

## ORIGIN-SYMMETRIC CONVEX BODIES WITH MINIMAL MAHLER VOLUME IN $\mathbb{R}^2$

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ABSTRACT. In this paper, a new proof of the following result is given:  
The product of the volumes of an origin-symmetric convex bodies  $K$  in  
 $\mathbb{R}^2$  and of its polar body is minimal if and only if  $K$  is a parallelogram.

### 1. Introduction

A well-known problem in the theory of convex sets is to find a lower bound for the product of volumes  $\mathcal{P}(K) = V(K)V(K^*)$ , which is called the *Mahler volume* of  $K$ , where  $K$  is an  $n$ -dimensional origin symmetric convex body and  $K^*$  is the polar body of  $K$  (see definition in Section 2). Is it true that we always have

$$(1.1) \quad \mathcal{P}(K) \geq \mathcal{P}(B_\infty^n),$$

where  $B_\infty^n = \{x \in \mathbb{R}^n : |x_i| \leq 1, 1 \leq i \leq n\}$ ? It was conjectured by K. Mahler in [8] and K. Mahler himself in [9] proved the conjecture for  $n = 2$  and S. Reisner in [12] showed that parallelograms are the only minimizers. The  $n = 3$  case is still open. J. Saint-Raymond in [15], for  $n > 3$ , exhibited convex bodies, different than  $B_\infty^n$  and its polar body, for which (1.1) is an equality.

For some particular classes of convex symmetric bodies in  $\mathbb{R}^n$ , a sharper estimate for the lower bound of  $\mathcal{P}(K)$  has been obtained. If  $K$  is an affine image of convex body symmetric with respect to a coordinate hyperplane, J. Saint-Raymond in [15] proved that  $\mathcal{P}(K) \geq 4^n/n!$ ; the equality case, obtained for  $1 - \infty$  spaces, is discussed in [10] and [13]. When  $K$  is a zonoid (limits of finite Minkowski sums of line segments), it was proved by S. Reisner in [11], [12] and [5] that the same inequality holds, with equality if and only if  $K$  is an  $n$ -cube.

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Using the theory of cotype (which roughly speaking controls the size of long random sums in a normed vector space), J. Bourgain and V. D. Milman in [1] proved that there exists a universal constant  $c > 0$  such that  $\mathcal{P}(K) \geq c^n \mathcal{P}(B_\infty^n)$ , which is now known as the reverse Santaló inequality. A slightly weaker “low-tech” bound of  $\mathcal{P}(K) \geq (\log_2 n)^{-n} \mathcal{P}(B_2^n)$ ,  $n \geq 4$ , was given by G. Kuperberg in [6], using only elementary methods. The best result currently known is again by G. Kuperberg in [7], who showed that the lower bound is roughly  $(\pi/4)^n$  times the product-volume of a cube by using some Gauss-type linking integrals associated to a Minkowski metric in  $\mathbb{R}^{2n}$ .

The Mahler conjecture is still open even in the three-dimensional case. Terence Tao in [17] made an excellent remark about the open question.

In the paper, we give a new proof of Mahler conjecture in  $\mathbb{R}^2$ .

**Theorem 1.1.** *If  $K \subset \mathbb{R}^2$  is an origin-symmetric convex body, then*

$$\mathcal{P}(K) \geq 8,$$

*with equality if and only if  $K$  is a square or diamond.*

*Remark 1.* In [3], for  $L_p$ -zonotopes, Campi and Gronchi considered the two quantities  $\frac{V(Z_1)}{V(Z_p)}$  and  $V(Z_1)V(Z_p^*)$  and showed that both are minimized when  $Z_1$  is a parallelotope, the second fact gave an interesting new proof of Mahler’s conjecture in  $\mathbb{R}^2$ . The technique used by Campi and Gronchi is based on a method introduced by Rogers and Shephard in [14] and previous results proved by Campi and Gronchi in [2, 4] (They used the shadow system introduced by Rogers and Shephard). Our proof is different from the previous proof done by Campi and Gronchi, we first reduce to the problem to the case when  $K$  has an axis of symmetry  $D$  and uses as a parameter the coordinate on  $D$  of the point of  $K$  with maximal distance to  $D$ .

## 2. Definitions, notation, and preliminaries

As usual,  $S^{n-1}$  denotes the unit sphere,  $B^n$  the unit ball centered at the origin,  $o$  the origin and  $\|\cdot\|$  the norm in Euclidean  $n$ -space  $\mathbb{R}^n$ . If  $x, y \in \mathbb{R}^n$ , then  $\langle x, y \rangle$  is the inner product of  $x$  and  $y$ .

