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# CONGRUENCES ON ABUNDANT SEMIGROUPS WITH QUASI-IDEAL $S$-ADEQUATE TRANSVERSALS

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Abstract. In this paper, we give congruences on an abundant semigroup with a quasi-ideal $S$-adequate transversal $S^\circ$ by the congruence pair abstractly which consists of congruences on the structure component parts $R$ and $\Lambda$. We prove that the set of all congruences on this kind of semigroups is a complete lattice.

## 1. Introduction and preliminaries

The multiplicative inverse transversals of a regular semigroup were first introduced by T. S. Blyth and R. B. McFadden in 1982 [1]. As an analogue of an inverse transversal, an adequate transversal, was introduced by EI-Qallali in [3]. Afterwards, J. F. Chen and X. J. Guo studied the subclass of abundant semigroups with adequate transversals in [2] and [6, 7, 8], respectively. The congruence on regular semigroups with inverse transversals was studied by L. M. Wang and X. L. Tang (see [10], [11], [12]). Recently, X. J. Kong and P. Wang gave the structure theory of abundant semigroups having quasi-ideal $S$-adequate transversals in [8]. In this paper, we give congruences on abundant semigroups having quasi-ideal $S$-adequate transversals by the congruence triple and prove that the set of all congruences on this kind of semigroups is a complete lattice.

Let $S$ be a semigroup. By $a\mathcal{R}^*b$ we mean that $xa = ya$ if and only if $xb = yb$ for all $x, y \in S^1$. The relation $\mathcal{L}^*$ is defined dually. $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence on $S$. It is easy to show that if $a$ and $b$ are regular elements of $S$ then $a\mathcal{R}^*b$ if and only if $a\mathcal{R}b$. We call a semigroup $S$ abundant if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class contains an idempotent. For the sake of convenience, for $a \in S$, we denote the typical idempotent $\mathcal{L}^*$-related to $a$ by $a^*$ and we use $a^\dagger$ to denote the idempotent which is $\mathcal{R}^*$-related to $a$. An abundant semigroup in which idempotents commute is called adequate. It is clear that regular semigroups are abundant and that inverse semigroups are adequate.

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**Lemma 1.1** ([5]). *Let $e \in E(S)$ and $a \in S$. Then $e\mathcal{R}^*a$ if and only if $ea = a$ and for all $x, y \in S^1$, $xa = ya$ implies $xe = ye$.*

**Lemma 1.2** ([4]). *A semigroup $S$ is adequate if and only if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class contains a unique idempotent and the subsemigroup generated by $E(S)$ is regular.*

It is easy to show that if $S$ is adequate and $a, b \in S$, then $a\mathcal{R}^*b$ if and only if $a^\dagger = b^\dagger$ and $a\mathcal{L}^*b$ if and only if $a^* = b^*$.

**Lemma 1.3** ([4]). *If $S$ is an adequate semigroup, then for all $a, b \in S$, $(ab)^* = (a^*b)^*$ and $(ab)^\dagger = (ab^\dagger)^\dagger$.*

Notice that we can then immediately deduce that for all $a, b \in S$, $(ab)^* = (ab)^*b^*$ and $(ab)^\dagger = a^\dagger(ab)^\dagger$.

Let $S$ be an abundant semigroup and $U$ is an abundant subsemigroup of $S$. We say that $U$ is a $*$-subsemigroup of $S$ if

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U),\ \mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U).$$

It can be show that $U$ is a $*$-subsemigroup of $S$ if and only if for all $a \in U$ there exist $e, f \in E(U)$ such that $e \in L^*_a(S),\ f \in R^*_a(S)$ (see [3]).

Suppose that $S^\circ$ is an adequate $*$-subsemigroup of an abundant semigroup $S$. $S^\circ$ is called an adequate transversal of $S$ if for any element $x \in S$, there exists a unique element $x^\circ \in S^\circ$ and idempotents $e, f \in E$, such that $x = ex^\circ f$, where $e\mathcal{L}^*x^{\circ\dagger}$ and $f\mathcal{R}^*x^{\circ *}$ for $x^{\circ\dagger}, x^{\circ *} \in E^\circ$. It is straightforward to show [3], that such $e$ and $f$ are uniquely determined by $x$.

