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SHADOWING, EXPANSIVENESS AND STABILITY OF DIVERGENCE-FREE VECTOR FIELDS

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Abstract. Let X be a divergence-free vector field defined on a closed, connected Riemannian manifold. In this paper, we show the equivalence between the following conditions:

- X is a divergence-free vector field satisfying the *shadowing property*.
- X is a divergence-free vector field satisfying the *Lipschitz shadowing property*.
- X is an *expansive* divergence-free vector field.
- X has no singularities and is *Anosov*.

1. Introduction and results' statement

Let M denote a d-dimensional ($d \geq 3$), compact, boundary-less, connected and smooth Riemannian manifold, endowed with a volume form, which has associated a measure μ, called the Lebesgue measure. Also, denote by $dist$ the Riemannian distance and consider, for $\epsilon > 0$ and $p \in M$, the open balls $B_\epsilon(p) = \{x \in M : dist(x,p) < \epsilon\}$.

Denote by $\mathfrak{X}^r(M)$ the set of vector fields defined on M and endowed with the C^r Whitney topology ($r \geq 1$). If the divergence of a C^r-vector field X is zero, then we call X a C^r-divergence-free vector field. Let $\mathfrak{X}^r_\mu(M)$ denote the set of divergence-free vector fields defined on M endowed with the induced C^r Whitney topology. A C^r-vector field $X : M \to TM$ generates the C^r-*flow* $X^t : M \to M$ for $t \in \mathbb{R}$. If X is a divergence-free vector field, then X^t is called a *conservative* flow, that is, the measure μ is X^t-invariant. From now on, we set $r = 1$.

In this paper, we want to relate expansiveness, shadowing and Lipschitz shadowing properties with the uniform hyperbolicity in the divergence-free setting. So, we start by stating these concepts.

The set of *singularities* of a vector field X is denoted by

$$Sing(X) = \{p \in M : X(p) = \vec{0}\}.$$

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If $p \notin Sing(X)$, then p is called a *regular* point and a subset of M is said *regular* if it has no singularities.

Given X in $\mathfrak{X}^1(M)$ and a regular point x in M, let $N_x = X(x)^{\perp} \subset T_xM$ denote the $(\dim(M)-1)$–dimensional *normal bundle* of X at x and define $N_{x,\epsilon} = N_x \cap \{u \in T_xM : \|u\| < \epsilon\}$ for $\epsilon > 0$. Since, in general, N_x is not DX^t_x-invariant, we define the *linear Poincaré flow*

$$P^t_X(x) = \Pi_{X^t(x)} \circ DX^t_x,$$

where $\Pi_{X^t(x)} : T_{X^t(x)}M \to N_{X^t(x)}$ is the canonical orthogonal projection.

Definition 1.1. Fix $X \in \mathfrak{X}^1(M)$. An X^t-invariant, compact and regular set $\Lambda \subset M$ is *uniformly hyperbolic* if N_Λ admits a P^t_X-invariant splitting $N^s_\Lambda \oplus N^u_\Lambda$ such that there is $\ell > 0$ satisfying

$$\|P^\ell_X(x)|_{N^s_x}\| \le \frac{1}{2} \text{ and } \|P^{-\ell}_X(X^\ell(x))|_{N^u_{X^\ell(x)}}\| \le \frac{1}{2} \text{ for any } x \in \Lambda.$$

A vector field X is called *Anosov* if the manifold M is uniformly hyperbolic. Let $\mathcal{A}^1_\mu(M)$ denote the set of Anosov C^1-divergence-free vector fields.

Since Λ is assumed to be compact, these definitions are equivalent to the usual definitions of hyperbolic set of a flow and of Anosov vector field (see [7, Proposition 1.1]).

Now, we want to state the definition of *shadowing* for continuous-time systems. First, define *Rep* as the set of the increasing homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$, called *reparametrizations*, satisfying $\alpha(0) = 0$. Fixing $\epsilon > 0$, define the set

$$Rep(\epsilon) = \Big\{\alpha \in Rep : \Big|\frac{\alpha(t)}{t} - 1\Big| < \epsilon, t \in \mathbb{R}\Big\}.$$

When we choose a reparametrization α in the previous set, we want $\alpha(t)$ to be taken arbitrarily close to the identity.

Definition 1.2. Fix $T > 0$ and $\delta > 0$. A map $\psi : \mathbb{R} \to M$ is a (δ, T)-*pseudo-orbit* of a flow X^t if $dist(X^t(\psi(\tau)), \psi(\tau + t)) < \delta$ for any $\tau \in \mathbb{R}$ and any $|t| \le T$. A pseudo-orbit ψ of a flow X^t is said to be ϵ-*shadowed* by some orbit of X^t if there is $x \in M$ and a reparametrization $\alpha \in Rep(\epsilon)$ such that $dist(X^{\alpha(t)}(x), \psi(t)) < \epsilon$ for every $t \in \mathbb{R}$.

