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GRADIENT RICCI SOLITONS WITH SEMI-SYMMETRY

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Abstract. We prove that a semi-symmetric 3-dimensional gradient Ricci soliton is locally isometric to a space form $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{R}^3$ (Gaussian soliton); or a product space $\mathbb{R}\times\mathbb{S}^2$, $\mathbb{R}\times\mathbb{H}^2$, where the potential function depends only on the nullity.

1. Introduction

A *Ricci soliton* is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$\frac{1}{2}\pounds_V g + \mathrm{Ric} - \lambda g = 0, \tag{1}$$

where V is a vector field (the potential vector field), λ a constant on M. Compact Ricci solitons are the fixed points of the Ricci flow: $\frac{\partial}{\partial t}g = -2\,\mathrm{Ric}$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, respectively. Hamilton [4] and Ivey [5] proved that a Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3, respectively. If the vector field $V = Dh$, the gradient of a potential function h, then g is called a gradient Ricci soliton and equation (1) assumes the form

$$\mathrm{Hess\ h} + \mathrm{Ric} = \lambda g, \tag{2}$$

Trivial examples are given by Einstein metrics with $\mathrm{Hess\ h} = 0$. Another interesting soliton occurs on Euclidean space $\mathbb{R}^n$ (with a flat metric). Indeed, assuming $h = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^n$ then we have $\mathrm{Hess\ h} = \lambda g$. Therefore it yields a gradient Ricci soliton, which is called a *Gaussian soliton*. Due to Perelman's result (Remark 3.2 in [6]), we find that the potential vector field in a compact Ricci soliton is written as the sum of a gradient and a Killing vector field. For details we refer to [3] or [2] about the Ricci flows and their solitons.

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Let R denote its Riemann curvature tensor on a Riemannian manifold (M, g). Cartan [1] investigated M satisfying the condition $R(X,Y)\cdot R = 0$ for all vector fields X, Y on M. Here the linear endomorphism $R(X,Y)$ acts as a derivation on R. Such a Riemannian manifold is called a semi-symmetric space. In dimension three, Sekigawa [8] proved that a complete and irreducible semi-symmetric space is of constant curvature if it has a finite volume. Other than locally symmetric spaces, the fundamental examples of semi-symmetric spaces are real cones, Kählerian cones, and spaces foliated by Euclidean spaces of codimension 2. Their local and global structures are intensively investigated by Szabo [9], [10].

In the present paper, we prove that a *semi-symmetric 3-manifold M admitting a gradient Ricci soliton is locally isometric to one of the following*: $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{R}^3$; $\mathbb{S}^2\times\mathbb{R}$, *or* $\mathbb{H}^2\times\mathbb{R}$, *where the potential function depends only on the nullity distribution for the latter two cases.*

All manifolds in the present paper are assumed to be connected and smooth.

2. Ricci solitons with semi-symmetry

First, we remind a result due to Petersen and Wylie [7].

Proposition 1. *If a gradient Ricci soliton is Einstein, then* Hess h $= 0$ *or it is a Gaussian soliton.*

Now, we prove:

Theorem 2. *If a semi-symmetric 3-manifold M admits a gradient Ricci soliton, then M is locally isometric to $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{R}^3$ (Gaussian soliton) or a product space $M_1^2\times\mathbb{R}$ almost everywhere.*

Proof. Let (M, g) be a semi-symmetric 3-manifold. Then M satisfies

$$R(X,Y) = SX\wedge Y + X\wedge SY - (r/2)X\wedge Y,$$

where $(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$, $\mathrm{Ric}(X,Y) = g(SX,Y)$ and r is the scalar curvature. Let (μ_1,μ_2,μ_3) be eigenvalues of the Ricci operator S at $p\in M$. Then the semi-symmetry condition $R(X,Y)\cdot R = 0$ is equivalent to

$$(\mu_i-\mu_j)(2(\mu_i+\mu_j)-R) = 0.$$

This implies only the three cases to be considered (at each point) (cf. [8]):

$$(\mu,\mu,\mu),\ (\mu,\mu,0),\ (0,0,0).$$

If there is an open neighborhood U of p with (μ,μ,μ), $\mu\neq 0$ (or $(0,0,0)$, respectively), then we can see that M is Einstein. Then by Proposition 1 we see that M is a space of constant curvature $\frac{\mu}{2}$ or locally a Gaussian soliton. So, we now assume the case $(\mu,\mu,0)$, $\mu\neq 0$ almost everywhere (in open and dense subset in M). Then, there exists a local orthonormal frame field $\{e_0 = u, e_1, e_2\}$ around p and a differentiable function μ on M such that

$$\begin{cases} R(e_1, e_2) = \mu e_1 \wedge e_2 \\ \text{all others } R(e_i, e_j) \text{ being zero,} \end{cases}$$

where $e_1 \wedge e_2$ denotes $g(\cdot, e_2)e_1 - g(\cdot, e_1)e_2$. The tangent space T_pM at p is decomposed as follows:

$$T_pM = D_0(p) \oplus D_1(p),$$

where $D_1(p) = \mathrm{span}_{\mathbb{R}}\{e_1, e_2\}$ and $D_0(p) = \mathrm{span}_{\mathbb{R}}\{e_0\}$. We put

$$\nabla_{e_i} e_j = \sum_k B_{ijk} e_k \text{ for } i, j, k = 0, 1, 2,$$

where ∇ denotes the Levi-Civita connection. Then we easily see that $B_{ijk} = -B_{ikj}$. Moreover, we get

$$\begin{aligned} &(\nabla_{e_0} R)(e_1, e_2) = \mu_0 \; e_1 \wedge e_2 + \mu(B_{010} e_0 \wedge e_2 + B_{020} e_1 \wedge e_0), \\ &(\nabla_{e_1} R)(e_2, e_0) = \mu B_{101} e_1 \wedge e_2, \\ &(\nabla_{e_2} R)(e_0, e_1) = \mu B_{202} e_1 \wedge e_2. \end{aligned}$$

By the second Bianchi identity, we have

$$\begin{aligned} &(\nabla_{e_0} R)(e_1, e_2) + (\nabla_{e_1} R)(e_2, e_0) + (\nabla_{e_2} R)(e_0, e_1) \\ &= \big(\mu_0 + \mu(B_{101} + B_{202})\big) e_1 \wedge e_2 + \mu(B_{010} e_0 \wedge e_2 + B_{020} e_1 \wedge e_0) = 0 \end{aligned}$$

and it implies that

$$B_{001} = B_{002} = 0 \tag{3}$$

and

$$\mu_0 - \mu(B_{110} + B_{220}) = 0. \tag{4}$$

Thus, from (3) we see that each integral curve of $e_0 = u$ is a totally geodesic leaf. Such a manifold is called a foliated space of totally geodesic leaves and the class of such manifolds is denoted by FOL_1^3.

For the class, we have the Ricci curvature:

$$\begin{cases} \mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_2, e_2) = \mu, \\ \mathrm{Ric}(e_1, e_2) = \mathrm{Ric}(u, e_j) = 0. \end{cases}$$

Suppose that M admits a gradient Ricci soliton with a potential vector field $Dh = h_0 e_0 + h_1 e_1 + h_2 e_2$, where $h_i = g(Dh, e_i)$, $i = 0, 1, 2$. Then from (2) we have

$$g(\nabla_X Dh, Y) + \mathrm{Ric}(X, Y) - \lambda g(X, Y) = 0. \tag{5}$$

Now, let $\gamma(t)$ be an integral curve of u through p, i.e., $\gamma_p(t) = \exp_p tu$. From (5) we can derive

$$R(X, Y)Dh = (\nabla_Y S)X - (\nabla_X S)Y. \tag{6}$$

From $Se_0 = 0$, $Se_1 = \mu e_1$ and $Se_2 = \mu e_2$, we compute

$$\begin{aligned}(\nabla_{e_1} S)e_2 =&(e_1\mu)e_2 + \mu\nabla_{e_1}e_2 - S\nabla_{e_1}e_2\\ =&(e_1\mu)e_2 + \mu B_{120}e_0.\end{aligned}$$

We put $X = e_0$ and $Y = e_1$ in (6). Then we have

$$(\mu B_{110} - \mu_0)e_1 + \mu B_{120}e_2 = 0$$

and it implies that

$$\mu B_{110} = \mu_0 \tag{7}$$

and

$$\mu B_{120} = 0. \tag{8}$$

Similarly, putting $X = e_0$ and $Y = e_2$ in (6), we have

$$\mu B_{210}e_1 + (\mu B_{220} - \mu_0)e_2 = 0$$

and it implies that

$$\mu B_{220} = \mu_0 \tag{9}$$

and

$$\mu B_{210} = 0. \tag{10}$$

Hence, from (4), (7) and (9) we have

$$\mu_0 = 0, \tag{11}$$

where $\mu_0 = \frac{\partial h}{\partial t}$.

Since $\mu \neq 0$, from (7), (9) and (11) we have $B_{110} = B_{220} = 0$. From (8) and (10) we also find that $B_{120} = B_{210} = 0$. From those facts we see that each conullity distribution D_1 is integrable and forms a totally geodesic submanifold. After all, we conclude that M is a local product space of a 1-dimensional $\mathbb{R}$ and a 2-dimensional manifold M_1^2 almost everywhere.