A subset $S^\circ$ of $S$ is called a quasi-ideal of $S$ if $S^\circ S S^\circ \subseteq S^\circ$. An adequate transversal $S^\circ$ of an abundant semigroup $S$ is called quasi-ideal adequate transversal if $S^\circ$ is a quasi-ideal of $S$. It is well known that the set $I = \{e \in S : ee^\circ = e\}$ is a left regular band and $\Lambda = \{f \in S : f^\circ f = f\}$ is a right regular band, and they play an important role in the study of regular semigroups with inverse transversals. In [2], Chen introduced two important idempotent subsets $I$ and $\Lambda$ and though they play a role similar to those in the regular case, in general, they are not subsemigroups of $S$. Following [9], if $S^\circ$ is an adequate transversal of an abundant semigroup $S$ and $I$, $\Lambda$ are both subsemigroups of $S$, then $S^\circ$ is called an $S$-adequate transversal of $S$.

**Lemma 1.4** ([2]). *Let $S$ be an abundant semigroup with a quasi-ideal adequate transversal $S^\circ$. Then for any $x, y \in S$*

(1) $\overline{xy} = \overline{x} f_x e_y \overline{y}$;
(2) $e_{xy} = e_x(\overline{x} f_x e_y)^\dagger$;
(3) $f_{xy} = (f_x e_y \overline{y})^* f_y$.

**Lemma 1.5** ([8]). *Let $S$ be an abundant semigroup with an adequate transversal $S^\circ$. If $S^\circ$ is a right ideal of $S$, then $f_x \in E^\circ$ for every $x \in S$ and $E = I$. Consequently, $f_x = \overline{x}^*$ and thus $x = e_x\overline{x}$.*

*Dually, if $S^\circ$ is a left ideal of $S$, then $e_a \in E^\circ$ for every $a \in S$ and $E = \Lambda$. Consequently, $e_a = \overline{a}^+$ and thus $a = \overline{a} f_a$.*

**Lemma 1.6** ([8])**.** *Let $R$ be a quasi-adequate semigroup with a right ideal adequate transversal $S^\circ$ and $\Lambda$ a right normal band with a left ideal semilattice transversal $E^\circ$. Suppose that the set of idempotents of $S$ coincides with $E^\circ$. Let $\Lambda \times R \longrightarrow S^\circ$ described by $(\lambda, x) \longrightarrow \lambda * x$ be a mapping, such that for any $x, y \in R$ and for any $\lambda, \mu \in \Lambda$:*

*(1) $(\lambda * x)y = \lambda * (xy)$ and $\mu(\lambda * x) = (\mu\lambda) * x$;*

*(2) if $x \in E^\circ$ or $\lambda \in E^\circ$, then $\lambda * x = \lambda x$;*

*(3) if $x_1 \mathcal{R}^* x_2$ in $R$, then for all $\mu_1, \mu_2 \in \Lambda^1$, $y_1, y_2 \in R^1$, $y_1(\mu_1 * x_1) = y_2(\mu_2 * x_1)$ if and only if $y_1(\mu_1 * x_2) = y_2(\mu_2 * x_2)$.*

*Define a multiplication on the set*

$$\Gamma \equiv R| \times |\Lambda = \{(x, \lambda) \in R \times \Lambda : f_x = \lambda^\circ\}$$

*by*

$$(x, \lambda)(y, \mu) = (x(\lambda * y), f_{\lambda * y}\mu).$$

*Then $\Gamma$ is an abundant semigroup with a quasi-ideal $S$-adequate transversal which is isomorphic to $S^\circ$.*

*Conversely, every abundant semigroup with a quasi-ideal $S$-adequate transversal can be constructed in this way.*

**Lemma 1.7** ([8])**.** *Let $(x, \lambda) \in \Gamma$. Denote $\mu = (e_x, \overline{x}^\dagger)$ and $\nu = (f_x, \lambda)$. Then $\mu, \nu \in E(\Gamma)$ and $\mu \mathcal{R}^* (x, \lambda) \mathcal{L}^* \nu$.*

**Lemma 1.8** ([8])**.** *Let $(x_1, \lambda_1), (x_2, \lambda_2) \in \Gamma$. Then*

*(1) $(x_1, \lambda_1)\mathcal{R}^*(x_2, \lambda_2)$ if and only if $e_{x_1} = e_{x_2}$ and $\overline{x_1}^\dagger = \overline{x_2}^\dagger$.*

*(2) $(x_1, \lambda_1)\mathcal{L}^*(x_2, \lambda_2)$ if and only if $f_{x_1} = f_{x_2}$ and $\lambda_1 = \lambda_2$.*

## 2. The main results

In this section, we describe a congruence on abundant semigroups having a quasi-ideal $S$-adequate transversal. Let $S$ be a semigroup, $\mathcal{E}(S)$ the lattice of all equivalences on $S$. For any $\sigma \in \mathcal{E}(S)$, we say that $A \subseteq S$ a *subset saturated* by $\sigma$, if $A$ is a union of some $\sigma$-classes of $S$.

