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ON GORENSTEIN COTORSION DIMENSION OVER GF-CLOSED RINGS

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Abstract. In this article, we introduce and study the Gorenstein cotorsion dimension of modules and rings. It is shown that this dimension has nice properties when the ring in question is left GF-closed. The relations between the Gorenstein cotorsion dimension and other homological dimensions are discussed. Finally, we give some new characterizations of weak Gorenstein global dimension of coherent rings in terms of Gorenstein cotorsion modules.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. For any left R-module M, as usual, $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ will denote the projective, injective and flat dimensions of M, respectively. We use $l.D(R)$ (resp., $wD(R)$) to stand for the left global dimension (resp., weak global dimension) of a ring R. For unexplained concepts and notations, we refer the reader to [1, 15, 23, 24].

As a generalization of the notion of projective dimension of modules, Auslander and Bridger [2] introduced the notion of G-dimension, $\text{G-dim}_R(M)$, for every finitely generated R-module M over a two-sided Noetherian ring. They proved the inequality $\text{G-dim}_R(M) \leqslant \mathrm{pd}_R(M)$, and equality holds if $\mathrm{pd}_R(M)$ is finite. Several decades later, Enochs, Jenda and Torrecillas in [14, 18] extended the G-dimension to modules that are not necessarily finitely generated and introduced three homological dimensions, called Gorenstein projective, injective and flat dimensions. These have been studied extensively by many authors (see, for example, [4, 6, 8, 11, 15, 17, 21, 22, 26]).

Let R be a ring. Recall that a left R-module M is called *Gorenstein flat* in [18], if there exists an exact sequence of flat left R-modules $\mathbf{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ with $M = \ker(F^0 \to F^1)$ such that $I \otimes_R -$ leaves the sequence $\mathbf{F}$ exact whenever I is an injective right R-module. The *Gorenstein flat dimension* of a left R-module M, denoted by $\mathrm{Gfd}_R(M)$, is defined that

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$\mathrm{Gfd}_R(M) \leqslant n$ if and only if M has a Gorenstein flat resolution of length n ([15, 21]). In [6], Bennis and Mahdou introduced the *left weak Gorenstein global dimension* of R, $l.w\mathrm{Ggldim}(R)$, which is defined as the supremum of the Gorenstein flat dimensions of all left R-modules.

It is well known that the class of cotorsion modules plays an important role in the progress of settling the "flat cove conjecture", which was conjectured by Enochs in [12]. A left R-module M is said to be *cotorsion* [26] if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat left R-modules F. The cotorsion dimension of M, denoted by $\mathrm{cd}_R(M)$, is defined to be the smallest integer $n \geqslant 0$ such that $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for any flat right R-module F (if no such n exists, set $\mathrm{cd}_R(M) = \infty$), and the left global cotorsion dimension $l.\mathrm{cot}.D(R)$ of R is defined as $\sup\{\mathrm{cd}_R(M) |\ M \text{ is a left } R\text{-module}\}$ ([23]).

It has been shown that there are many nice results about Gorenstein flat modules over (special) coherent rings. In [11], Ding and Chen characterized n-FC rings (i.e., R is left and right coherent with $\text{FP-id}({}_RR) \leqslant n$ and $\text{FP-id}(R_R) \leqslant n$ for an integer $n \geqslant 0$) in terms of Gorenstein flat modules. Holm proved in [21] that the class of all Gorenstein flat modules over a right coherent ring R is closed under direct sums, direct summands and direct limits ([21, Proposition 3.2, Theorem 3.7]). Inspired by [7] (that is, Bican, Bashir and Enochs in [7] proved the existence of flat covers for every module over any ring), Enochs et al. proved in [16, 17] that the class of Gorenstein flat left R-modules and its right orthogonal class (called Gorenstein cotorsion modules) over right coherent rings form a complete hereditary cotorsion theory, thus all left R-modules over a right coherent ring R have Gorenstein flat covers (see [16, Theorem 2.12] and [17, Theorem 3.1.9]). In the same direction, Yang and Liu [27] generalized the principal results of [16] to a larger class of rings, called left GF-closed rings (i.e., R satisfies the class of Gorenstein flat left R-modules is closed under extensions), they showed all left R-modules over left GF-closed rings have Gorenstein flat covers.

Naturally, many literature take the role of Gorenstein cotorsion modules into consideration. Recall from [17] that a left R-module M is called *Gorenstein cotorsion* if $\mathrm{Ext}^1_R(G, M) = 0$ for all Gorenstein flat left R-modules G. Clearly, every Gorenstein cotorsion module is cotorsion. Many basic properties of the Gorenstein cotorsion modules have been given in [13, 16, 17, 26, 27]. Especially, in [17], Enochs and López-Ramos gave some characterizations on such modules (for more details, see [17, Sections 3.3 and 3.4]). It is an important question to establish the corresponding Gorenstein versions of the numerous interesting results about cotorsion dimension.