Note that ψ is not assumed to be continuous.

Definition 1.3. A C^1-vector field X satisfies the *shadowing property* if for any $\epsilon > 0$ and any $T > 0$, there is $\delta > 0$ such that any (δ, T)-pseudo-orbit ψ is ϵ-shadowed by some orbit of X. Let $\mathcal{S}^1(M)$ and $\mathcal{S}^1_\mu(M)$ denote the sets of vector fields in $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$, respectively, satisfying the shadowing property.

In [16], Smale proved that a diffeomorphism in the C^1-interior of the set of diffeomorphisms with the shadowing property is C^1-structurally stable. More recently, Lee and Sakai proved, in [9], that if X belongs to the interior of the set

$\mathcal{S}^1(M)$ and has no singularities, then X satisfies the Axiom A and the strong transversality conditions.

The *Lipschitz shadowing property* is a stronger definition of shadowing.

Definition 1.4. A C^1-vector field X satisfies the *Lipschitz shadowing property* if there are positive constants ℓ and δ_0 such that any (δ, T)-pseudo-orbit ψ, with $T > 0$ and $\delta \leq \delta_0$, is $\ell\delta$-shadowed by an orbit of X. Let $\mathcal{LS}^1(M)$ and $\mathcal{LS}^1_\mu(M)$ denote the sets of vector fields in $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$, respectively, satisfying the Lipschitz shadowing property.

By definition, it is immediate that the set $\mathcal{LS}^1(M)$ is a subset of $\mathcal{S}^1(M)$ and that the set $\mathcal{LS}^1_\mu(M)$ is a subset of $\mathcal{S}^1_\mu(M)$.

In [17], Tikhomirov proved that a vector field in the C^1-interior of the set of vector fields with the Lipschitz shadowing property is structurally stable. Recently, Pilyugin and Tikhomirov proved that a C^1-diffeomorphism having the Lipschitz shadowing property is structurally stable (see [15]).

The following definition is the notion of *expansive vector field*, introduced by Bowen and Walters, in [6].

Definition 1.5. A C^1-vector field X is *expansive* if, for any $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in M$ satisfy $dist(X^t(x), X^{\alpha(t)}(y)) \leq \delta$ for any $t \in \mathbb{R}$ and for some continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X^s(x)$, where $|s| \leq \epsilon$. Denote by $\mathcal{E}^1(M) \subset \mathfrak{X}^1(M)$ the set of *expansive vector fields* and by $\mathcal{E}^1_\mu(M) \subset \mathfrak{X}^1_\mu(M)$ the set of *expansive divergence-free vector fields*, both endowed with the C^1 Whitney topology.

Observe that the reparametrization α, in the previous definition, is not assumed to be close to identity and that the expansiveness property does not depend on the choice of the metric on M. This definition asserts that any two points whose orbits remain indistinguishable, up to any continuous time displacement, must be in the same orbit.

In 1970's, Mañé proved that a diffeomorphism f in the C^1-interior of the set of expansive diffeomorphisms is Axiom A and satisfies the quasi-transversality condition (see [10]). Later, Moriyasu, Sakai and Sun proved the same result for vector fields, in [11]. Moreover, the authors proved that if X belongs to the C^1-interior of the set $\mathcal{E}^1(M)$ and has the shadowing property, then X is Anosov. Recently, Pilyugin and Tikhomirov proved that an expansive diffeomorphism having the Lipschitz shadowing property is Anosov (see [15]).

The expansiveness and the shadowing properties play an essential role in the investigation of the stability theory and the ergodic theory of Axiom A diffeomorphisms (see [5]). It is well-known that Anosov systems are expansive and satisfy the shadowing and the Lipschitz shadowing properties (see [1, 14]). In this paper, we prove the following result.

Theorem 1. $\mathcal{S}^1_\mu(M) = \mathcal{LS}^1_\mu(M) = \mathcal{E}^1_\mu(M) = \mathcal{A}^1_\mu(M)$.

2. Definitions and auxiliary results

In this section, we state some extra definitions on divergence-free dynamics, as well as some auxiliary results.