If  $K$  is a set,  $\partial K$  is its boundary,  $\text{int } K$  is its interior, and  $\text{conv } K$  denotes its convex hull. Let  $\mathbb{R}^n \setminus K$  denote the complement of  $K$ , i.e.,  $\mathbb{R}^n \setminus K = \{x \in \mathbb{R}^n : x \notin K\}$ . If  $K$  is a  $n$ -dimensional convex subset of  $\mathbb{R}^n$ , then  $V(K)$  is its volume  $V_n(K)$ .

Let  $\mathcal{K}^n$  denote the set of convex bodies (compact, convex subsets with non-empty interiors) in  $\mathbb{R}^n$ . Let  $\mathcal{K}_o^n$  denote the subset of  $\mathcal{K}^n$  that contains the origin in its interior. Let  $h(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$ , denote the support function of  $K \in \mathcal{K}_o^n$ ; i.e.,

$$(2.1) \quad h(K, u) = \max\{u \cdot x : x \in K\}, u \in S^{n-1},$$

and let  $\rho(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$ , denote the radial function of  $K \in \mathcal{K}_o^n$ ; i.e.,

$$(2.2) \quad \rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}, u \in S^{n-1}.$$

If  $K \in K_o^n$ , we define the polar body  $K^*$  of  $K$  by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K\}.$$

It is easy to verify that (see p. 44 in [16])

$$(2.3) \quad h(K^*, u) = \frac{1}{\rho(K, u)} \text{ and } \rho(K^*, u) = \frac{1}{h(K, u)}.$$

### 3. Main result and its proof

**Lemma 3.1.** *For any origin symmetric convex body  $K \subset \mathbb{R}^n$ ,  $\mathcal{P}(K)$  is linear invariant, that is, for every linear transformation  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we have  $\mathcal{P}(AK) = \mathcal{P}(K)$ .*

*Proof.* For any  $u \in S^{n-1}$ , we have

$$\rho((AK)^*, u) = \frac{1}{h(AK, u)} = \frac{1}{h(K, A^t u)} = \rho(K^*, A^t u) = \rho(A^{-t} K^*, u).$$

Hence,  $(AK)^* = A^{-t} K^*$ , therefore

$$\begin{aligned} \mathcal{P}(AK) &= V(AK)V((AK)^*) = V(AK)V(A^{-t} K^*) \\ &= |A| |A^{-t}| V(K)V(K^*) = V(K)V(K^*) = \mathcal{P}(K). \end{aligned} \quad \square$$

**Lemma 3.2.** *In  $\mathbb{R}^2$ , for any origin symmetric convex body  $K_0$ , there exists a convex body  $K$ , symmetric with respect to X-axis  $OX$  and  $K \cap OX = [-a, a]$ , satisfying  $\mathcal{P}(K) = \mathcal{P}(K_0)$ .*

*Proof.* Since  $K_0$  is origin symmetric, any line through origin  $O$  cuts the boundary of  $K_0$  into two points with parallel support lines to  $K_0$ . Since  $\mathcal{P}(K_0)$  is affinely invariant, we can suppose that these parallel support lines are orthogonal to X-axis. We define now

$$K = \{(x, y) \mid (x, |y|) \in K_0\}.$$

It is clear that  $K$  is symmetric with respect to X-axis and  $V(K) = V(K_0)$ , thus we just need to prove  $V(K^*) = V(K_0^*)$ .

$$\begin{aligned} V(K_0^*) &= \frac{1}{2} \int_{S^1} \rho(K_0^*, u)^2 du \\ &= \frac{1}{2} \int_{S^1} h(K_0, u)^{-2} du \\ &= \int_{S^1_+} h(K_0, u)^{-2} du \\ &= \int_{S^1_+} h(K, u)^{-2} du \\ &= \frac{1}{2} \int_{S^1} h(K, u)^{-2} du \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{S^1} \rho(K^*, u)^2 du \\
(3.1) \quad &= V(K^*),
\end{aligned}$$

where  $S_+^1 = S^1 \cap \{(x, y) \mid y \geq 0\}$ , which completes the proof.  $\square$

We define function  $f(x) = \max\{y \mid (x, y) \in K\}$ , then  $f(x)$  is a concave non-negative function on  $[-a, a]$  and  $K = \{(x, y) \mid x \in [-a, a], |y| \leq f(x)\}$ . We have the following theorem.