The aim of this article is to introduce a concept of Gorenstein cotorsion dimensions of modules and rings and generalize some known results for cotorsion dimension in [23] to Gorenstein cotorsion dimension. In Section 2, the definition and some basic results are given. Then in Section 3, we present some general properties of the Gorenstein cotorsion dimension. It is shown that the Gorenstein cotorsion dimension has nice properties when the ring in question is left GF-closed. The relations between the Gorenstein cotorsion dimension and other homological dimensions are discussed. In Section 4, some applications of Gorenstein cotorsion modules are discussed. It is given some new characterizations of weak Gorenstein global dimension of coherent rings in terms of Gorenstein cotorsion modules. As a corollary, we also obtain some characterizations of FC rings.

Next we recall some known notions and facts needed in the article.

Let $\mathcal{C}$ be a class of left R-modules and M a left R-module. Following [15], we say that a homomorphism $\phi : M \to C$ is a $\mathcal{C}$-preenvelope if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(\phi, C') : \mathrm{Hom}_R(C, C') \to \mathrm{Hom}_R(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a $\mathcal{C}$-envelope if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover. In general, $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist, but if they exist, they are unique up to isomorphism. A $\mathcal{C}$-envelope $\varphi : M \to C$ is said to have the unique mapping property [9] if for any homomorphism $f : M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g : C \to C'$ such that $g\varphi = f$. The concept of an $\mathcal{C}$-cover with the unique mapping property can be defined similarly.

For a class $\mathcal{L}$ of left R-modules, we denote by $\mathcal{L}^{\perp} = \{C : \mathrm{Ext}^1_R(L, C) = 0 \text{ for all } L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by ${}^{\perp}\mathcal{L} = \{C : \mathrm{Ext}^1_R(C, L) = 0 \text{ for all } L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$. A pair $(\mathcal{L}, \mathcal{C})$ of classes of left R-modules is called a cotorsion theory [15] if $\mathcal{L}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{L}$. A cotorsion theory $(\mathcal{L}, \mathcal{C})$ is called perfect [16] if every left R-module has a $\mathcal{C}$-envelope and an $\mathcal{L}$-cover. A cotorsion theory $(\mathcal{L}, \mathcal{C})$ is said to be hereditary in [16] if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{L}$, then L' is also in $\mathcal{L}$.

2. Preliminaries

We begin with the following definition.

Definition 2.1. Let R be a ring and M a left R-module.

The *Gorenstein cotorsion dimension* of M, denoted by $\text{G-cd}_R(M)$, is defined to be smallest integer $n \geqslant 0$ such that $\mathrm{Ext}^{n+1}_R(N, M) = 0$ for all Gorenstein flat left R-modules N. If no such n exists, set $\text{G-cd}_R(M) = \infty$.

Put $l.\text{G-cD}(R) = \sup\{\text{G-cd}_R(M) \mid M \text{ is a left } R\text{-module}\}$ and call $l.\text{G-cD}(R)$ the *left global Gorenstein cotorsion dimension* of R. Similarly, we have $r.\text{G-cD}(R)$ (when R is a commutative ring, we drop the unneeded letters r and l).

Remark 2.2. (1) Clearly, the modules of Gorenstein cotorsion dimension 0 are just the well-known Gorenstein cotorsion modules. Moreover, if M is a Gorenstein cotorsion left R-module and N is a Gorenstein flat left R-module, then $\mathrm{Ext}^i_R(N, M) = 0$ for all $i \geqslant 1$.

(2) Since every flat module is Gorenstein flat, it follows that $\mathrm{cd}_R(M) \leqslant \text{G-cd}_R(M) \leqslant \mathrm{id}_R(M)$ for any left R-module M by [17, Remark 2.3]. If R is

von Neumann regular, then the equalities $\mathrm{cd}_R(M) = \text{G-cd}_R(M) = \mathrm{id}_R(M)$ hold.

Proposition 2.3. *Let $\{N_i\}_{i\in I}$ be a family of left R-modules. Then we have*

$$G\text{-}cd_R(\prod_{i\in I} N_i) = \sup\{G\text{-}cd_R(N_i)|\, i \in I\}.$$

In particular, $\prod_{i\in I} N_i$ is Gorenstein cotorsion if and only if each N_i is Gorenstein cotorsion.

Proof. This follows from the isomorphism:

$$\mathrm{Ext}^n_R(F, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Ext}^n_R(F, N_i)$$

for all $n \geqslant 1$ and all left R-modules F by [24, Theorem 7.14]. □

In what follows, we write $\mathcal{GF}$ and $\mathcal{GC}$ for the categories of all Gorensein flat left R-modules and all Gorenstein cotorsion left R-modules, respectively. For a left R-module M, we use $GC(M)$ and $GF(M)$ to denote the Gorenstein cotorsion envelope and Gorenstein flat cover of M, respectively. By the Wakamutsu's Lemmas ([26, Lemmas 2.1.1 and 2.1.2]), we easily get the next two results.