A *closed orbit* γ of X is a non-constant integral curve $\gamma : [a,b] \to M$ of X such that $\gamma(a) = \gamma(b)$. We define b as the smallest number greater than a satisfying $\gamma(a) = \gamma(b)$. Observe that the period of γ is $b - a$. For simplicity, sometimes we call $p \in \gamma$ a closed orbit. So, the set of *closed orbits* associated to the vector field X is denoted by

$$Per(X) = \{p \in M : \exists\, t > 0\, , X^t(p) = p\}.$$

Fix a closed orbit γ and $p \in \gamma$. If $\pi > 0$ is the least number such that $X^\pi(p) = p$, then γ is a closed orbit with *period* π. Denote by Σ a $(\dim(M)-1)$-*transversal section* to X at p. Poincaré defined a map f from $\tilde{\Sigma} \subset \Sigma$ to Σ, called *the Poincaré first return map* of the trajectories on Σ, such that, for any point $x \in \Sigma$ in a small neighborhood of p, the ω-trajectory of x will intersect Σ again at some point y at some time t close to b. A closed orbit γ of X is *hyperbolic* if $p \in \gamma$ is a hyperbolic fixed point of the Poincaré first return map.

Singularities and closed orbits of X are called *critical points* and are denoted by

$$Crit(X) = Sing(X) \cup Per(X).$$

A singularity p is called *hyperbolic* if the eigenvalues of DX_p are not purely imaginary. We say that any element of $Crit(X)$ is hyperbolic, if any singularity and any closed orbit of X is hyperbolic.

Definition 2.1. A C^1-vector field X is a *star vector field* if there exists a C^1-neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1(M)$ such that, for any $Y \in \mathcal{U}$, any element of the set $Crit(Y)$ is hyperbolic. Moreover, a vector field $X \in \mathfrak{X}^1_\mu(M)$ is a *divergence-free star vector field* if there exists a C^1-neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1_\mu(M)$ such that, for any $Y \in \mathcal{U}$, any element of the set $Crit(Y)$ is hyperbolic. Note that if $X \in \mathfrak{X}^1_\mu(M)$ is a star vector field, then X is a divergence-free star vector field. The set of star vector fields is denoted by $\mathcal{G}^1(M)$ and the set of divergence-free star vector fields is denoted by $\mathcal{G}^1_\mu(M)$.

By definition, $\mathcal{G}^1(M)$ and $\mathcal{G}^1_\mu(M)$ are C^1-open subsets of $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$, respectively. Observe that, in the previous definition, the hyperbolicity imposed at the critical points is not uniform. So, the hyperbolicity constants depend on the critical point.

In [8], it is proved that divergence-free star vector fields are Anosov.

Theorem 2.1 ([8, Theorem 1])**.** *If $X \in \mathcal{G}^1_\mu(M)$, then $Sing(X) = \emptyset$ and X is Anosov.*

A 3-dimensional proof of this result is presented in [4] and a version for 4-dimensional symplectic Hamiltonian vector fields can be found in [2].

In [8], it is also proved that a C^1-structurally stable divergence-free vector fields is Anosov.

Theorem 2.2 ([8, Theorem 2]). *If $X \in \mathfrak{X}^1_\mu(M)$ is C^1-structurally stable, then X is Anosov.*

Now, we state a more relaxed definition of hyperbolicity.

Definition 2.2. Let $X \in \mathfrak{X}^1(M)$ and let $\Lambda \subset M$ be a compact, X^t-invariant and regular set. Assume that there exists a P^t_X-invariant splitting $N = N^1 \oplus \cdots \oplus N^k$ over Λ, for $1 \leq k \leq \dim(M) - 1$, such that all the subbundles have constant dimension. This splitting is *dominated* if there exists $\ell > 0$ such that, for any $0 \leq i < j \leq k$,

$$\|P^\ell_X(x)|_{N^i_x}\| \cdot \|P^{-\ell}_X(X^\ell(x))|_{N^j_{X^\ell(x)}}\| \leq \frac{1}{2} \text{ for any } x \in \Lambda.$$

Note that a vector field with a dominated splitting structure is not necessarily uniformly hyperbolic.

The next result corresponds to a dichotomy for C^1-divergence-free vector fields.

Theorem 2.3. *Let $X \in \mathfrak{X}^1_\mu(M)$ and let $\mathcal{U}$ be a small C^1-neighborhood of X. Then, for any $\epsilon > 0$, there exist $l > 0$ and $\tau > 0$ such that, for any $Y \in \mathcal{U}$ and any $x \in Per(Y)$, with period greater than τ,*

- *either P^t_Y admits an l-dominated splitting over the Y^t-orbit of x, or else*
- *for any neighborhood U of x, there exists an ϵ-C^1-perturbation $\tilde{Y}$ of Y, coinciding with Y outside U and along the orbit of x, such that $P^{\pi(x)}_{\tilde{Y}}(x)$ has only eigenvalues equal to 1 or -1, where $\pi(x)$ stands for the period of x.*

The proof of this result follows the ideas stated in the proof of [3, Proposition 2.4].