**Lemma 3.3.** *If  $K = \{(x, y) \mid x \in [-a, a], |y| \leq f(x)\}$ , where  $f(x)$  is a concave non-negative function on  $[-a, a]$ , then*

$$K^* = \{(x', y') \mid x' \in [-\frac{1}{a}, \frac{1}{a}], |y'| \leq f^*(x')\},$$

where

$$f^*(x') = \inf\left\{\frac{1 - xx'}{f(x)} \mid x \in [-a, a]\right\}.$$

*Proof.* We firstly prove that  $f^*$  is also a concave non-negative function on  $[-\frac{1}{a}, \frac{1}{a}]$ . Because that  $f(x) \geq 0$  and  $1 - xx' \geq 0$ , thus  $f^*(x') \geq 0$  for any  $x' \in [-\frac{1}{a}, \frac{1}{a}]$ . By the definition of  $f^*(x')$ , for any  $-\frac{1}{a} \leq x'_1, x'_2 \leq \frac{1}{a}$  and  $0 < \lambda < 1$ , we have

$$\begin{aligned}
&f^*((1 - \lambda)x'_1 + \lambda x'_2) \\
&= \inf\left\{\frac{1 - x((1 - \lambda)x'_1 + \lambda x'_2)}{f(x)} \mid x \in [-a, a]\right\} \\
&= \inf\left\{\frac{(1 - \lambda)(1 - xx'_1) + \lambda(1 - xx'_2)}{f(x)} \mid x \in [-a, a]\right\} \\
&\geq (1 - \lambda) \inf\left\{\frac{1 - xx'_1}{f(x)} \mid x \in [-a, a]\right\} + \lambda \inf\left\{\frac{1 - xx'_2}{f(x)} \mid x \in [-a, a]\right\} \\
(3.2) \quad &= (1 - \lambda)f^*(x'_1) + \lambda f^*(x'_2),
\end{aligned}$$

which implies that  $f^*$  is a concave function.

Next, we let

$$K_1 = \{(x', y') \mid x' \in [-\frac{1}{a}, \frac{1}{a}], |y'| \leq f^*(x')\}.$$

On the one hand, we prove  $K_1 \subset K^*$ , for any  $(x', y') \in K_1$  and  $(x, y) \in K$ , we have

$$\begin{aligned}
&xx' + yy' \\
&\leq xx' + yf^*(x') \\
&\leq xx' + y\frac{1 - xx'}{f(x)} \\
&\leq xx' + f(x)\frac{1 - xx'}{f(x)}
\end{aligned}$$

$$(3.3) \quad \leq 1,$$

which implies that  $K_1 \subset K^*$ .

On the other hand, we prove  $K^* \subset K_1$ . Let  $(x', y') \in K^*$  and  $-a < x < a$ , then  $(x, 0) \in K$ , we have  $xx' \leq 1$  for any  $-a < x < a$ , hence  $-\frac{1}{a} \leq x' \leq \frac{1}{a}$ . For any  $(x, f(x)) \in K$ , we have  $xx' + f(x)y' \leq 1$ , then  $|f(x)y'| \leq 1 - xx'$ , thus  $|y'| \leq \frac{1-xx'}{f(x)}$ , thus  $|y'| \leq \inf\{\frac{1-xx'}{f(x)} \mid x \in [a, b]\} = f^*(x')$ , which implies that  $K^* \subset K_1$ .  $\square$

From the above lemma, we have

$$(3.4) \quad \mathcal{P}(K) = 4 \left( \int_{-a}^a f(x) dx \right) \cdot \left( \int_{-\frac{1}{a}}^{\frac{1}{a}} f^*(x) dx \right).$$

Let now  $t_0 \in [-a, a]$ , such that  $f(t_0) = M$  is maximal on  $[-a, a]$ ; then for any  $x' \in [0, \frac{1}{a}]$ , since

$$(3.5) \quad \frac{1 - x'x}{f(x)} \geq \frac{1 - x't_0}{f(t_0)}$$

for any  $x \in [-a, t_0]$ , thus

$$(3.6) \quad f^*(x') = \inf\left\{\frac{1 - x'x}{f(x)} \mid x \in [t_0, a]\right\},$$

similarly for  $x' \in [-\frac{1}{a}, 0]$ ,

$$(3.7) \quad f^*(x') = \inf\left\{\frac{1 - x'x}{f(x)} \mid x \in [-a, t_0]\right\}.$$