Suppose $\rho^\Lambda$ and $\rho^R$ are congruences on $\Lambda$ and $R$, respectively, and $\rho^R$ is saturated by $\mathcal{L}^*$ and $\mathcal{R}^*$. Then $(\rho^R, \rho^\Lambda)$ is called a congruence pair on $\Gamma$ if the following conditions hold:

(C.1) $\rho^R|_{E^\circ} = \rho^\Lambda|_{E^\circ}$;

(C.2) $(\forall\ \nu \in \Lambda)(\forall\ x, y \in R)\ x\rho^R y \Rightarrow (\nu * x)\rho^R(\nu * y)$;

(C.3) $(\forall\ z \in R)(\forall\ \lambda, \mu \in \Lambda)\ \lambda\rho^\Lambda\mu \Rightarrow f_{\lambda * z}\rho^\Lambda f_{\mu * z}$.

Define $\rho^{(\rho^R, \rho^\Lambda)}$ on $\Gamma$ by

$$(x, \lambda)\rho^{(\rho^R, \rho^\Lambda)}(y, \mu) \Leftrightarrow x\rho^R y, \lambda\rho^\Lambda\mu.$$

**Theorem 2.1.** *Let $\Gamma$ be an abundant semigroup having a quasi-ideal $S$-adequate transversal as in Lemma 1.6, and $(\rho^R, \rho^\Lambda)$ be a congruence pair on $\Gamma$. Then $\rho^{(\rho^R, \rho^\Lambda)}$ is a congruence on $\Gamma$.*

*Conversely, every congruence on $\Gamma$ can be constructed in the above manner.*

*Proof.* Let $(\rho^R, \rho^\Lambda)$ be a congruence pair on $\Gamma$. Obviously, $\rho^{(\rho^R,\rho^\Lambda)}$ is an equivalence on $\Gamma$. Suppose that

$$(x, \lambda), (y, \mu) \in \Gamma, \text{ and } (x, \lambda)\rho^{(\rho^R,\rho^\Lambda)}(y, \mu).$$

Then

$$x\rho^R y, \ \lambda\rho^\Lambda\mu.$$

For any $(z, \nu) \in \Gamma$, by $x\rho^R y$ and C.2, we have

$$(\nu * x)\rho^R(\nu * y).$$

Since $\rho^R$ is saturated by $\mathcal{L}^*$ and C.1, then

$$f_{\nu*x} = (\nu * x)^*\rho^\Lambda(\nu * y)^* = f_{\nu*y}.$$

It follows that

$$z(\nu * x)\rho^R z(\nu * y) \text{ and } f_{\lambda*z}\lambda\rho^\Lambda f_{\mu*z}\mu.$$

Hence

$$(z(\nu * x), f_{\lambda*z}\lambda)\rho^{(\rho^R,\rho^\Lambda)}(z(\nu * y), f_{\mu*z}\mu).$$

That is,

$$(z, \nu)(x, \lambda)\rho^{(\rho^R,\rho^\Lambda)}(z, \nu)(y, \mu).$$

For any $(z, \nu) \in \Gamma$, by $\lambda\rho^\Lambda\mu$ and C.3, we have $f_{\lambda*z}\rho^\Lambda f_{\mu*z}$. Since $\lambda * z, \mu * z \in S^\circ$, then

$$f_{\lambda*z} = (\lambda * z)^*, f_{\mu*z} = (\mu * z)^*.$$

By C.1, $(\lambda * z)^*\rho^R(\mu * z)^*$. Hence, by $\rho^R$ is saturated by $\mathcal{L}^*$, $(\lambda * z)\rho^R(\mu * z)$. Furthermore, we have

$$x(\lambda * z)\rho^R y(\mu * z) \text{ and } f_{\lambda*z}\nu\rho^\Lambda f_{\mu*z}\nu.$$

Hence

$$(x(\lambda * z), f_{\lambda*z}\nu)\rho^{(\rho^R,\rho^\Lambda)}(y(\mu * z), f_{\mu*z}\nu).$$

Thus $(x, \lambda)(z, \nu)\rho^{(\rho^R,\rho^\Lambda)}(y, \mu)(z, \nu)$ and so $\rho^{(\rho^R,\rho^\Lambda)}$ is a congruence.

