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IDENTITIES WITH ADDITIVE MAPPINGS IN SEMIPRIME RINGS

Ajda Fošner and Nadeem ur Rehman

Abstract. The aim of this paper is to prove the next result. Let $n > 1$ be an integer and let R be a $n!$-torsion free semiprime ring. Suppose that $f : R \to R$ is an additive mapping satisfying the relation $[f(x), x^n] = 0$ for all $x \in R$. Then f is commuting on R.

1. Introduction and the main theorem

Throughout, R will represent an associative ring with a center $Z(R)$. Let $n > 1$ be an integer. A ring R is n-torsion free if $nx = 0$, $x \in R$, implies $x = 0$. The Lie product (or a commutator) of elements $x, y \in R$ will be denoted by $[x, y]$ (i.e., $[x, y] = xy - yx$). Recall that a ring R is prime if $aRb = \{0\}$, $a, b \in R$, implies that either $a = 0$ or $b = 0$. Furthermore, a ring R is called semiprime if $aRa = \{0\}$, $a \in R$, implies $a = 0$. We will denote by C and Q the extended centroid and the maximal right ring of quotients of a semiprime ring R, respectively. For the explanation of the extended centroid as well as the maximal right ring of quotients of a semiprime ring we refer the reader to [4]. As usual, the socle of a ring R will be denoted by $\mathrm{soc}(R)$.

An additive mapping $D : R \to R$ is called a derivation on R if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. An additive mapping $f : R \to R$ is called centralizing on R if $[f(x), x] \in Z(R)$ holds for all $x \in R$. In a special case, when $[f(x), x] = 0$ for all $x \in R$, the mapping f is said to be commuting on R. A classical result of Posner [21] (Posner's second theorem) states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Posner's second theorem in general cannot be proved for semiprime rings as shows the following example. Let R_1 and R_2 be prime rings with R_1 commutative and set $R = R_1 \oplus R_2$. Further, let $D_1 : R_1 \to R_1$ be a nonzero derivation. A mapping $D : R \to R$ defined by

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$D((r_1, r_2)) = (D_1(r_1), 0)$ is then a nonzero commuting derivation. It is also easy to show that if $D : R \to R$ is a commuting derivation on a semiprime ring R, then D maps R into $Z(R)$ (see, for example, the end of the proof of Theorem 2.1 in [25]). Furthermore, Brešar [7] proved that every additive commuting mapping of a prime ring R is of the form $x \mapsto \lambda x + \zeta(x)$, where λ is an element of the extended centroid C and $\zeta : R \to C$ is an additive mapping. For results concerning commuting mappings, centralizing mappings and related problems we refer the reader to [1,5–13,18,22–28] where further references can be found.

In [18] Vukman and the first named author generalized the result proved by Brešar and Hvala for prime rings [9].

Theorem 1 ([18, Theorem 2])**.** *Let R be a 2-torsion free semiprime ring. Suppose that an additive mapping $f : R \to R$ satisfies the relation*

$$\left[f(x), x^2\right] = 0$$

for all $x \in R$. Then f is commuting on R.

This result motivated us to prove our main theorem.

Main Theorem. *Let $n > 1$ be a fixed integer and R a $n!$-torsion free semiprime ring. Suppose that an additive mapping $f : R \to R$ satisfies the relation*

$$[f(x), x^n] = 0 \tag{1}$$

for all $x \in R$. Then f is commuting on R.

Let us point out that the above theorem might be of some interest from the functional analysis point of view as well since C^*-algebras (moreover, semisimple Banach algebras) are semiprime.

2. Proof of the main theorem

Let $n > 1$ be a fixed integer. Before proving our main theorem, let us fix some notation and write two results (Lemma 1 and Proposition 1) which we will need in the following. Let $m > 1$ be an integer and $\mathbb{F}$ an arbitrary field. Then $M_m(\mathbb{F})$ denotes the algebra of all $m \times m$ matrices over the field $\mathbb{F}$. Recall that $Z(M_m(\mathbb{F})) = \mathbb{F}I$, where $I \in M_m(\mathbb{F})$ is the identity matrix. By $E_{ij} \in M_m(\mathbb{F})$, $1 \le i, j \le m$, we will denote the matrix with (i, j)-entry equal to one and all the others equal to zero.

Lemma 1. *Let $R = M_m(\mathbb{F})$, $m > 1$, and $A \in R$. Suppose that*

$$[A, X^n] = 0 \tag{2}$$

for all $X \in R$. Then $A \in \mathbb{F}I$.

Proof. Let P be an idempotent matrix in $M_m(\mathbb{F})$. Setting $X = P$ in (2) and multiplying left side by $(I - P)$, we see that $(I - P)AP = 0$ for any idempotent matrix P. Thus, A is a diagonal matrix. Note that UAU^{-1} must be diagonal for each invertible element $U \in M_m(\mathbb{F})$, since $[UAU^{-1}, X^n] = 0$ for all $X \in M_m(\mathbb{F})$. Write $A = \sum_{i=1}^{m} \alpha_i E_{ii}$, where $\alpha_i \in \mathbb{F}$. Then, for each $j > 1$ the $(1, j)$-entry of $(I + E_{1j})A(I + E_{1j})^{-1}$ equals 0. That is, $\alpha_j = \alpha_1$ for $j > 1$. Hence, $A \in \mathbb{F}I$, as desired. □

Proposition 1. *Let R be a non-commutative prime ring and $a \in R$ such that*

$$[a, x^n] = 0$$

for all $x \in R$. Then $a \in Z(R)$.