It follows that if we set

$$(3.8) \quad F_1 = \{(x, y) \in K \mid y \geq 0, x \geq t_0\}$$

$$(3.9) \quad F_2 = \{(x, y) \in K \mid y \geq 0, x \leq t_0\}$$

$$(3.10) \quad F_1^* = \{(x, y) \in K^* \mid y \geq 0, x \geq 0\}$$

$$(3.11) \quad F_2^* = \{(x, y) \in K^* \mid y \geq 0, x \leq 0\}$$

and let  $u_i = \frac{2|F_i|}{M}$ ,  $i = 1, 2$ , we have

$$(3.12) \quad Mu_1 \geq (x - t_0)M + f(x)(a - t_0) \text{ for } x \in [t_0, a]$$

$$(3.13) \quad Mu_2 \geq (t_0 - x)M + f(x)(a + t_0) \text{ for } x \in [-a, t_0].$$

**Lemma 3.4.** *Let  $v_i = 2|F_i^*|M$ ,  $i = 1, 2$ . Then we can get the following three inequalities:*

$$(3.14) \quad v_1(u_1 + t_0) \geq 2 - \frac{t_0}{a}, \quad v_2(u_2 - t_0) \geq 2 + \frac{t_0}{a};$$

$$(3.15) \quad a - t_0 \leq u_1 \leq 2(a - t_0), \quad a + t_0 \leq u_2 \leq 2(a + t_0);$$

$$(3.16) \quad \frac{1}{a} \leq v_1 \leq \frac{1}{a}(2 - \frac{t_0}{a}), \quad \frac{1}{a} \leq v_2 \leq \frac{1}{a}(2 + \frac{t_0}{a}).$$

*Proof.* Let

$$A_1 = (x_1, y_1) = \left( \frac{1}{u_1 + t_0}, \frac{a - t_0}{M(u_1 + t_0)} \right)$$

and

$$A_2 = (x_2, y_2) = \left( \frac{-1}{u_2 - t_0}, \frac{t_0 + a}{M(u_2 - t_0)} \right),$$

next, we prove  $A_i \in F_i^*$  and  $0 \leq y_i \leq f^*(x_i)$   $i = 1, 2$ . Because  $2|F_1| > M(a - t_0)$  and  $x_1 = \frac{1}{u_1 + t_0} = \frac{M}{2|F_1| + Mt_0}$ , hence  $0 < x_1 < \frac{1}{a}$  and  $y_1 > 0$ . By  $y_1 = \frac{a - t_0}{M(u_1 + t_0)}$  and  $f^*(x_1) = \inf\{\frac{1 - xx_1}{f(x)} \mid x \in [t_0, a]\}$ , we prove that  $\frac{a - t_0}{M(u_1 + t_0)} \leq \frac{1 - xx_1}{f(x)}$  for any  $x \in [t_0, a]$ .

$$\begin{aligned} & \frac{a - t_0}{M(u_1 + t_0)} \leq \frac{1 - xx_1}{f(x)} \\ \Leftrightarrow & f(x)(a - t_0) \leq (1 - xx_1)M(u_1 + t_0) \\ \Leftrightarrow & f(x)(a - t_0) \leq (1 - xx_1)M\left(\frac{2|F_1|}{M} + t_0\right) \\ \Leftrightarrow & f(x)(a - t_0) \leq (1 - xx_1)(2|F_1| + Mt_0) \\ \Leftrightarrow & f(x)(a - t_0) \leq \left(1 - \frac{x}{u_1 + t_0}\right)(2|F_1| + Mt_0) \\ \Leftrightarrow & f(x)(a - t_0) \leq \left(1 - \frac{x}{\frac{2|F_1|}{M} + t_0}\right)(2|F_1| + Mt_0) \\ \Leftrightarrow & f(x)(a - t_0) \leq \left(1 - \frac{Mx}{2|F_1| + Mt_0}\right)(2|F_1| + Mt_0) \\ \Leftrightarrow & f(x)(a - t_0) \leq 2|F_1| + Mt_0 - Mx \\ (3.17) \quad \Leftrightarrow & 2|F_1| \geq M(x - t_0) + f(x)(a - t_0) \end{aligned}$$

which is (3.12), hence  $y_1 \leq f^*(x_1)$ , hence  $A_1 \in F_1^*$ . Similarly, we can get  $A_2 \in F_2^*$ .

According to Figure 3, we can get  $2|F_1^*| \geq x_1 f^*(0) + f^*(x_1) \frac{1}{a}$ , and because  $f^*(0) = \frac{1}{M}$  and  $y_1 \leq f^*(x_1)$ , hence

$$(3.18) \quad \frac{v_1}{M} \geq \frac{x_1}{M} + \frac{y_1}{a}.$$

Similarly, we can get

$$(3.19) \quad \frac{v_2}{M} \geq -\frac{x_2}{M} + \frac{y_2}{a}.$$

The above two inequalities are equivalent to (3.14).

From Figure 2, we can get (3.15) easily.