Lemma 2.4. *If $f : GF(M) \to M$ is a Gorenstein flat cover of a left R-module M, then* $\ker(f)$ *is Gorenstein cotorsion. Thus if M is Gorenstein cotorsion, so is $GF(M)$.*

Lemma 2.5. *If $g : M \to GC(M)$ is a Gorenstein cotorsion envelope of a left R-module M, then* $\mathrm{coker}(g)$ *is Gorenstein flat. Moreover, if M is Gorenstein flat, so is $GC(F)$.*

The following example illustrates cotorsion modules need not be Gorenstein cotorsion in general.

Example 2.6. Let N be any right R-module, then N^+ is pure injective by [15, Proposition 5.3.7], and so N^+ is cotorsion since every pure injective module is cotorsion. We claim that N^+ is not Gorenstein cotorsion. If not, let G be a Gorenstein flat left R-module, but not flat. One easily gets the isomorphism: $\mathrm{Tor}^R_1(N, G)^+ \cong \mathrm{Ext}^1_R(G, N^+)$. Then $\mathrm{Ext}^1_R(G, N^+) = 0$ since N^+ is Gorenstein cotorsion. It follows from [24, Lemma 3.51] that $\mathrm{Tor}^R_1(N, G) = 0$, and so G is flat, a contradiction.

Proposition 2.7. *Let R be a commutative ring. The following are equivalent:*
(1) *M is Gorenstein cotorsion.*
(2) $\mathrm{Hom}_R(F, M)$ *is Gorenstein cotorsion for any flat R-module F.*
(3) $\mathrm{Hom}_R(F, M)$ *is Gorenstein cotorsion for any projective R-module F.*

Proof. (1) ⇒ (2) Let N be a Gorenstein flat R-module and F a flat R-module. Then there exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective, which yields the exactness of

$$0 \to K \otimes_R F \to P \otimes_R F \to N \otimes_R F \to 0.$$

Note that $N \otimes_R F$ is Gorenstein flat by [22, Proposition 2.11] since R is commutative, it follows that

$$\mathrm{Hom}_R(P \otimes_R F, M) \to \mathrm{Hom}_R(K \otimes_R F, M) \to \mathrm{Ext}^1_R(N \otimes_R F, M) = 0$$

is exact, which gives rise to the exactness of

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(F, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M)) \to 0.$$

On the other hand, the sequence

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(F, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M)) \to$$
$$\mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M)) \to \mathrm{Ext}^1_R(P, \mathrm{Hom}_R(F, M)) = 0$$

is exact. Thus $\mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M)) = 0$, which implies $\mathrm{Hom}_R(F, M)$ is Gorenstein cotorsion.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) holds by putting $F = R$. □

We shall say that a ring R is *left Gorenstein cotorsion* if ${}_RR$ is Gorenstein cotorsion. Let $\mathcal{A}$ be a nonempty collection of left ideals of a ring R. Following [25], a left R-module X is called *$\mathcal{A}$-injective* if each R-homomorphism $f : A \to X$ with $A \in \mathcal{A}$ extends to R, or equivalently $\mathrm{Ext}^1_R(R/A, X) = 0$ for any $A \in \mathcal{A}$.

Proposition 2.8. *Let R be a left GF-closed ring and $\mathcal{A}$ a nonempty collection of left ideals of R. Then the following are equivalent*:

(1) *Every Gorenstein cotorsion left R-module is $\mathcal{A}$-injective.*

(2) *R/A is a Gorenstein flat left R-module for any $A \in \mathcal{A}$.*

Moreover, if R is a left Gorenstein cotorsion ring, then the above equivalent conditions imply that $GC(A)$ is a direct summand of ${}_RR$ for any $A \in \mathcal{A}$.

Proof. (1) ⇒ (2) For any $A \in \mathcal{A}$, let M be any Gorenstein cotorsion left R-module. The exactness of $0 \to A \to R \to R/A \to 0$ gives rise to an exact sequence

$$\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(R/A, M) \to 0.$$

Since $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(A, M) \to 0$ is exact by (1), we have

$$\mathrm{Ext}^1_R(R/A, M) = 0.$$

By the arbitrariness of the Gorenstein cotorsion M, we get R/A is Gorenstein flat by [27, Theorem 3.4] since R is left GF-closed.

(2) ⇒ (1) Let M be a Gorenstein cotorsion left R-module. For any $A \in \mathcal{A}$, the exactness of $0 \to A \to R \to R/A \to 0$ induces an exact sequence

$$\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(A, M) \to \mathrm{Ext}^1_R(R/A, M) \to 0.$$

Note that $\mathrm{Ext}^1_R(R/A, M) = 0$ by (2), and hence M is $\mathcal{A}$-injective. Thus (1) follows.

