

A NOTE ON EXPONENTIAL ALMOST SURE STABILITY OF STOCHASTIC DIFFERENTIAL EQUATION

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ABSTRACT. Our goal is to relax a sufficient condition for the exponential almost sure stability of a certain class of stochastic differential equations. Compared to the existing theory, we prove the almost sure stability, replacing Lipschitz continuity and linear growth conditions by the existence of a strong solution of the underlying stochastic differential equation. This result is extendable for the regime-switching system. An explicit example is provided for the illustration purpose.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a filtered probability space satisfying general conditions, on which $B(t) \in \mathbb{R}^m$ is a standard m -dimensional Brownian motion. Consider a stochastic differential equation (SDE) of

$$(1) \quad dX(t) = f(X(t), t)dt + g(X(t), t)dB(t), \quad X(t_0) = x_0,$$

where $X(t) \in \mathbb{R}^d$, $f : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times m}$. It is well known that SDE (1) has a unique strong solution under the following assumption:

Assumption 1. f and g obey the local Lipschitz continuity and Linear growth condition.

In the past a few decades, the stability of such a system has been well studied as an important feature, see more details in [1, 3] and the references therein. Our interest of this note is to relax a sufficient condition for the almost sure exponential stability based on the existing result of Theorem 3.3 of the book [3].

For convenience, we define an operator L by, for all $\varphi \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$,

$$L\varphi(x, t) = \varphi_t(x, t) + \varphi_x(x, t) \cdot f(x, t) + \frac{1}{2} \text{Trace} \left(\varphi_{xx}(x, t)(gg^T)(x, t) \right).$$

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Theorem 3.3 of the book [3] provides a sufficient condition for the exponential almost sure stability under the following assumptions, and its proof is referred to [3].

Theorem 1 (Theorem 3.3 of [3]). *Suppose SDE (1) satisfies Assumption 1 and*

$$(2) \quad f(0, t) = g(0, t) = 0, \quad \forall t \in [t_0, \infty).$$

Also, assume that there exist a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, and constants $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$ such that for all $x \neq 0$ and $t \geq t_0$,

$$(H1) \quad c_1|x|^p \leq V(x, t),$$

$$(H2) \quad LV(x, t) \leq c_2V(x, t),$$

$$(H3) \quad |V_x(x, t) \cdot g(x, t)|^2 \geq c_3V^2(x, t).$$

Then,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |X(t; t_0, x_0)| \leq -\frac{c_3 - 2c_2}{2p} \quad a.s.$$

In particular, SDE (1) is exponentially stable in almost sure sense, if $c_3 > 2c_2$.

Together with Assumption 1, equation (2) implies zero is an absorbing state for $X(t)$. Moreover, one can also conclude $X(t) > 0$ almost surely if the initial x_0 is given strictly positive.

In this below, we will present a stability result with weaker sufficient condition. In particular, we will remove Assumption 1 and (2). Instead, we impose the existence of the strong solution as follows.

Assumption 2. SDE (1) has a strong solution for any initial x_0 .

The proof of the next result is relegated to the next section for better illustration.

Theorem 2 (Main stability result). *If SDE (1) satisfies Assumption 2, and assume that there exist a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, and constants $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$ such that for $t \geq t_0$, (H1)-(H3) are satisfied. Then*

$$(3) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log |X(t; t_0, x_0)| \leq -\frac{c_3 - 2c_2}{2p} \quad a.s.$$

In the above, we extend the definition to $\log 0 = -\infty$.

It is well known that Assumption 1 is only a sufficient condition for the existence of a strong solution, but not a necessary condition (see more discussions in Chapter 5 of [2]). In the next section after the proof of Theorem 2, we will provide an example, which can be verified to be almost surely exponential stable by Theorem 2, but not by Theorem 1. Section 3 includes an extension of the result to regime-switching diffusion and other possible generalizations.

2. Proof of the main result and an example

In this section, we will provide a proof of Theorem 2 and an example for the illustration purpose.

