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HYBRID *d*-ARY TREES AND THEIR GENERALIZATION

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Abstract. We enumerate black and white colored d-ary trees with no leftmost $\diagup$-edges, which is a generalization of hybrid binary trees. Then the multi-colored hybrid d-ary trees with the same condition is studied.

1. Introduction

In 1994, J. Pallo [3] introduced hybrid binary trees as equivalence classes with respect to associative property of internal nodes, which was to construct an easier data process in computer systems. Mansour et al. [1] in 2008 mentioned that $\diagdown$-free two colored binary trees are hybrid binary trees, and considered several types of "X-free" bicolored binary trees and enumerated them. In 2009, Panholzer and Prodinger [4] studied d-ary trees with no rightmost $\diagdown$-edges and found a closed formula as a generalized Catalan numbers. They also enumerate k colored d-ary trees with $\{1, 2, \ldots, k\}$ colors, where there is no internal vertex i of the rightmost child j with $i > j$. There are some other results on special cases of two colored binary trees [2] and [6].

We examined J. Pallo's [3] hybrid binary trees with associativity property and found that it could be generalized to the hybrid d-ary trees with a simple representation rather than the associativity. We also consider coloring vertices with more than two colors for the hybrid d-ary trees. We remark Pallo's hybrid binary trees on the last section.

In this paper we first study two colored d-ary trees with no leftmost $\diagup$-edges for $d \geq 1$ and enumerate them. In the section 3 the multi-colored d-ary trees with set $\{1, 2, \ldots, p+q\}$ of colors, with no leftmost (i, i)-edges for $i \in \{1, 2, \ldots, p\}$ is also studied and enumerated.

2. Hybrid *d*-ary trees

We define hybrid trees as follows:

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Definition 2.1. A hybrid d-ary tree is a d-ary tree where every internal vertex is labeled with either 1 or 2 and every leaf with 0, but with no leftmost $(1,1)$-labeled edges, i.e., an edge consisted of 1 labeled internal vertex with 1 labeled leftmost child of it.

By applying the preorder traversal (i.e., visit the root and then visit subtrees from left to right) to a hybrid d-ary tree we obtain a word of alphabet $\{0,1,2\}$. We say the subword separated by 0's a *block*. Then we have the following proposition derived straightforwardly from the definition.

Proposition 2.2. *Every block of a hybrid d-ary tree is a Fibonacci word which is a word of $\{1,2\}$ with no consecutive 1's.*

Example 2.3. Consider the following hybrid ternary tree (Red numbered edges will be used for decomposition in the next example).

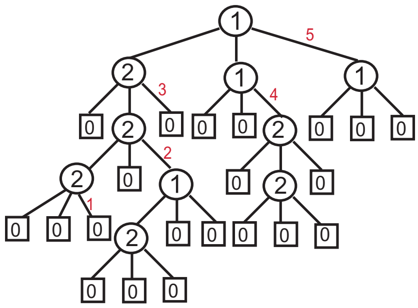

FIGURE 1. A hybrid ternary tree

By the preorder traversal we obtain the word

$$120\ 220\ 0\ 0\ 0\ 120\ 0\ 0\ 0\ 0\ 0\ 10\ 0\ 20\ 20\ 0\ 0\ 0\ 10\ 0\ 0,$$

where each block, the word separated by 0, is a Fibonacci word of $\{1,2\}$.

By replacing every 1 by black color and every 2 by white in a hybrid d-ary tree, we have the -free d-ary tree. If we switch the colors or the left-right order of the children we have the same number of -free d-ary trees, -free d-ary trees, -free d-ary trees, and -free d-ary trees.

We now consider factorizing hybrid d-ary trees to obtain a functional relation of a generating function for the hybrid trees as follows:

For the decomposition of a hybrid d-ary tree with n internal vertices we take the following steps:

(1) Apply the preorder traversal.

(2) Every time the traversal reads the rightmost edge of an internal vertex, remove the edge and the subtree attached below it, if there is.

(3) Arrange the removed subtrees from Step (2) in a row from left to right.

(4) If the traversal is over, stop.

Then the output is a sequence of a hybrid $(d-1)$-ary tree followed by right most edges along with their d-ary subtrees, if there exist. The following figure shows the decomposition of the ternary tree in Example 2.3.

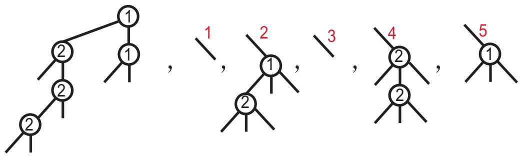

FIGURE 2. Decomposition of a hybrid ternary tree

The recovery is fairly straightforward. While traversing the $(d-1)$-ary tree, every time an internal vertex needs the dth child, take the corresponding edge with the subtree in the sequence for the child. From this observation we have the following proposition.