Assume that R is a left Gorenstein cotorsion ring, let R/A be Gorenstein flat for any $A \in \mathcal{A}$, then the inclusion map $A \to {}_RR$ is a Gorenstein cotorsion preenvelope of A. It follows from [15, Proposition 6.1.2] that $GC(A)$ is a direct summand of ${}_RR$, as desired. □

3. The Gorenstein cotorsion dimension of modules and rings

In this section, we study the Gorenstein cotorsion dimension of modules and rings. The relations between the Gorenstein cotorsion dimension and other homological dimensions are discussed. Some known results for cotorsion dimension of [23] are extended to Gorenstein cotorsion dimension.

The following proposition gives a characterization of the Gorenstein cotorsion dimension.

Proposition 3.1. *Let R be a ring. The following are equivalent for a left R-module M and an integer $n \geqslant 0$:*

(1) *G-$cd_R(M) \leqslant n$;*

(2) $\mathrm{Ext}^{n+1}_R(N, M) = 0$ *for any Gorenstein flat left R-module N;*

(3) $\mathrm{Ext}^{n+j}_R(N, M) = 0$ *for any Gorenstein flat left R-module N and all $j \geqslant 1$;*

(4) *If the sequence $0 \to M \to G^0 \to G^1 \to \cdots \to G^{n-1} \to G^n \to 0$ is exact with $G^0, \dots, G^{n-1}$ Gorenstein cotorsion, then also G^n is Gorenstein cotorsion;*

(5) *There is an exact sequence $0 \to M \to G^0 \to G^1 \to \cdots \to G^{n-1} \to G^n \to 0$ is exact with $G^0, \dots, G^n$ Gorenstein cotorsion;*

Moreover, if R is a left GF-closed ring, then the above conditions are equivalent to:

(6) *G-$cd(GF(M)) \leqslant n$.*

Proof. The proof consists entirely of standard arguments, and so it is omitted. □

Yang and Liu proved in [27, Corollary 3.6] that if R is left GF-closed, then every module has a Gorenstein cotorsion envelope. The following corollary is a description of Gorenstein cotorsion dimension of a left R-module over such rings, which is a generalization of [23, Corollary 19.2.2].

Corollary 3.2. *Let R be left GF-closed. Then the following are identical for a left R-module M :*

(1) *G-$cd_R(M)$.*

(2) $\inf\{k$ *: there is an exact sequence $0 \to M \to G^0 \to G^1 \to \cdots \to G^k \to 0$, where $G^0, \dots, G^k$ are Gorenstein cotorsion*$\}$.

(3) *The integer n such that M admits a minimal Gorenstein cotorsion resolution, that is, an exact sequence $0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0$, where each G^i is Gorenstein cotorsion, $L^i = \mathrm{coker}(G^{i-2} \to G^{i-1}) \to G^i$ is a Gorenstein cotorsion envelope of L^i, $G^i \neq 0$, $i = 0, 1, \dots, n$, $G^{-2} = 0, G^{-1} = M$.*

Proof. Similar to that of [23, Corollary 19.2.2]. □

Proposition 3.3. *Let R be a ring and $0 \to A \to B \to C \to 0$ a short exact sequence of left R-modules. Then we have*

(1) *$G\text{-}cd_R(A) \leqslant \sup\{G\text{-}cd_R(B), G\text{-}cd_R(C)+1\}$ with equality if $G\text{-}cd_R(B) \neq G\text{-}cd_R(C)$.*

(2) *$G\text{-}cd_R(B) \leqslant \sup\{G\text{-}cd_R(A), G\text{-}cd_R(C)\}$ with equality if $G\text{-}cd_R(A) \neq G\text{-}cd_R(C)+1$.*

(3) *$G\text{-}cd_R(C) \leqslant \sup\{G\text{-}cd_R(B), G\text{-}cd_R(A)-1\}$ with equality if $G\text{-}cd_R(B) \neq G\text{-}cd_R(A)$.*

Proof. The proof is similar to [19, Lemma 3.3] and omitted. □

The next proposition is the Gorenstein version of [5, Lemma 2.5].

Proposition 3.4. *Let R be a left GF-closed ring and M a left R-modules. Then we have*

$$G\text{-}cd_R(M) \leqslant id_R(M) \leqslant G\text{-}cd_R(M) + w\,Ggldim(R).$$

Proof. By Remark 2.2(2), it suffices to show the inequality

$$\mathrm{id}_R(M) \leqslant \mathrm{G\text{-}cd}_R(M) + w\mathrm{Ggldim}(R).$$

In order to do so, we may assume that $\mathrm{G\text{-}cd}_R(M) = n < \infty$ and $w\mathrm{Ggldim}(R) = m < \infty$. For any left R-module N, take a partial projective resolution of N:

$$0 \to G \to P_{m-1} \to \cdots \to P_1 \to P_0 \to N \to 0.$$

Then G is Gorenstein flat by [3, Theorem 2.8] since $w\mathrm{Ggldim}(R) = m$. Applying $\mathrm{Hom}_R(-, M)$ to the above exact sequence, we get

$$\mathrm{Ext}_R^{n+m+1}(N, M) \cong \mathrm{Ext}_R^{n+1}(G, M) = 0$$

by Proposition 3.1, which implies that $\mathrm{id}_R(M) \leqslant n+m$, as desired. □

The following theorem gives a generalization of [23, Theorem 19.2.5].