Next, we prove (3.16). Because

$$f^*(x) = \inf\left\{\frac{1 - xt}{f(t)} \mid t \in [-a, a]\right\} \leq \frac{1 - xt_0}{M}$$

$$\begin{aligned}
 &\implies \int_0^{\frac{1}{a}} f^*(x) \leq \int_0^{\frac{1}{a}} \frac{1-xt_0}{M} dx \\
 (3.20) \quad &\implies |F_1| \leq \frac{1}{2M} \cdot \frac{1}{a} (2 - \frac{t_0}{a})
 \end{aligned}$$

and  $|F_1| \geq \frac{1}{2} \frac{1}{M} \frac{1}{a}$ , hence

$$\frac{1}{a} \leq v_1 \leq \frac{1}{a} (2 - \frac{t_0}{a}).$$

Similarly,

$$\begin{aligned}
 &\int_{-\frac{1}{a}}^0 f^*(x) \leq \int_{-\frac{1}{a}}^0 \frac{1-xt_0}{M} dx \\
 (3.21) \quad &\implies |F_2| \leq \frac{1}{2M} \cdot \frac{1}{a} (2 + \frac{t_0}{a})
 \end{aligned}$$

and  $|F_2| \geq \frac{1}{2} \frac{1}{M} \frac{1}{a}$ , hence

$$\frac{1}{a} \leq v_1 \leq \frac{1}{a} (2 + \frac{t_0}{a}).$$

We have proved (3.16).  $\square$

**Theorem 3.5.** *If  $K_0 \subset \mathbb{R}^2$  is an origin-symmetric convex body, then*

$$\mathcal{P}(K) \geq 8,$$

*with equality if and only if  $K$  is a square or diamond.*

*Proof.* Let  $K \subset \mathbb{R}^2$  be a convex body symmetric with respect to X-axis OX and  $K \cap OX = [-a, a]$  and  $\mathcal{P}(K) = \mathcal{P}(K_0)$ . By (3.4), (3.14) and let  $q = \frac{v_2}{v_1}$ , we have

$$\begin{aligned}
 \mathcal{P}(K) &= (u_1 + u_2)(v_1 + v_2) \\
 &= u_1v_1 + u_1v_2 + u_2v_1 + u_2v_2 \\
 &\geq (2 - \frac{t_0}{a} - v_1t_0) + u_1v_2 + u_2v_1 + (2 + \frac{t_0}{a} + v_2t_0) \\
 &= 4 - v_1t_0 + u_1v_2 + u_2v_1 + v_2t_0 \\
 &= 4 + \frac{v_2}{v_1}(u_1v_1 + v_1t_0) + \frac{v_1}{v_2}(u_2v_2 - v_2t_0) \\
 &\geq 4 + q(2 - \frac{t_0}{a}) + \frac{1}{q}(2 + \frac{t_0}{a}) \\
 (3.22) \quad &= 4 + \frac{1}{a}(\frac{1}{q} - q)t_0 + 2(q + \frac{1}{q}).
 \end{aligned}$$

If  $v_2 > v_1$ , then  $q > 1$  and  $\frac{1}{q} - q < 0$ , then

$$(3.23) \quad \mathcal{P}(K) \geq 4 + \frac{1}{a}(\frac{1}{q} - q)a + 2(q + \frac{1}{q}) = 4 + q + \frac{3}{q} \geq 4 + 2\sqrt{3},$$

and it follows from (3.16) that

$$(3.24) \quad 1 < q = \frac{v_2}{v_1} \leq 2 + \frac{t_0}{a} \leq 3.$$

If  $v_2 < v_1$ , then  $q < 1$  and  $\frac{1}{q} - q > 0$ , then

$$(3.25) \quad \mathcal{P}(K) \geq 4 + \frac{1}{a}\left(\frac{1}{q} - q\right)(-a) + 2\left(q + \frac{1}{q}\right) = 4 + 3q + \frac{1}{q} \geq 4 + 2\sqrt{3},$$

and it follows from (3.16) that

$$(3.26) \quad 1 > q = \frac{v_2}{v_1} \geq \frac{1}{2 - \frac{t_0}{a}} \geq \frac{1}{3}.$$

If  $v_2 = v_1$ , we have

$$(3.27) \quad \mathcal{P}(K) \geq 4 + 2\left(q + \frac{1}{q}\right) \geq 8.$$

In the following, we prove the case of equality in (3.12). If  $K$  is a square or a diamond, it is clear that the equality holds. On the other hand, if the equality holds, then equalities in (3.14) hold, which implies that  $K$  is a square or a diamond.  $\square$

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