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STABILITY OF ZEROS OF POWER SERIES EQUATIONS

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Abstract. We prove that if $|a_1|$ is large and $|a_0|$ is small enough, then every approximate zero of power series equation $\sum_{n=0}^{\infty} a_n x^n = 0$ can be approximated by a true zero within a good error bound. Further, we obtain Hyers-Ulam stability of zeros of the polynomial equation of degree n, $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ for a given integer $n > 1$.

1. Introduction

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. Such a problem was formulated by Ulam [16] in 1940 and solved in the next year for the case of approximately additive functions by Hyers [6]. In this case, it gave rise to the Hyers-Ulam stability for functional equations. Later, Hyers' result was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [12] for linear mappings by allowing the Cauchy difference to be unbounded.

Since then, the stability of several functional equations has been extensively investigated by several mathematicians (see [4, 5, 7, 8, 10, 13, 14] and references therein). The terminology Hyers-Ulam stability can also be applied to the case of other mathematical objects.

Recently, Li and Hua [11] discussed and proved the Hyers-Ulam stability of the zeros of the polynomial equation given in the form, $z^n + \alpha z + \beta = 0$. Moreover, they raised an open problem whether the Hyers-Ulam stability also holds true for zeros of polynomial equations given in the general form (see also [2])

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0. \tag{1.1}$$

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The aim of this work is to investigate the generalized of and the Hyers-Ulam stability of zeros of the power series equation (see also [3])

$$\sum_{n=0}^{\infty} a_n x^n = 0 \tag{1.2}$$

by using the fixed point theory and an idea from [9, 11]. More precisely, we will prove that if $|a_1|$ is large and $|a_0|$ is small enough, then the zeros of the power series equation (1.2) are stable in the sense of Hyers and Ulam. As a corollary, we obtain the Hyers-Ulam stability of zeros of the polynomial equation (1.1).

2. Generalized Hyers-Ulam stability

In this section, we will investigate the generalized Hyers-Ulam stability of zeros of the power series equation (1.2).

Theorem 2.1. *For given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, assume that B_r is a closed ball in $\mathbb{K}$ centered at 0 and of radius $r > 1/2$, and p is a real number. Let the constants $a_0, a_1, \ldots, a_n, \ldots \in \mathbb{K}$ satisfy $|a_1|^p \geq |a_n|^p$ for all $n \in \{0, 1, \ldots\}$,*

$$|a_1| > \sum_{n=2}^{\infty} n r^{n-1} |a_n| \text{ and } |a_0| < \sum_{n=2}^{\infty} (n-1) r^n |a_n|. \tag{2.1}$$

If a $y \in B_r$ satisfies the inequality

$$\left| \sum_{n=0}^{\infty} a_n y^n \right| \leq \varepsilon \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{(2r)^n} \right) \tag{2.2}$$

for some $\varepsilon > 0$, then there exists a zero $x \in B_r$ of the power series equation (1.2) such that

$$|y - x| \leq \frac{2r\varepsilon}{|a_1|^{1-p}(2r-1)(1-\lambda)}, \tag{2.3}$$

where $\lambda = (1/|a_1|) \sum_{n=2}^{\infty} n r^{n-1} |a_n|$ is a positive constant less than 1 and it is independent of ε, x and y.

Proof. First, we define a function $\varphi : B_r \to B_r$ by

$$\varphi(w) = -\frac{1}{a_1} \left(\sum_{n=0,\, n \neq 1}^{\infty} a_n w^n \right)$$

for all $w \in B_r$. It follows from (2.1) that

$$\begin{aligned} \varphi(w) &\leq \frac{1}{|a_1|} \sum_{n=2}^{\infty} r^n |a_n| + \frac{|a_0|}{|a_1|} \\ &\leq \frac{1}{|a_1|} \sum_{n=2}^{\infty} r^n |a_n| + \frac{1}{|a_1|} \sum_{n=2}^{\infty} (n-1) r^n |a_n| \end{aligned}$$

$$= \frac{1}{|a_1|}\sum_{n=2}^{\infty} nr^n|a_n|$$
$$< r$$

for all $w \in B_r$, i.e., the range of φ is included in B_r.