Theorem 3.5. *Let R be a left GF-closed ring. Then*

(1)

$$\begin{aligned} l.G\text{-}cD(R) &= \sup\{pd_R(M) \mid M \text{ is a Gorenstein flat left } R\text{-module}\} \\ &= \sup\{G\text{-}cd_R(M) \mid M \text{ is a Gorenstein flat left } R\text{-module}\}. \end{aligned}$$

(2) *If $l.G\text{-}cD(R) < \infty$, then*

$$\begin{aligned} l.G\text{-}cD(R) &= \sup\{pd_R(M) \mid M \text{ is a Gorenstein flat G-cotorsion left } R\text{-module}\} \\ &= \sup\{pd_R(GC(M)) \mid M \text{ is a Gorenstein flat left } R\text{-module}\} \\ &= \sup\{G\text{-}cd_R(P) \mid P \text{ is a projective left } R\text{-module}\}. \end{aligned}$$

Proof. (1) First we show that $l.\text{G-cD}(R) \leqslant \sup\{\text{pd}_R(F) \mid F$ is a Gorenstein flat left R-module$\}$. We may assume that $\sup\{\text{pd}_R(F) \mid F$ is a Gorenstein flat left R-module$\} = m < \infty$. Let N be any left R-module. It follows that $\text{Ext}_R^{m+1}(F, N) = 0$ for any Gorenstein flat left R-module F since $\text{pd}_R(F) \leqslant m$, and so $\text{G-cd}_R(N) \leqslant m$. Thus $l.\text{G-cD}(R) \leqslant m$.

Clearly, $\sup\{\text{G-cd}_R(F) \mid F$ is a Gorenstein flat left R-module$\} \leqslant l.\text{G-cD}(R)$. Next we show that $\sup\{\text{pd}_R(F) \mid F$ is a Gorenstein flat left R-module$\} \leqslant \sup\{\text{G-cd}_R(F) \mid F$ is a Gorenstein flat left R-module$\}$. In fact, we can assume that $\sup\{\text{G-cd}_R(F) \mid F$ is a Gorenstein flat left R-module$\} = n < \infty$. Let N be a Gorenstein flat left R-module and M any left R-module. Since R is left GF-closed, there exists an exact sequence $0 \to K \to GF(M) \to M \to 0$. By Lemma 2.4, K is Gorenstein cotorsion. So we have the following exact sequence

$$\text{Ext}_R^{n+1}(N, GF(M)) \to \text{Ext}_R^{n+1}(N, M) \to \text{Ext}_R^{n+2}(N, K) = 0.$$

Note that $\text{Ext}_R^{n+1}(N, GF(M)) = 0$ since $\text{G-cd}_R(GF(M)) \leqslant n$. Thus

$$\text{Ext}_R^{n+1}(N, M) = 0,$$

which shows $\text{pd}_R(N) \leqslant n$, as desired.

(2) We note that Gorenstein cotorsion envelopes of Gorenstein flat left R-modules are always Gorenstein flat by Lemma 2.5, it follows that

$$\begin{aligned}
&l.\text{G-cD}(R)\\
&= \sup\{\text{pd}_R(M) \mid M \text{ is a Gorenstein flat left } R\text{-module}\}\\
&\geqslant \sup\{\text{pd}_R(M) \mid M \text{ is a Gorenstein flat and Gorenstein cotorsion}\\
&\qquad\qquad \text{left } R\text{-module}\}\\
&\geqslant \sup\{\text{pd}_R(GC(M)) \mid M \text{ is a Goresntein flat left } R\text{-module}\}.
\end{aligned}$$

Next, we will prove that $l.\text{G-cD}(R) \leqslant \sup\{\text{pd}_R(GC(M)) \mid M$ is a Gorenstein flat left R-module$\}$. Suppose that $\sup\{\text{pd}_R(GC(M)) \mid M$ is a Gorenstein flat left R-module$\} = m < \infty$. For every Gorenstein flat left R-module M, $\text{G-cd}_R(M) = t < \infty$ since $l.\text{G-cD}(R) < \infty$. By Corollary 3.2, M admits a minimal Gorenstein cotorsion resolution

$$0 \to M \to G_0 \to G_1 \to \cdots \to G_{t-1} \to G_t \to 0.$$

Since each G_i is a Gorenstein cotorsion envelope of the Gorenstein flat left R-module L_i, for $i = 0, 1, \ldots, t$, we have $\text{pd}_R(G_i) \leqslant m$ $(i = 0, 1, \ldots, t)$ by hypothesis, and so $\text{pd}_R(M) \leqslant m$. Thus we get $l.\text{G-cD}(R) \leqslant m$ by (1).