Conversely, assume that $\rho$ is a congruence on $\Gamma$. We define the following equivalences on $R$ and $\Lambda$, respectively,

$(\forall\ x, y \in R)\ x\rho_R y \Leftrightarrow (e_x, \overline{x}^\dagger)\rho(e_y, \overline{y}^\dagger)$; $(\forall\ \lambda, \mu \in \Lambda)\ \lambda\rho_\Lambda\mu \Leftrightarrow (f_x, \lambda)\rho(f_y, \mu)$.

Since $\rho$ is a congruence on $\Gamma$, we have $\rho_R, \rho_\Lambda$ are equivalences on $R$ and $\Lambda$, respectively.

Let $(x, \lambda), (y, \mu), (x_1, \lambda_1), (y_1, \mu_1) \in \Gamma$. If $x\rho_R y$ and $x_1\rho_R y_1$, then

$$(e_x, \overline{x}^\dagger)\rho(e_y, \overline{y}^\dagger) \text{ and } (e_{x_1}, \overline{x_1}^\dagger)\rho(e_{y_1}, \overline{y_1}^\dagger).$$

Now we immediately get

$$(e_x, \overline{x}^\dagger)(e_{x_1}, \overline{x_1}^\dagger)\rho(e_y, \overline{y}^\dagger)(e_{y_1}, \overline{y_1}^\dagger).$$

And this implies that

$$(e_x(\overline{x}^\dagger * e_{x_1}), f_{\overline{x}^\dagger * e_{x_1}}\overline{x_1}^\dagger)\rho(e_y(\overline{y}^\dagger * e_{y_1}), f_{\overline{y}^\dagger * e_{y_1}}\overline{y_1}^\dagger).$$

Since

$$e_x(\overline{x}^\dagger * e_{x_1})\overline{x x_1} = e_x(\overline{x}^\dagger * e_{x_1})\overline{x}^\dagger\overline{x} f_x e_{x_1}\overline{x_1} = e_x\overline{x} f_x e_{x_1}\overline{x_1} = x x_1$$

and

$$xx_1 f_{\overline{x}^\dagger * e_{x_1}} \overline{x_1}^\dagger = xx_1(\overline{x}^\dagger * e_{x_1})^* \overline{x_1}^\dagger = xe_{x_1}\overline{x_1}\ \overline{x_1}^\dagger(\overline{x}^\dagger * e_{x_1})^* \overline{x_1}^\dagger = xx_1.$$

Then

$$(e_{xx_1}, f_{\overline{x_1}^\dagger * e_{x_1}} \overline{x}^\dagger)\rho(e_{yy_1}, f_{\overline{y}^\dagger * e_{y_1}} \overline{y_1}^\dagger).$$

So we have proved that $xx_1 \rho_R yy_1$.

Suppose that $\lambda \rho_\Lambda \mu$ and $\lambda_1 \rho_\Lambda \mu_1$, then we have

$$(f_x, \lambda)\rho(f_y, \mu) \text{ and } (f_{x_1}, \lambda_1)\rho(f_{y_1}, \mu_1).$$

Hence

$$(f_x, \lambda)(f_{x_1}, \lambda_1)\rho(f_y, \mu)(f_{y_1}, \mu_1).$$

That is,

$$(f_x(\lambda * f_{x_1}), f_{\lambda * f_{x_1}} \lambda_1)\rho(f_y(\mu * f_{y_1}), f_{\mu * f_{y_1}} \mu_1).$$

It follows from $f_{\lambda * f_{x_1}} = f_{\lambda * \lambda_1^\circ} = \lambda\lambda_1^\circ$ and $f_{\mu * f_{y_1}} = f_{\mu * \mu_1^\circ} = \mu\mu_1^\circ$ that

$$(f_x(\lambda * f_{x_1}), \lambda\lambda_1^\circ \lambda_1)\rho(f_y(\mu * f_{y_1}), \mu\mu_1^\circ \mu_1).$$

Hence $\lambda\lambda_1 \rho_\Lambda \mu\mu_1$.

And we have the following cases:

(1) $\rho_R|_{E^\circ} = \rho_\Lambda|_{E^\circ}$ is obvious. So (C.1) holds.