We now consider the Banach space $(\mathbb{K}, |\cdot|)$. Then, B_r is a closed subset of the Banach space $\mathbb{K}$, and φ maps B_r into B_r. We assert that φ is a contraction from B_r into B_r. Indeed, it holds true that

$$\begin{aligned}
|\varphi(w_1) - \varphi(w_2)| &= \left|\frac{1}{a_1}(-a_0 - a_2w_1^2 - \cdots) - \frac{1}{a_1}(-a_0 - a_2w_2^2 - \cdots)\right| \\
&\le \frac{1}{|a_1|}\sum_{n=2}^{\infty} |nr^{n-1}a_n||w_1 - w_2| \\
&\le \frac{1}{|a_1|}\sum_{n=2}^{\infty} nr^{n-1}|a_n||w_1 - w_2| \\
&= \lambda|w_1 - w_2| \qquad (2.4)
\end{aligned}$$

for all $w_1, w_2 \in B_r$, where $\lambda = (1/|a_1|)\sum_{n=2}^{\infty} nr^{n-1}|a_n|$. In view of (2.1), λ is a positive constant less than 1.

According to the Banach fixed point theorem (see [15, Theorem 19.39]), there exists a unique fixed point x of φ, i.e., $\varphi(x) = x$, or equivalently, x satisfies (1.2). Moreover, it follows from (2.4) that

$$\begin{aligned}
|y - x| &= |y - \varphi(y) + \varphi(y) - x| \le |y - \varphi(y)| + |\varphi(y) - x| \\
&= |y - \varphi(y)| + |\varphi(y) - \varphi(x)| \\
&\le \left|y - \left(-\frac{1}{a_1}\sum_{n=0,\ n\neq 1}^{\infty} a_ny^n\right)\right| + \lambda|y - x| \\
&= \frac{1}{|a_1|}\left|\sum_{n=0}^{\infty} a_ny^n\right| + \lambda|y - x|.
\end{aligned}$$

Thus, by (2.2), we have

$$\begin{aligned}
|y - x| &\le \frac{1}{|a_1|(1-\lambda)}\left|\sum_{n=0}^{\infty} a_ny^n\right| \le \frac{\varepsilon}{|a_1|(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{|a_n|^p}{(2r)^n}\right) \\
&\le \frac{\varepsilon}{|a_1|(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{|a_1|^p}{(2r)^n}\right) \\
&\le \frac{2r\varepsilon}{|a_1|^{1-p}(2r-1)(1-\lambda)}
\end{aligned}$$

and so the results follows. □

Theorem 2.2. *For given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, assume that B_r is a closed ball in $\mathbb{K}$ centered at 0 and of radius $r > 1/2$, and p is a real number. Let the constants*

$a_0, a_1, \ldots, a_n, \ldots, b_0, b_1, \ldots, b_n, \ldots \in \mathbb{K}$ satisfy $|a_1 - b_1|^p \geq |a_n - b_n|^p$ for all $n \in \{0, 1, \ldots\}$,

$$|a_1 - b_1| > \sum_{n=2}^{\infty} n r^{n-1} |a_n - b_n| \text{ and } |a_0 - b_0| < \sum_{n=2}^{\infty} (n-1) r^n |a_n - b_n|. \tag{2.5}$$

If an $x \in B_r$ satisfies the inequality

$$\left| \sum_{n=0}^{\infty} (a_n - b_n) y^n \right| \leq \varepsilon \left(\sum_{n=0}^{\infty} \frac{|a_n - b_n|^p}{(2r)^n} \right) \tag{2.6}$$

for some $\varepsilon > 0$, then there exists a zero $x \in B_r$ such that $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ and

$$|y - x| \leq \frac{2r\varepsilon}{|a_1 - b_1|^{1-p}(2r-1)(1-\lambda)}, \tag{2.7}$$

where $\lambda = (1/|a_1 - b_1|) \sum_{n=2}^{\infty} n r^{n-1} |a_n - b_n|$ is a positive constant less than 1 and it is independent of ε, x and y.