Finally, we prove that $l.\text{G-cD}(R) = \sup\{\text{G-cd}_R(P) \mid P$ is a projective left R-module$\}$. It is clear that the inequality "$\geqslant$" holds by Definition 2.1. In order to show the converse inequality "$\leqslant$", let $\sup\{\text{G-cd}_R(P) \mid P$ is a projective left R-module$\} = n < \infty$. For any Gorenstein flat left R-module F, we may suppose $\text{pd}_R(F) = m < \infty$ since $l.\text{G-cD}(R) < \infty$. Then, we have an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to F \to 0$$

with each P_i ($0 \leqslant i \leqslant m$) projective, and $\text{G-cd}_R(P_i) \leqslant n$ by the hypothesis. Hence $\text{G-cd}_R(F) \leqslant n$ by Proposition 3.3, and so $l.\text{G-cD}(R) \leqslant \sup\{\text{G-cd}_R(P) \mid P$ is a projective left R-module$\}$. This completes the proof. □

Proposition 3.6. *Let R be a left GF-closed ring. Then*

$$l.cot.D(R) \leqslant l.G\text{-}cd(R) \leqslant l.D(R).$$

Proof. It is clear that the inequality $l.\text{G-cd}(R) \leqslant l.D(R)$ holds by Theorem 3.5(1). Now we shall show that $l.\text{cot}.D(R) \leqslant l.\text{G-cd}(R)$. We may assume that $l.\text{G-cd}(R) = n < \infty$. Let M be any left R-module. Then $\text{Ext}_R^{n+1}(G, M) = 0$ for all Gorenstein flat left R-modules G since $\text{G-cd}_R(M) \leqslant n$. It follows that $\text{Ext}_R^{n+1}(F, M) = 0$ for any flat left R-module F. Thus $\text{cd}_R(M) \leqslant n$, and hence $l.\text{cot}.D(R) \leqslant n$, as desired. □

Following [11], a ring R is said to be n-FC if R is a left and right coherent ring with self-FP-injective dimension at most n on either side. A ring R is called FC if it is 0-FC.

Remark 3.7. We note that a ring R is FC if and only if every R-module (left and right) is Gorenstein flat by [11, Theorem 6], it follows that $l.\text{G-cD}(R) = l.D(R)$ when R is an FC ring.

It is well known that a ring R is left perfect if and only if every flat left R-module is projective if and only if every (flat) left R-module is cotorsion if and only if every (flat) left R-module has a cotorsion envelope with the unique mapping property. Here, we have:

Proposition 3.8. *The following are equivalent for any ring R :*

(1) *$l.G\text{-}cD(R) = 0$.*

(2) *All Gorenstein flat left R-modules are Gorenstein cotorsion.*

(3) *All Gorenstein flat left R-modules are projective.*

Moreover, if R is a left GF-closed ring, then the above conditions are equivalent to:

(4) *Every left R-module has a Gorenstein cotorsion envelope with the unique mapping property.*

(5) *Every Gorenstein flat left R-module has a Goresntein cotorsion envelope with the unique mapping property.*

(6) *For any left R-homomorphism $f : M_1 \to M_2$ with M_1, M_2 Gorenstein cotorsion,* $\ker(f)$ *is Gorenstein cotorsion.*

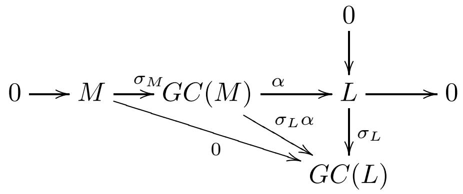
Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (3) Assume that G is a Gorenstein flat left R-module and $0 \to N \to P \to G \to 0$ is a short exact sequence, where P is projective. It is easy to see that N is Gorenstein flat. Hence $\text{Ext}_R^1(G, N) = 0$ by (2), and thus G is projective, as desired.

(3) ⇒ (1) Since $(\mathcal{P}roj, {}_R\mathcal{M})$ are cotorsion pairs, this result holds directly by (3).

(1) ⇒ (4) ⇒ (5) and (1) ⇒ (6) are trivial. Now it remains to show that (5) ⇒ (2) and (6) ⇒ (1).

(5) ⇒ (2) Let M be a Gorenstein flat left R-module. Then, we have the following exact commutative diagram:



Note that $\sigma_L \alpha \sigma_M = 0 = 0\sigma_M$, so $\sigma_L \alpha = 0$ by (5). It follows that $L = \mathrm{im}(\alpha) \subseteq \ker(\sigma_L) = 0$, and so M is Gorenstein cotorsion. Thus (2) holds.

(6) ⇒ (1) Let M be any Gorenstein flat left R-module. The above commutative diagram implies $M = \ker(\alpha) = \ker(\sigma_L \alpha)$ is Gorenstein cotorsion by (6), as desired. □

4. Applications of Gorenstein cotorsion modules

In this section, we give some new characterizations of weak Gorenstein global dimension of coherent rings in terms of Gorenstein cotorsion modules.