2.1. Proof of Theorem 2

In the above Assumption 2 we do not exclude possibly zero solutions. Therefore, although $\log V(x(t), t)$ is well defined for all $t \in (0, \infty)$ under extended logarithm, it is not directly applicable to Ito's formula to calculate $d \log V(x(t), t)$ due to possible value $\log 0$. For this reason, we define a new strictly positive function

$$V^\eta(x, t) = V(x, t) + e^{-\eta t}$$

for some constant $\eta > 0$ to be given by (9). By Ito's formula on V^η , we obtain

$$dV^\eta(X(t), t) = (LV(X(t), t) - \eta e^{-\eta t})dt + (V_x \cdot g)(X(t), t)dB_t.$$

For the strictly positive function V^η , we can apply Ito's formula on $\log V^\eta(X(t), t)$ to obtain

$$\begin{aligned} & d \log V^\eta(X(t), t) \\ &= \frac{1}{V^\eta(X(t), t)} dV^\eta(X(t), t) - \frac{1}{2} \frac{1}{V^\eta(X(t), t)^2} d\langle V^\eta, V^\eta \rangle(X(t), t) \\ (4) \quad &= \left(\frac{LV}{V^\eta}(X(t), t) - \frac{1}{2} \left| \frac{V_x \cdot g}{V^\eta} \right|^2(X(t), t) - \frac{\eta e^{-\eta t}}{V^\eta}(X(t), t) \right) dt \\ &+ \frac{V_x \cdot g}{V^\eta}(X(t), t) dB_t. \end{aligned}$$

Rewrite it into integral form

$$\begin{aligned} & \log V^\eta(X(t), t) \\ &= \log V^\eta(x_0, t_0) \\ (5) \quad &+ \int_{t_0}^t \left(\frac{LV}{V^\eta}(X(s), s) - \frac{1}{2} \left| \frac{V_x \cdot g}{V^\eta} \right|^2(X(s), s) - \frac{\eta e^{-\eta s}}{V^\eta}(X(s), s) \right) ds \\ &+ \int_{t_0}^t \frac{V_x \cdot g}{V^\eta}(X(s), s) dB_s. \end{aligned}$$

Denote the last term by

$$M(t) = \int_{t_0}^t \frac{V_x \cdot g}{V^\eta}(X(s), s) dB_s.$$

Note that $M(t)$ is a continuous martingale with initial value $M(t_0) = 0$. Assign $\epsilon \in (0, 1)$, by exponential martingale inequality (see [5, 7])

$$\mathbb{P} \left(\sup_{0 \leq t \leq n} [M(t) - \frac{\epsilon}{2} \int_{t_0}^t \frac{|V_x \cdot g|^2}{|V^\eta|^2}(X(s), s) ds] > \frac{2}{\epsilon} \log n \right) \leq \frac{1}{n^2}.$$

Applying Borel-Cantelli lemma, we will see that for almost all $\omega \in \Omega$, there is an integer $n_0 = n_0(\omega)$ such that when $n \geq n_0$

$$(6) \quad M(t) \leq \frac{2}{\epsilon} \log n + \frac{\epsilon}{2} \int_{t_0}^t \frac{|V_x \cdot g|^2}{|V^\eta|^2}(X(s), s) ds$$

for all $0 \leq t \leq n$.

Using (H2) and (H3) on (5) and (6), it leads to the inequality

$$(7) \quad \log V^\eta(X(t), t) \leq \log V^\eta(x_0, t_0) + \int_{t_0}^t I(s) ds + \frac{2}{\epsilon} \log n,$$

where

$$I(s) = \frac{c_2 V}{V^\eta}(X(s), s) - \frac{\frac{1}{2}(1-\epsilon)c_3|V|^2}{|V^\eta|^2}(X(s), s) - \frac{\eta e^{-\eta s}}{V^\eta}(X(s), s).$$

Define $G : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$G(y) = \frac{c_2 y}{y+1} - \frac{(1-\epsilon)c_3 y^2}{2(y+1)^2} - \frac{\eta}{y+1}.$$