Proposition 2.4. *Let $h_d(x)$ be a generating function for the number of Hybrid d-ary trees each of whose internal vertex is weighted by an x for $d \geq 2$. Then the initial condition for $d = 1$ becomes $h_1(x) = \sum_{n\geq 0} F_{n+1}x^n$, where F_n is the nth Fibonacci number ($F_0 = F_1 = 1$). We also have*

$$\begin{aligned} h_d(x) &= h_{d-1}(xh_d(x)) \\ &= h_1(x \cdot h_d(x)^{d-1}). \end{aligned}$$

Proof. Since the decomposition of hybrid d-ary tree T is unique, we can express the decomposition as

$$T_0,\ e_1T_1,\ e_2T_2,\ \ldots,\ e_kT_k,$$

where T_0 is a hybrid $(d-1)$-ary tree with k internal vertices and e_iT_i's are ith removed rightmost edges in traversal along with T_i hybrid d-ary subtrees, if there is. From the recovery process, the hybrid d-ary tree is obtained by attaching hybrid d-ary subtrees e_iT_i's into the internal vertices of T_0 as the rightmost children. In other words, we get $h_d(x) = h_{d-1}(xh_d(x))$, which gives by iterating the equation $h_1(x \cdot h_d(x)^{d-1})$. □

By Proposition 2.4 and the Lagrange Inversion Formula we obtain the following theorem.

Theorem 2.5. *The number of hybrid d-ary trees with n internal vertices is*

$$\frac{1}{(d-1)n+1} \sum_{i=0}^{n} \binom{(d-1)n+i}{i} \binom{(d-1)n+i+1}{n-i}.$$

Proof. From the above Proposition 2.4, we have $h_d(x) = h_1(x(h_d(x))^{d-1})$, and taking $(d-1)$-th powers yields

$$x(h_d(x))^{d-1} = x\left(h_1(x(h_d(x))^{d-1})\right)^{d-1}.$$

Let $f = x(h_d(x))^{d-1}$, we have $f = x(h_1(f))^{d-1}$. Applying Lagrange Inversion Formula (LIF) [Chap 5. in [5]],

$$\begin{aligned}
[x^n]h_d(x) &= [x^n]h_1(f(x)) \\
&= \frac{1}{n}[x^{n-1}]h_1'(x)\,(h_1(x))^{(d-1)n} \\
&= \frac{1}{n}[x^{n-1}]\left(\frac{h_1(x)^{(d-1)n+1}}{(d-1)n+1}\right)' \\
&= \frac{1}{(d-1)n+1}[x^n]\,(h_1(x))^{(d-1)n+1} \\
&= \frac{1}{(d-1)n+1}[x^n]\left(\frac{x+1}{1-x-x^2}\right)^{(d-1)n+1} \\
&= \frac{1}{(d-1)n+1}[x^n](x+1)^{(d-1)n+1}\sum_{l\geq 0}\binom{(d-1)n+l}{l}x^l(1+x)^l \\
&= \frac{1}{(d-1)n+1}\sum_{i=0}^{n}\binom{(d-1)n+i}{i}\binom{(d-1)n+i+1}{n-i}
\end{aligned}$$

where $[x^n]g(x)$ is the coefficient of x^n in $g(x)$. □

Example 2.6. The generating function $h_3(x)$ for the hybrid ternary trees is

$$\begin{aligned}
h_3(x) &= h_1(xh_3(x)^2) \\
&= \sum_{n\geq 0} F_{n+1}\cdot x\cdot h_3(x)^2.
\end{aligned}$$

Thus we have

$$x^2h_3(x)^5 + xh_3(x)^3 + xh_3(x)^2 - h_3(x) - 1 = 0.$$

Therefore, the number of the hybrid ternary trees with n internal vertices becomes

$$\frac{1}{2n+1}\sum_{i=0}^{n}\binom{2n+i}{i}\binom{2n+i+1}{n-i}.$$

The following table shows first few values of the coefficients of x^n in $h_d(x)$.

$d_k(n)$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	OEIS
$d=1$	1	2	3	5	8	13	21	A000045
$d=2$	1	2	7	31	154	820	4575	A007863
$d=3$	1	2	11	81	684	6257	60325	A215654
$d=4$	1	2	15	155	1854	24124	331575	-
$d=5$	1	2	19	253	3920	66221	1183077	-
$d=6$	1	2	23	375	7138	148348	3262975	-

3. The (p, q)-hybrid d-ary trees

We generalize hybrid d-ary trees further by considering more labelings.

Definition 3.1. A (p, q)-hybrid d-ary tree is a d-ary tree where every internal vertex is labeled with $\{1, 2, \ldots, p+q\}$ and every leaf with 0, but with no leftmost (i, i)-labeled edges for $i \in \{1, 2, \ldots, p\}$, i.e., there is no edge consisted of i labeled internal vertex with i labeled leftmost child of it.

By applying the preorder traversal again to a (p, q)-hybrid d-ary tree we obtain a word of alphabet $\{0, 1, 2, \ldots, p+q\}$. We still use a *block* for a subword separated by 0's. Then the (p, q)-hybrid d-ary trees have the following property from Definition 3.1.