Bennis in [4, Theorem 2.8] proved $l.w\mathrm{Ggldim}(R) = r.w\mathrm{Ggldim}(R)$ for two-sided coherent rings. The following theorem gives a generalization of [26, Theorem 3.2.2].

Theorem 4.1. *Let R be a two-sided coherent ring. Then the following are equivalent for any integer $n \geqslant 0$:*

(1) *$w\mathit{Ggldim}(R) \leqslant n$.*

(2) *For all Gorenstein cotorsion left R-modules L, $id_R(L) \leqslant n$.*

(3) *For all Gorenstein cotorsion right R-modules N, $Gfd_R(N) \leqslant n$.*

(4) *For all injective right R-modules N, $Gfd_R(N) \leqslant n$.*

To prove this theorem, we need the following lemma.

Lemma 4.2 ([4, Corollary 2.3])**.** *Let R be a GF-closed ring. Then $fd_R(E) = Gfd_R(E)$ for any injective R-module E.*

Proof of Theorem 4.1. (1) ⇒ (2) Let M be a left R-module. Then, by (1), there exists an exact sequence

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

with each F_i $(0 \leqslant i \leqslant n)$ Gorenstein flat by [21, Theorem 3.14]. For any Gorenstein cotorsion left R-module L, applying $\mathrm{Hom}_R(-, L)$ to the above exact sequence, one easily gets $\mathrm{Ext}_R^{n+1}(M, L) \cong \mathrm{Ext}_R^1(F_n, L) = 0$ since F_n is Gorenstein flat, which implies that $\mathrm{id}_R(L) \leqslant n$.

(2) $\Rightarrow$ (1) Let $\cdots \to F_1 \to F_0 \to N \to 0$ be a flat resolution of a left R-module N and K_n the nth $\mathcal{F}lat$-syzygy. Then we have the following exact sequence

$$0 \to K_n \to F_{n-1} \cdots \to F_1 \to F_0 \to N \to 0.$$

For any Gorenstein cotorsion left R-module L, it follows that $\mathrm{Ext}^1_R(K_n, L) \cong \mathrm{Ext}^{n+1}_R(N, L) = 0$ since $\mathrm{id}_R(L) \leqslant n$ by (2). Note that $(\mathcal{GF}, \mathcal{GC})$ is cotorsion theory by [16, Theorem 2.11], and so K_n is Gorenstein flat, as desired.

(1) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. Next it remains to prove that (4) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) Let E be an injective right R-module. Then $\mathrm{Gfd}_R(E) \leqslant n$ by (4), and so $\mathrm{fd}_R(E) \leqslant n$ by Lemma 4.2. It follows from [10, Theorem 3.8] that FP-id$({}_RR) \leqslant n$ since R is left coherent. Similarly, we have FP-id$(R_R) \leqslant n$. Consequently, R is an n-FC ring. So the desired result follows from [11, Theorem 7], and completing the proof. □

Specializing Theorem 4.1 to the case $n = 0$, we get the following characterizations of FC rings.

Corollary 4.3. *Let R be a two-sided coherent ring. Then the following are equivalent*:

(1) *R is an FC ring* (*or equivalently $wGgldim(R) = 0$ by* [11, Theorem 6]).

(2) *Every Gorenstein cotorsion R-module* (*left and right*) *is injective.*

(3) *Every Gorenstein cotorsion R-module* (*left and right*) *is Gorenstein flat.*

(4) *Every injective R-module* (*left and right*) *is Gorenstein flat.*

It is known that the Gorenstein cotorsion envelope of any Gorenstein flat left R-module is always Gorenstein flat by Lemma 2.5. Next, we discuss when the Gorenstein cotorsion envelope of any left R-module is Gorenstein flat.

Proposition 4.4. *Let R be a two-sided coherent ring. Then the following are equivalent*:

(1) *R is an FC ring.*

(2) *Every R-module* (*left and right*) *has a G-cotorsion envelope which is Gorenstein flat.*

(3) *Every R-module* (*left and right*) *has a G-flat cover with the unique mapping property.*

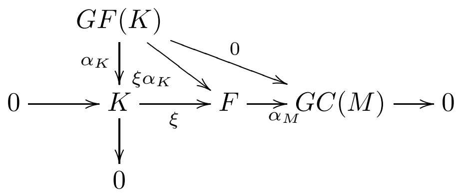
(4) *Every G-cotorsion R-module* (*left and right*) *has a Gorenstein flat cover with the unique mapping property.*

(5) *For any left R-homomorphism $f : M_1 \to M_2$ with M_1 and M_2 G-flat,* coker(f) *is G-flat.*

Proof. (1) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) are clear by [11, Theorem 6]. Next it suffices to show that (4) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1).