Proof. Suppose on the contrary that $a \notin Z(R)$. Then

$$f(X) = [a, X^n]$$

is a nontrivial generalized polynomial identity (in the following referred as GPI) for R. Using [14], $f(X)$ is also a GPI for Q. Denote by F either the algebraic closure of C or C itself according to the cases when C is either infinite or finite dimensional, respectively. Then, using a standard argument (e.g., see [19, Proposition]), $f(X)$ is also a GPI for $Q \oplus_C F$. Since $Q \oplus_C F$ is a centrally closed prime F-algebra [15, Theorem 2.5 and Theorem 3.5], by replacing R and C with $Q \oplus_C F$ and F, respectively, we may assume that R is centrally closed and C is either finite dimensional or algebraically closed. In a view of Martindale's theorem [20], R is a primitive ring having a non-zero socle with C as its associated division ring.

Since $a \notin C$, we have $[a, x] \neq 0$ for some $x \in \mathrm{soc}(R)$. By Litoff's theorem [16], there exists an idempotent $e \in \mathrm{soc}(R)$ such that $x, ax, xa \in eRe$. Note that $ef(eXe)e$ is a GPI for R. Thus, $[(eae), X^n]$ is a GPI for eRe. Since $eRe \cong M_m(C)$ for some $m \geq 1$, eae is central in eRe by Lemma 1. It follows that there exists $c \in C$ such that $ce = eae$. Hence, $cx = eaex = eax = ax$. Similarly, $xc = xeae = xae = exae = xa$. So $[a, x] = 0$, a contradiction. Therefore, $a \in Z(R)$, as desired. □

Remark. Let us point out that in Proposition 1 we have no restriction on the characteristic of a non-commutative ring R. But if R is $2n!$-torsion free, then the above proposition is a direct consequence of Theorem 2.1 in [25] (see also Theorem 3 in [17] for the generalization). Namely, if we define an inner derivation $D : R \to R$ by $D(x) = [a, x]$, then $D(x^n) = [a, x^n]$. Therefore, if $[a, x^n] = 0$, then $D(x^n)x + xD(x^n) = 0$ for all $x \in R$ and, by [25, Theorem 2.1], $D(x) = [a, x] = 0$ for all $x \in R$. Thus, $a \in Z(R)$.

Now we are ready to prove our main theorem. In the proof we will use some ideas similar to those used in [28].

Proof of Main Theorem. By semiprimeness of R, there exists a family of prime ideals $\{P_\alpha \ : \ \alpha \in I\}$ such that $\cap_{\alpha \in I} P_\alpha = \{0\}$. Without loss of generality, we may assume that prime rings R/P_α, $\alpha \in I$, are 2-torsion free (see [2, p. 459]).

Now, let us fix an arbitrary $\alpha \in I$. It is sufficient to show that $[f(x), x] \in P_\alpha$ for all $x \in R$. Denote by C the extended centroid of a prime ring R/P_α and by A the central closure of R/P_α. One can consider A as a vector space over the field C which can be regarded as a subspace of A. Thus, there exists a subspace B of A such that $A = B + C$. Let π be the canonical projection of A onto B. For $x \in R$ we shall write $\overline{x}$ for the coset $x + P_\alpha \in R/P_\alpha$. Replacing x by $x + p$ in (1) we obtain

$$[f(p), x^n] \in P_\alpha$$

for all $x \in R$ and $p \in P_\alpha$. Therefore, $[\overline{f(p)}, \overline{x}^n] = 0$ for all $x \in R$. Using Proposition 1, it follows that $\overline{f(p)}$ lies in the center of R/P_α, which means that $[\overline{f(p)}, \overline{x}] = 0$ for all $x \in R$, $p \in P_\alpha$. In particular, we have $\pi\overline{f(p)} = 0$. This yields that the mapping $\overline{f} : R/P_\alpha \to A$, $\overline{f}(\overline{x}) = \pi\overline{f(x)}$, is well defined. It is easy to verify that $\overline{f}$ is additive and satisfies $[\overline{f}(\overline{x}), \overline{x}^n] = 0$ for all $x \in R$. Using [3, Theorem 1.1] it follows that $[\overline{f}(\overline{x}), \overline{x}] = 0$ which in turn implies $[f(x), x] \in P_\alpha$. The proof is completed. □

In [8], Brešar proved that there are no nonzero skew-commuting additive mappings on a 2-torsion free semiprime ring R. In other words, if R is a 2-torsion free semiprime ring and $f : R \to R$ an additive mapping such that $f(x)x + xf(x) = 0$ for all $x \in R$, then $f = 0$. Motivated by this result, we conclude our paper with the following conjecture.

Conjecture. *Let $n \geq 1$ be some fixed integer and let R be a semiprime ring with suitable torsion restrictions. Suppose that an additive mapping $f : R \to R$ satisfies the relation*

$$f(x)x^n + x^n f(x) = 0$$

for all $x \in R$. Then $f = 0$.

In the case $n = 1$, the above conjecture has been proved by Brešar in [8].

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Ajda Fošner
Faculty of Management
University of Primorska
Cankarjeva 5, SI-6104 Koper, Slovenia
E-mail address: ajda.fosner@fm-kp.si

Nadeem ur Rehman
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
E-mail address: rehman100@gmail.com