Proposition 3.2. *Every block of a (p, q)-hybrid d-ary tree is a (p, q)-generalized Fibonacci word, which is a word of $\{1, 2, \ldots, p+q\}$ with no consecutive i's for any element $i \in \{1, 2, \ldots, p\}$.*

Let $G_{p,q}(n)$ be the set of all (p, q)-generalized Fibonacci words of length n and let $g_{p,q}(n)$ be the cardinality of $G_{p,q}(n)$. Then it is not difficult to see that the number $g_{p,q}(n)$ satisfies the following recurrence relation for $n \geq 2$:

$$g_{p,q}(n) = (p+q-1) \cdot g_{p,q}(n-1) + q \cdot g_{p,q}(n-2),$$

where $g_{p,q}(0) = 1$ and $g_{p,q}(1) = p+q$.

Since the decomposition of a (p, q)-hybrid d-ary tree is uniquely determined in a similiar fashion as we have seen in the previous section, we get the following proposition.

Proposition 3.3. *Let $h_d^{(p,q)}(x)$ be a generating function for the number of (p, q)-hybrid d-ary trees with n internal vertices for $d \geq 2$. For $d = 1$, we have the initial condition $h_1^{(p,q)}(x) = \sum_{n \geq 0} g_{p,q}(n)x^n = \frac{1+x}{1-(p+q-1)x-qx^2}$. Then*

$$qx^2 \ h_d^{(p,q)}(x)^{2d-1} + (p+q-1)xh_d^{(p,q)}(x)^d + xh_d^{(p,q)}(x)^{d-1} - h_d^{(p,q)}(x) + 1 = 0.$$

Proof. Since the structure is the same as the hybrid d-ary trees, we have the following equation.

$$\begin{aligned} h_d^{(p,q)}(x) &= h_{d-1}^{(p,q)}(xh_d^{(p,q)}(x)), \\ &= h_1^{(p,q)}(x \cdot h_d^{(p,q)}(x)^{d-1}). \end{aligned} \tag{1}$$

That is,

$$h_d^{(p,q)}(x) = \frac{1 + x \cdot h_d^{(p,q)}(x)^{d-1}}{1 - (p+q-1)x \cdot h_d^{(p,q)}(x)^{d-1} - qx^2 \cdot h_d^{(p,q)}(x)^{2d-2}}.$$

Thus

$$qx^2\ h_d^{(p,q)}(x)^{2d-1} + (p+q-1)xh_d^{(p,q)}(x)^d + xh_d^{(p,q)}(x)^{d-1} - h_d^{(p,q)}(x) + 1 = 0.$$

□

Since $h_d^{(p,q)}(x) = h_1^{(p,q)}(xh_d^{(p,q)}(x)^{d-1})$, by taking $(d-1)$-th powers and substitute $f = x(h_d^{(p,q)}(x))^{d-1}$, we have $f = x(h_1^{(p,q)}(f))^{d-1}$. By LIF [Chap 5. in [5]], the number of (p,q)-hybrid d-ary trees with n internal vertices is equal to $\frac{1}{(d-1)n+1}$ times the coefficient of x^n in $h_1^{(p,q)}(x)^{(d-1)n+1}$. Thus we have the following formula.

Theorem 3.4. *The number of (p,q)-hybrid d-ary trees with n internal vertices is*

$$\frac{1}{(d-1)n+1}\sum_{k=0}^{n}\sum_{i=\lceil \frac{k}{2}\rceil}^{k}\binom{(d-1)n+1}{n-k}\binom{(d-1)n+i}{i}\binom{i}{k-i}(p+q-1)^{2i-k}q^{k-i}.$$

If $p = 1$ in Proposition 3.3, then the generating function can be factored out, which enables us to find the function. In other words:

Corollary 3.5. *Let $f(x)$ be a formal power series with $f(0) = 1$ and $f'(0) = 1 + q$ for a positive integer q, and satisfy*

$$f(x) = (1 + x\ f(x)^{d-1})(1 + q\ x\ f(x)^d).$$

Then the coefficient of x^n in $f(x)$ is

$$\frac{1}{(d-1)n+1}\sum_{k=0}^{n}\sum_{i=\lceil \frac{k}{2}\rceil}^{k}\binom{(d-1)n+1}{n-k}\binom{(d-1)n+i}{i}\binom{i}{k-i}q^i.$$

Proof. Since $h_d^{(1,q)}$ satisfy the functional equation $f(x) = (1 + x\ f(x)^{d-1})(1 + q\ x\ f(x)^d)$, the statement is true. □

Remark 3.6. J. Pallo [3] defined hybrid binary tree with internal vertices in $\{a, n\}$ where a denotes for an associative vertices and n for non-associative one. The associativity of internal vertex a imply to $[[T_1, a, T_2], a, T_3]$ is equivalent to $[T_1, a, [T_2, a, T_3]]$ where $[A, a, B]$ means A and B are the left and right trees of internal vertex a. The equivalent classes by associative property in a $\{a, n\}$ labelled binary trees is to be defined as hybrid binary trees.

The generalized hybrid d-ary trees is based on the unique decomposition of $(d-1)$-ary trees by right most edges of each internal vertices as in Figure 2. By recursive using decompositions up to each component being a binary tree, the equivalence classes could be determined by associativity property of internal vertex a in hybrid binary trees.

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