(4) $\Rightarrow$ (2) Let M be a left R-module. There is an exact sequence $0 \to M \to GC(M) \to L \to 0$ with L Gorenstein flat by Lemma 2.5. Then, we have the

commutative diagram with exact row:



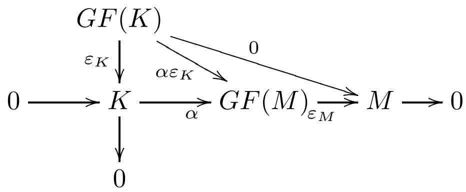
where $F \to GC(M)$ and $GF(K) \to K$ are Gorenstein flat covers of $GC(M)$ and K, respectively. But $\sigma_M \xi \sigma_K = 0 = \sigma_M 0$, it follows that $\xi \sigma_K = 0$ by (4), and so $K = \mathrm{im}(\sigma_K) \subseteq \ker(\xi) = 0$. Therefore $GC(M)$ is Gorenstein flat. Similarly, we can prove the case of right R-modules, and thus (2) follows.

(2) $\Rightarrow$ (1) We only prove the case of left R-modules. Let N be a Gorenstein cotorsion left R-module and M any left R-module. There is an exact sequence $0 \to M \to GC(M) \to L \to 0$ with L Gorenstein flat. Then $GC(M)$ is Gorenstein flat by (2). Thus we have the exactness of

$$\mathrm{Ext}^1_R(GC(M), N) \to \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^2_R(L, N).$$

Note that $\mathrm{Ext}^1_R(GC(M), N) = \mathrm{Ext}^2_R(L, N) = 0$ by Proposition 3.1, we have $\mathrm{Ext}^1_R(M, N) = 0$, and so N is injective. Hence (1) follows by Corollary 4.3.

(5) $\Rightarrow$ (1) Assume that M is a Gorenstein cotorsion left R-module. We consider the following commutative diagram with exact row:



which implies that $M = \mathrm{coker}(\alpha) = \mathrm{coker}(\alpha\varepsilon_K)$ is Goresntein flat by (5). Similarly, we have the case of right R-modules. Consequently, the desired result follows directly from Corollary 4.3. □

The next proposition gives a description of weak Gorenstein global dimension of communicative coherent rings in terms of Goresntein cotorsion modules.

Theorem 4.5. *Let R be a commutative coherent ring and n a nonnegative integer. Then the following are equivalent*:

(1) $wGgldim(R) \leqslant n$.

(2) $Gid_R\mathrm{Hom}_R(A, B) \leqslant n$ *for any Gorenstein cotorsion module A and any injective module B.*

(3) $Gfd_R(A \otimes_R F) \leqslant n$ *for all Gorenstein cotorsion modules A and all flat modules F.*

(4) $Gfd_R\mathrm{Hom}_R(F, B) \leqslant n$ *for all flat modules F and for all Gorenstein cotorsion modules B.*

Proof. (1) $\Rightarrow$ (2) Let A be a Goresntein cotorsion left R-module and B an injective left R-module. Then $\mathrm{Gfd}_R(A) \leqslant n$ by (1), and so we have a Gorenstein flat resolution of A:

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to A \to 0,$$

which gives rise to the exactness of the sequence

$$0 \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(F_0, B) \to \cdots \to \mathrm{Hom}_R(F_n, B) \to 0.$$

Note that each F_i is Gorenstein flat and B is injective, it follows that $\mathrm{Hom}_R(F_i, B)$ is Gorenstein injective by [20, Lemma 4.6]. Consequently, $\mathrm{Gid}_R\mathrm{Hom}_R(A, B) \leqslant n$, and so (2) follows.

(2) $\Rightarrow$ (1) Let A be a Gorenstein cotorsion R-module and let $\cdots \to F_1 \to F_0 \to A \to 0$ be a flat resolution of A. Take $K = \mathrm{im}(F_n \to F_{n-1})$. Then we get the exactness of

$$0 \to K \to F_{n-1} \to \cdots \to F_0 \to A \to 0.$$

For any injective R-module B, the sequence

$$0 \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(F_0, B) \to \cdots \to \mathrm{Hom}_R(F_{n-1}, B) \to \mathrm{Hom}_R(K, B) \to 0$$

is exact. Since $\mathrm{Gid}_R\mathrm{Hom}_R(A, B) \leqslant n$ and each $\mathrm{Hom}_R(F_i, B)$ is injective by [24, Theorem 3.44], it follows that $\mathrm{Hom}_R(K, B)$ is Gorenstein injective for all injective R-modules B. Therefore, K is Gorenstein flat by [20, Lemma 4.6]. Thus $\mathrm{Gfd}_R(A) \leqslant n$, and so (1) holds by Theorem 4.1.

(1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are obvious.

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) follow by taking $F = R$ and Theorem 4.1. □

Corollary 4.6. *The following are equivalent for a commutative coherent ring R :*

(1) *R is an FC ring.*

(2) *For each G-cotorsion R-module A, $\mathrm{Hom}_R(A, B)$ is G-injective for all injective modules B.*

(3) *For each G-cotorsion R-module A, $A \otimes_R F$ is G-flat for all flat R-modules F.*

(4) *For each flat R-module F, $\mathrm{Hom}_R(F, B)$ is G-flat for all G-cotorsion R-modules B.*

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