In terms of function $G(\cdot)$, $I(s)$ has representation of

$$(8) \quad I(s) = G\left(V(X(s), s) \cdot e^{\eta s}\right).$$

Now, by taking special η of

$$(9) \quad \eta := (1-\epsilon)c_3 - c_2,$$

one can observe that

$$G'(y) = \frac{(1-\epsilon)c_3}{(y+1)^3} \geq 0, \quad \forall y \in (0, \infty)$$

which means $G(\cdot)$ is an non-decreasing function on $(0, \infty)$. Hence, we can find an upper bound of G by calculating its limit at infinity, that is

$$G(y) \leq c_2 - \frac{1}{2}(1-\epsilon)c_3 = \lim_{y \rightarrow \infty} G(y), \quad \forall y \in \mathbb{R}_+.$$

Together with (8), we have

$$I(s) \leq c_2 - \frac{1}{2}(1-\epsilon)c_3.$$

With this inequality, we rewrite (7) as

$$(10) \quad \log V^\eta(X(t), t) \leq \log V^\eta(x_0, t_0) + \left(c_2 - \frac{1}{2}(1-\epsilon)c_3\right)(t-t_0) + \frac{2}{\epsilon} \log n$$

for all $0 \leq t \leq n$. Therefore, if t is chosen as $n-1 \leq t \leq n$, then

$$\frac{1}{t} \log V^\eta(X(t), t) \leq \frac{1}{t} \log V^\eta(x_0, t_0) + \left(c_2 - \frac{1}{2}(1-\epsilon)c_3\right) \frac{t-t_0}{t} + \frac{2}{\epsilon} \frac{\log(t+1)}{t}.$$

Passing the limit $t \rightarrow \infty$, we obtain

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log V^\eta(X(t), t) \leq \left(c_2 - \frac{1}{2}(1-\epsilon)c_3\right).$$

Finally, we reach the desired result (3) as a consequence of the arbitrariness of $\epsilon \in (0, 1)$, (H1), and $V \leq V^\eta$.

2.2. Example

Consider SDE given by

$$(11) \quad dX(t) = -X(t)dt + 2e^{-t/2}X^{1/2}(t)dB(t), \quad X(0) \geq 0.$$

SDE (11) violates Assumption 1 due to its coefficient of $dB(t)$. On the other hand, SDE (11) satisfies Assumption 2, since we can find a strong solution of the form $X(t) = e^{-t}Y(t)$, where $Y(t)$ is a squared Bessel process with dimension $\delta = 0$, namely $BESQ^0$, which satisfies

$$dY(t) = 2\sqrt{Y(t)}dB(t), \quad Y_0 = x_0$$

(see Chapter XI of [6]). In fact, the above $X(t)$ is the unique solution of SDE (11) by Theorem IX.3.5 of [6].

Set Liapunov function $V(x)$ by $V(x) = x$ on $(x, t) \in \mathcal{O} := \mathbb{R}_+ \times \mathbb{R}_+$. We only define V on \mathcal{O} , since $X(t) > 0$ almost surely provided that $x_0 > 0$. Then, we can verify that V satisfies (H1)-(H3) with the constants $p = 1, c_1 = 1, c_2 = -1, c_3 = 0$. Thus, $X(t)$ is exponentially stable in the almost sure sense, with

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |X(t)| \leq -1.$$

3. Further extensions

In this part, we will present two generalizations of the stability result. The proof can be completed using exactly the same methodologies discussed above.