(2) Let $x, y \in R$ and $x\rho_R y$. Then

$$(e_x, \overline{x}^\dagger)\rho(e_y, \overline{y}^\dagger).$$

Hence

$$(f_z, \nu)(e_x, \overline{x}^\dagger)\rho(f_z, \nu)(e_y, \overline{y}^\dagger).$$

That is,

$$(f_z(\nu * e_x), f_{\nu * e_x} \overline{x}^\dagger)\rho(f_z(\nu * e_y), f_{\nu * e_y} \overline{y}^\dagger).$$

It follows from $f_z(\nu * e_x)\overline{(\nu * x)} = \nu^\circ \nu e_x (\nu * x)^\dagger (\nu * x) = (\nu * x)$ that

$$(e_{\nu * x}, f_{\nu * e_x} \overline{x}^\dagger)\rho(e_{\nu * y}, f_{\nu * e_y} \overline{y}^\dagger).$$

Thus $(\nu * x)\rho_R(\nu * y)$ and so C.2 hold.

(3) Let $\lambda, \mu \in \Lambda$ and $\lambda \rho_\Lambda \mu$. Then

$$(f_x, \lambda)\rho(f_y, \mu).$$

Hence

$$(f_x, \lambda)(z, \nu)\rho(f_y, \mu)(z, \nu).$$

It follows that

$$(f_x(\lambda * z), f_{\lambda * z} \nu)\rho(f_y(\mu * z), f_{\mu * z} \nu).$$

Thus $f_{\lambda * z} \nu \rho_\Lambda f_{\mu * z} \nu$ and so $f_{\lambda * z} \nu f_z \rho_\Lambda f_{\mu * z} \nu f_z$. Moreover, $(\lambda * z)^* z^* \rho_\Lambda (\mu * z)^* z^*$. Then $f_{\lambda * z} \rho_\Lambda f_{\mu * z}$ and so C.3 hold.

Let $x \rho_R y$. Then

$$(e_x, \overline{x}^\dagger)\rho(e_y, \overline{y}^\dagger).$$

From $x\mathcal{R}^*x^*, y\mathcal{R}^*y^*$, we have $e_x = e_{x^*}, e_y = e_{y^*}$. Since $e_x = e_x \cdot \overline{x}^\dagger \cdot \overline{x}^\dagger = e_{x^*}$, we have $\overline{x}^\dagger = \overline{x^*}^\dagger$. Then

$$(e_{x^*}, \overline{x^*}^\dagger)\rho(e_{y^*}, \overline{y^*}^\dagger).$$

It follows that $x^*\rho_R y^*$ and so $\rho_R$ is saturated by $\mathcal{R}^*$. Similarly, we can proved $\rho_R$ is saturated by $\mathcal{L}^*$.

Now from the above proof, $(\rho_R, \rho_\Lambda)$ is a congruence pair on $\Gamma$.

By the direct part, $\rho^{(\rho_R,\rho_\Lambda)}$ is a congruence. If $(x, \lambda)\rho^{(\rho_R,\rho_\Lambda)}(y, \mu)$, then we have

$$x\rho_R y, \lambda\rho_\Lambda\mu.$$

And hence

$$(e_x, \overline{x}^\dagger)\rho(e_y, \overline{y}^\dagger), (f_x, \lambda)\rho(f_y, \mu).$$

From $\overline{x}\mathcal{R}^*\overline{x}^*\mathcal{R}f_x\mathcal{L}^*x$ and $\rho_R$ is saturated by $\mathcal{L}^*$ and $\mathcal{R}^*$, we have $\overline{x}\rho_R x\rho_R\overline{x}^*$. Similarly, we have $\overline{y}\rho_R y\rho_R\overline{y}^*$. Thus $\overline{x}\rho_R\overline{y}\rho_R\overline{x}^*\rho_R\overline{y}^*$. By $\overline{x}^*\rho_R\overline{y}^*$ and C.1, we have $\overline{x}^*\rho_\Lambda\overline{y}^*$. And hence

$$(\overline{x}, \overline{x}^*)\rho(\overline{y}, \overline{y}^*).$$

It follows from

$$(e_x, \overline{x}^\dagger)\rho(e_y, \overline{y}^\dagger), (f_x, \lambda)\rho(f_y, \mu) \text{ and } (\overline{x}, \overline{x}^*)\rho(\overline{y}, \overline{y}^*)$$

that

$$(e_x, \overline{x}^\dagger)(\overline{x}, \overline{x}^*)(f_x, \lambda)\rho(e_y, \overline{y}^\dagger)(\overline{y}, \overline{y}^*)(f_y, \mu).$$

That is,

$$(x, \lambda)\rho(y, \mu).$$

Thus, $\rho^{(\rho_R,\rho_\Lambda)} \subseteq \rho$. Since $\rho \subseteq \rho^{(\rho_R,\rho_\Lambda)}$ is obvious, $\rho^{(\rho_R,\rho_\Lambda)} = \rho$. □

We denote the set of all congruences on $\Gamma$ and the set of all congruence pairs on $\Gamma$ constructed as in Theorem 2.1 by $C(\Gamma)$ and $CT(\Gamma)$.