Proof. The proof runs similarly as the proof of Theorem 2.1. □

3. Hyers-Ulam stability

In this section, we will investigate the Hyers-Ulam stability of zeros of the power series equation (1.2).

Theorem 3.1. *For given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, assume that B_r is a closed ball in $\mathbb{K}$ centered at 0 and of radius $r > 0$. Let the constants $a_0, a_1, \ldots, a_n, \ldots \in \mathbb{K}$ satisfy*

$$|a_1| > \sum_{n=2}^{\infty} n r^{n-1} |a_n| \text{ and } |a_0| < \sum_{n=2}^{\infty} (n-1) r^n |a_n|. \tag{3.1}$$

If a $y \in B_r$ satisfies the inequality

$$\left| \sum_{n=0}^{\infty} a_n y^n \right| \leq \varepsilon \tag{3.2}$$

for some $\varepsilon > 0$, then there exists a zero $x \in B_r$ of the power series equations (1.2) such that

$$|y - x| \leq \frac{\varepsilon}{|a_1|(1-\lambda)}, \tag{3.3}$$

where $\lambda = (1/|a_1|) \sum_{n=2}^{\infty} n r^{n-1} |a_n|$ is a positive constant less than 1 and it is independent of ε, x and y.

Proof. The proof runs similarly as the proof of Theorem 2.1. □

Theorem 3.2. *For given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, assume that B_r is a closed ball in $\mathbb{K}$ centered at* 0 *and of radius $r > 0$. Let the constants $a_0, a_1, \ldots, a_n, \ldots, b_0, b_1, \ldots, b_n, \ldots \in \mathbb{K}$ satisfy*

$$|a_1 - b_1| > \sum_{n=2}^{\infty} nr^{n-1}|a_n - b_n| \text{ and } |a_0 - b_0| < \sum_{n=2}^{\infty}(n-1)r^n|a_n - b_n|. \tag{3.4}$$

If an $x \in B_r$ satisfies the inequality

$$\left|\sum_{n=0}^{\infty}(a_n - b_n)y^n\right| \le \varepsilon \tag{3.5}$$

for some $\varepsilon > 0$, then there exists a zero $x \in B_r$ such that $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ and

$$|y - x| \le \frac{\varepsilon}{|a_1 - b_1|(1-\lambda)}, \tag{3.6}$$

where $\lambda = (1/|a_1 - b_1|)\sum_{n=2}^{\infty} nr^{n-1}|a_n - b_n|$ is a positive constant less than 1 *and it is independent of ε, x and y.*

Proof. The proof runs similarly as the proof of Theorem 3.1. □

By a similar way as above, we can easily obtain the following corollary concerning the Hyers-Ulam stability of zeros of the polynomial of degree n, which is practically the same as a result of S.-M. Jung (see [9, Theorem 2.1]).

Corollary 3.1. *For given $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, assume that B_r is a closed ball in $\mathbb{K}$ centered at* 0 *and of radius $r > 0$. For a given integer $n > 1$, let the constants $a_0, a_1, \ldots, a_n \in \mathbb{K}$ satisfy*

$$|a_1| > \sum_{i=2}^{n} ir^{i-1}|a_i| \text{ and } |a_0| < \sum_{i=2}^{n}(i-1)r^i|a_i|.$$

If a $z \in B_r$ satisfies the inequality

$$|a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \le \varepsilon$$

for some $\varepsilon > 0$, then there exists a zero $z_0 \in B_r$ of the polynomial equation (1.1) *such that*

$$|z - z_0| \le \frac{\varepsilon}{|a_1|(1-\lambda)},$$

where $\lambda = (1/|a_1|)\sum_{i=2}^{n} ir^{i-1}|a_i|$ is a positive constant less than 1 *and it is independent of ε, z_0 and z.*

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