Moreover, we put $X = Y = e_0$ in (5). Then we have

$$\frac{\partial^2 h}{\partial t^2}(= h_{00}) = \lambda.$$

Hence, in this case, the potential function h is of the form

$$h(t, x^1, x^2) = \frac{\lambda}{2}t^2 + \tilde{h}(x^1, x^2)t + \hat{h}(x^1, x^2), \tag{12}$$

where functions $\tilde{h}$ and $\hat{h}$ depend on (x^1, x^2). Here, we assume that (x^1, x^2) is an isothermal coordinate system on M_1^2, i.e.,

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2 \end{pmatrix},$$

where $\{\partial_i := \frac{\partial}{\partial x^i}\}$ is a coordinate basis and $r = r(x^1, x^2)$ is a positive smooth function on M^2. We denote

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ijk}\partial_k \text{ for } i, j, k = 0, 1, 2.$$

Then we can easily show that

$$\begin{cases} \Gamma_{111} = \Gamma_{122} = \Gamma_{212} = -\Gamma_{221} = \frac{r_1}{r}, \\ -\Gamma_{112} = \Gamma_{121} = \Gamma_{211} = \Gamma_{222} = \frac{r_2}{r}, \\ \text{all others } \Gamma_{ijk} \text{ being zero,} \end{cases} \tag{13}$$

where $r_i = \frac{\partial r}{\partial x^i}$, $i = 1, 2$. Let $e_0 = \partial_0, e_1 = \frac{1}{r}\partial_1$ and $e_2 = \frac{1}{r}\partial_2$, then $\{e_i\}$ is an orthnormal frame field around p. Also, a simple calculation shows that

$$\nabla_{e_0} e_j = \nabla_{\partial_0}\frac{1}{r}\partial_j = \frac{1}{r}\sum_k \Gamma_{0jk}\partial_k \quad \text{for } j = 1, 2$$

and hence we have

$$B_{0ji} = \Gamma_{0ji} \quad \text{for } i, j = 1, 2. \tag{14}$$

We put $X = e_0$ and $Y = e_1$ in (5) then we have

$$\frac{\partial^2 h}{\partial t \partial x^1}(=: h_{10}) + h_2 B_{021} = 0$$

and with (13) and (14) it becomes

$$h_{10} = 0. \tag{15}$$

Similarly, putting $X = e_0$ and $Y = e_2$ in (5) we have

$$h_{20} = 0. \tag{16}$$

Consequently, from (15) and (16), we get $\tilde{h}(x^1, x^2) = A$ for some constant A and (12) becomes

$$h(t, x^1, x^2) = \frac{\lambda}{2}t^2 + At + \hat{h}(x^1, x^2).$$

□

Theorem 3. *If a product space $M^2 \times \mathbb{R}$ admits a gradient Ricci soliton whose potential function depends only on $\mathbb{R}$, then M^2 is of constant curvature.*

Proof. We follow the same notation to that of Theorem 2. For a product manifold $M^2 \times \mathbb{R}$, we already found

$$B_{001} = B_{002} = B_{110} = B_{120} = B_{210} = B_{220} = 0. \tag{17}$$

Suppose $M^2 \times \mathbb{R}$ admits a gradient Ricci soliton with a potential vector field $Dh = h_0 e_0$, that is, the gradient field of potential function h depends only on $\mathbb{R}$. We put $X = Y = e_1$ in (5). Then we have

$$\lambda = h_0 B_{101} + \mu$$

and with (17) it becomes

$$\mu = \lambda.$$

Since μ is a constant, M^2 is locally isometric to $\mathbb{S}^2$, $\mathbb{H}^2$ or $\mathbb{R}^2$. In this case, the potential function h is of the form

$$h = \frac{\lambda}{2}t^2 + At + B,$$

where A and B are constants. Such a gradient Ricci soliton is said to be *rigid* ([7]). □

Corollary 4. *A complete and simply-connected semi-symmetric 3-manifold M admitting a gradient Ricci soliton is isometric to* $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{R}^3$; $\mathbb{S}^2 \times \mathbb{R}$ *or* $\mathbb{H}^2 \times \mathbb{R}$, *where the potential function depends only on the nullity distribution for the latter two cases.*

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References

[1] E. Cartan, *Lecons sur la géométrie des espaces de Riemann*, Gauthier-Villars, Paris, 1946.

[2] B. Chow and D. Knopf, *The Ricci Flow: An introduction*, Mathematical Surveys and Monographs 110, American Mathematical Society, 2004.

[3] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[4] ———, *The Ricci flow on surfaces*, Mathematics and general relativity (santa Cruz, CA, 1986), 237–262, Contemp. Math. 71, American Math. Soc., 1988.

[5] T. Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. **3** (1993), no. 4, 301–307.

[6] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, http://arXiv.org/abs/math.DG/02111159.

[7] P. Petersen and W. Wylie, *Rigidity of gradient Ricci solitons*, Pacific J. Math. **241** (2009), no. 2, 329–345.

[8] K. Sekigawa, *On some 3-dimensional complete Riemannian manifolds satisfying* $R(X,Y) \cdot R = 0$, Tôhoku Math. J. **27** (1975), no. 4, 561–568.

[9] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$. *I. The Local version*, J. Diff. Geom. **17** (1982), no. 4, 531–582.

[10] ———, *Structure theorems on Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$. *II, Global versions*, Geom. Dedicata **19** (1985), no. 1, 65–108.

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