**Lemma 2.2.** *If* $(\rho_1^R, \rho_1^\Lambda), (\rho_2^R, \rho_2^\Lambda) \in CT(\Gamma)$, *then*

$$\rho^{(\rho_1^R,\rho_1^\Lambda)} \subseteq \rho^{(\rho_2^R,\rho_2^\Lambda)} \Leftrightarrow \rho_1^R \subseteq \rho_2^R, \rho_1^\Lambda \subseteq \rho_2^\Lambda.$$

*Proof.* Suppose $\rho^{(\rho_1^R,\rho_1^\Lambda)} \subseteq \rho^{(\rho_2^R,\rho_2^\Lambda)}$. Let $x\rho_1^R y$. By the proof of Theorem 2.1, there exist $(e_x, \overline{x}^\dagger), (e_y, \overline{y}^\dagger) \in E(\Gamma)$ and $\rho_1, \rho_2 \in C(\Gamma)$ such that

$$((e_x, \overline{x}^\dagger), (e_y, \overline{y}^\dagger)) \in \rho_1 \subseteq \rho_2.$$

Hence $x\rho_2^R y$, and immediately we get $\rho_1^R \subseteq \rho_2^R$. Similarly, we have $\rho_1^\Lambda \subseteq \rho_2^\Lambda$. The reverse implication is obvious. □

Define $\leq$ on $CT(\Gamma)$ by

$$(\rho_1^R, \rho_1^\Lambda) \subseteq (\rho_2^R, \rho_2^\Lambda) \Leftrightarrow \rho_1^R \subseteq \rho_2^R, \rho_1^\Lambda \subseteq \rho_2^\Lambda.$$

Then $CT(\Gamma)$ is a partial ordered set with respect to $\leq$. By Theorem 2.1 and Lemma 2.2, we can easily see that $C(\Gamma)$ and $CT(\Gamma)$ are isomorphic as partial ordered set.

**Proposition 2.3.** *Let* $\Omega \subseteq C(\Gamma)$ *and* $T_\rho = (\rho^R, \rho^\Lambda)$ *where* $\rho \in \Omega$. *Then*

$$T_{(\cap_{\rho\in\Omega}\rho)} = (\bigcap_{\rho\in\Omega} \rho^R, \bigcap_{\rho\in\Omega} \rho^\Lambda)$$

*and*

$$T_{(\bigvee_{\rho\in\Omega}\rho)} = (\bigvee_{\rho\in\Omega} \rho^R, \bigvee_{\rho\in\Omega} \rho^\Lambda).$$

*Proof.* The first equality is obvious, we only need to prove the second equality. Let $x, y \in R$ be such that $x(\bigvee_{\rho\in\Omega}\rho)^R y$. Then

$$i = (e_x, \overline{x}^\dagger) \bigvee_{\rho\in\Omega} \rho (e_y, \overline{y}^\dagger) = j.$$

Hence, there exist $\rho_i \in \Omega$ and $a_i = (e_{x_i}, \overline{x_i}^\dagger) \in \Gamma$ such that

$$i\rho_1 a_1 \rho_2 a_2 \cdots a_{n-1}\rho_n j.$$

This implies that

$$x\rho_1^R x_1 \rho_2^R x_2 \cdots x_{n-1}\rho_n^R y.$$

We have proved that

$$(\bigvee_{\rho\in\Omega}\rho)^R \subseteq (\bigvee_{\rho\in\Omega}\rho^R).$$

$(\bigvee_{\rho\in\Omega}\rho^R) \subseteq (\bigvee_{\rho\in\Omega}\rho)^R$ is obvious. The dually equality can be proved similarly. □

Now, by summing up the above results, we obtain the following theorem.

**Theorem 2.4.** *Let* $\Gamma$ *be can constructed in Lemma 1.6. The* $CT(\Gamma)$ *forms a complete lattice with respect to* $\leq$ *and* $C(\Gamma)$ *is isomorphic to* $CT(\Gamma)$ *as complete lattice.*

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