group homomorphism. Then, the vanishing of $\overline{\psi^*k}$ in $H^3(Q,\mathrm{Ker}d)$ is necessary and sufficient for there to exist an extension of B by Q of type $B\to D$ inducing ψ. Further, if $\overline{\psi^*k}$ vanishes, then the equivalence classes of such extensions are bijective with $H^2(Q,\mathrm{Ker}d)$.*

Proof. By the assumption $\overline{\psi^*k}=0$, it follows from Proposition 1 that there is a Gr-functor $(\Psi,\widetilde{\Psi}):\mathrm{Dis}Q\to S_{\mathbb{P}}$. Then the composition of $(\Psi,\widetilde{\Psi})$ and $(H,\widetilde{H}):S_{\mathbb{P}}\to\mathbb{P}$ is a Gr-functor $(F,\widetilde{F}):\mathrm{Dis}Q\to\mathbb{P}$, and hence, by Lemma 6 one obtains an associated extension $\mathcal{E}_F$.

Conversely, suppose that there is an extension making the diagram (3) commutative. Let $\mathbb{P}'$ be the category associated to the crossed module $B\to E$. By Lemma 3, there is a Gr-functor $F:\mathbb{P}'\to\mathbb{P}$. Since the reduced Gr-category of $\mathbb{P}'$ is $\mathrm{Dis}Q$, by Proposition 1 F induces a Gr-functor of type $(\psi,0)$ from $\mathrm{Dis}Q$ to $(\mathrm{Coker}d,\mathrm{Ker}d,k)$. Now, by Proposition 1, the obstruction of the pair $(\psi,0)$ vanishes in $H^3(Q,\mathrm{Ker}d)$, i.e., $\overline{\psi^*k}=0$.

The final assertion of the theorem is obtained from Theorem 7.

Firstly, there is a natural bijection

$$\mathrm{Hom}[\mathrm{Dis}\,Q,\mathbb{P}]\leftrightarrow\mathrm{Hom}[\mathrm{Dis}\,Q,S_{\mathbb{P}}].$$

Since $\pi_0(\mathrm{Dis}\,Q)=Q,\pi_1(S_{\mathbb{P}})=\mathrm{Ker}\,d$, it follows from Theorem 7 and Proposition 1 that there is a bijection

$$\mathrm{Ext}_{B\to D}(Q,B,\psi)\leftrightarrow H^2(Q,\mathrm{Ker}d).$$ □

Remark 2. Theorem 8 contains Proposition 8.3, Ch. IV [10]. Indeed, for the crossed module $B\overset{\mu}{\to}\mathrm{Aut}(B)$ one can see that $\psi:Q\to\mathrm{Aut}(B)/\mathrm{In}(B)$ and

$\operatorname{Ker}\mu = Z(B)$, hence each extension of the type of this crossed module is an ordinary group extension of B by Q. Thus, $\operatorname{Ext}(Q, B, \varphi) \leftrightarrow H^2(Q, Z(B))$.

Remark 3. If the homomorphism d of the crossed module $\mathcal{M}$ is an injection, then the diagram (3) shows that the extension $(\mathcal{E} : B \to E \to Q)$ is obtained from one $(\mathcal{D} : B \to D \to \operatorname{Coker} d)$ via ψ, that means $\mathcal{E} = \mathcal{D}\psi$ (see [10]). Since $\operatorname{Ker} d = 0$, from Theorem 7 one gets the following well-known result.

Corollary 9. *Let $(\mathcal{D} : B \to D \to C)$ be a group extension and $\psi : Q \to C$ is a group homomorphism. Then, the extension $\mathcal{D}\psi$ exists and it is uniquely defined up to equivalence.*

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