3.1. Regime-switching model

Let $\gamma(\cdot)$ be a continuous Markov chain taking value in a finite state space $\mathcal{M} = \{1, 2, \dots, m\}$ with generator $Q(x) = q_{ij}(x)$ satisfying $(q_{ij}(x))_{i,j \in \mathcal{M}} \geq 0$ for $j \neq i$ and $\sum_{j \in \mathcal{M}} q_{ij}(x) = 0$ for all $x \in \mathbb{R}^d$ and $i \in \mathcal{M}$. Associated to the Markov chain γ , we consider following stochastic differential equation:

$$(12) \quad dX(t) = f(X(t), \gamma(t), t)dt + g(X(t), \gamma(t), t)dB(t), \quad X(t_0) = x_0, \gamma(t) = \gamma_0.$$

The stochastic differential equation of the form (12) is often referred to regime-switching diffusion. The applications of such a system have been witnessed in various areas, such as signal processing, production planning, biological systems (see more in [4, 8, 9]). The generator of (12) can be written by, for $\varphi \in C^{2,1}(\mathbb{R}^d \times \mathcal{M} \times \mathbb{R}_+; \mathbb{R})$

$$\begin{aligned} L\varphi(x, i, t) &= \varphi_t(x, i, t) + \varphi_x(x, i, t) \cdot f(x, i, t) \\ &\quad + \frac{1}{2} \text{Trace} \left(\varphi_{xx}(x, i, t)(gg^T)(x, i, t) \right) \\ &\quad + \sum_{\mathcal{M} \ni j \neq i} q_{ij}(x) \left(\varphi(x, j, t) - \varphi(x, i, t) \right). \end{aligned}$$

If we go through similar steps in Section 2, we obtain the following result on the stability, and the proof is omitted to keep this note brief.

Theorem 3. *Suppose SDE (12) has the unique strong solution, and there exist a function $V \in C^{2,1}(\mathbb{R}^d \times \mathcal{M} \times [t_0, \infty); \mathbb{R}_+)$, and constants $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$ such that for all $x \neq 0, i \in \mathcal{M}$ and $t \geq t_0$,*

$$\begin{aligned} \text{(H1')} \quad & c_1|x|^p \leq V(x, i, t), \\ \text{(H2')} \quad & LV(x, i, t) \leq c_2V(x, i, t), \\ \text{(H3')} \quad & |V_x(x, i, t) \cdot g(x, i, t)|^2 \geq c_3V^2(x, i, t). \end{aligned}$$

Then,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |X(t)| \leq -\frac{c_3 - 2c_2}{2p} \quad a.s.$$

In particular, SDE (12) is exponentially stable in almost sure sense, if $c_3 > 2c_2$.

3.2. Delay systems

In this part, we consider stochastic differential delay equations (see [10] and the references therein):

$$(13) \quad dX(t) = f(X(t), X(t - \delta(t)), t)dt + g(X(t), X(t - \delta(t)), t)dB(t)$$

with initial data

$$X(t) = \psi(t), \quad \forall t \in [t_0 - \delta(t_0), t_0],$$

where $X(t) \in \mathbb{R}^d, f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^d, g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times m}$. We define an operator $L : C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}) \mapsto L^0(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}; \mathbb{R})$ by, for all $\varphi \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$,

$$L\varphi(x, y, t) = \varphi_t(x, t) + \varphi_x(x, t) \cdot f(x, y, t) + \frac{1}{2} \text{Trace} \left(\varphi_{xx}(x, t)(gg^T)(x, y, t) \right).$$

Assumption 3. SDE (13) has a strong solution for any initial x .

Theorem 4. *If SDE (13) satisfies Assumption 3 and*

$$(14) \quad f(0, y, t) = g(0, y, t) = 0, \forall (y, t) \in \mathbb{R}^d \times \mathbb{R}_+,$$

and assume that there exist a function $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}_+)$, and constants $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$ such that for $t \geq t_0$,

$$\begin{aligned} \text{(H1'')} \quad & c_1|x|^p \leq V(x, t), \\ \text{(H2'')} \quad & LV(x, y, t) \leq c_2V(x, t), \\ \text{(H3'')} \quad & |V_x(x, t)g(x, y, t)|^2 \geq c_3V^2(x, t). \end{aligned}$$

Then

$$(15) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log |x(t; t_0, x_0)| \leq -\frac{c_3 - 2c_2}{2p} \quad a.s.$$

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