Recall that a singularity p is *linear* if there exist smooth local coordinates around p such that X is linear and equal to DX_p in these coordinates (see [18, Definition 4.1]). The following result says that if the vector field has a linear hyperbolic singularity of saddle-type, then the linear Poincaré flow cannot admit a dominated splitting over the set of regular points of M.

Proposition 2.4 ([18, Proposition 4.1]). *If $X \in \mathfrak{X}^1(M)$ has a linear hyperbolic singularity of saddle-type, then P^t_X does not admit any dominated splitting over $M \backslash Sing(X)$.*

We remark that the proof of this proposition can be easily adapted to the divergence-free context. Hence, Proposition 2.4 remains valid for C^1-divergence-free vector fields.

The following lemma states that a singularity can be turned into a linear one, by performing a small perturbation on the vector field.

Lemma 2.5 ([3, Lemma 3.3]). *Let p be a singularity of $X \in \mathfrak{X}^1_\mu(M)$. For any $\epsilon > 0$, there exists $Y \in \mathfrak{X}^\infty_\mu(M)$ such that Y is ϵ-C^1-close to X and p is a linear hyperbolic singularity of Y.*

We end this section with a perturbation result due to Zuppa (see [19]). It allows us to C^1-approximate any divergence-free vector field by a smooth one, keeping the divergence-free property.

Theorem 2.6. *The set of C^∞-divergence-free vector fields is C^1-dense in $\mathfrak{X}^1_\mu(M)$.*

3. Proof of Theorem 1

In this section, we split the proof of Theorem 1 in some lemmas. We already know that $\mathcal{A}^1_\mu(M) \subset \mathcal{S}^1_\mu(M)$, that $\mathcal{A}^1_\mu(M) \subset \mathcal{E}^1_\mu(M)$ and that $\mathcal{LS}^1_\mu(M) \subset \mathcal{S}^1_\mu(M)$. So, by Theorem 2.1, it is enough to show that $int(\mathcal{S}^1_\mu(M)) \subset \mathcal{G}^1_\mu(M)$ and $int(\mathcal{E}^1_\mu(M)) \subset \mathcal{G}^1_\mu(M)$. Hence, given that a flow satisfying the Lipschitz shadowing property is structurally stable (see [13]), Theorem 2.2 concludes the proof of Theorem 1.

Let us firstly prove that any divergence-free vector field in the C^1-interior of the set $\mathcal{S}^1_\mu(M)$ has all the closed orbits hyperbolic. For this, we adapt the strategy described in [9], by Lee and Sakai. After this, we prove that a vector field with the described properties does not have singularities. Therefore, by Theorem 2.1, $int(\mathcal{S}^1_\mu(M)) = \mathcal{A}^1_\mu(M)$.

Lemma 3.1. *If $X \in int(\mathcal{S}^1_\mu(M))$, then any closed orbit of X is hyperbolic.*

Proof. Fix $X \in int(\mathcal{S}^1_\mu(M))$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{S}^1_\mu(M)$. Let p be a point in a closed orbit γ of X with period π and U_p a small neighborhood of p on M. By contradiction, assume that there is an eigenvalue σ_0 of $P^\pi_X(p)$ satisfying $|\sigma_0| = 1$. Applying Zuppa's Theorem (Theorem 2.6), there exists a smooth vector field $Y \in \mathcal{U}$ such that $Y^\pi(p) = p$ and $P^\pi_Y(p)$ has an eigenvalue σ with $|\sigma| = 1$.

Remark 3.1. In fact, $P^\pi_Y(p)$, in the proof, may not have an eigenvalue σ with modulus 1. In this case, there exists $\mathcal{W} \subset \mathcal{U}$ and $Z \in \mathcal{W}$, chosen C^1-arbitrarily close to Y and having an eigenvalue with modulus arbitrarily close to 1. So, by the Franks Lemma ([3, Lemma 3.2]), we can perform an ϵ-C^1-perturbation $\tilde{Z} \in \mathcal{W}$ of Z, with arbitrarily small $\epsilon > 0$, such that $P^\pi_{\tilde{Z}}(p)$ has an eigenvalue $\bar{\sigma}$ with $|\bar{\sigma}| = 1$.

Accordingly with Moser's Theorem (see [12]), there is a smooth conservative change of coordinates $\varphi_p : U_p \to T_pM$ such that $\varphi_p(p) = \vec{0}$. Recall that $f_Y : \varphi_p^{-1}(N_p) \to \Sigma$ denotes the Poincaré map associated to Y^t, where Σ is a Poincaré section through p. Let $\mathcal{V}$ be a C^1-neighborhood of f_Y. By the Franks Lemma ([3, Lemma 3.2]), taking $\mathcal{T}$ a small flowbox of $Y^{[0,t_0]}(p)$, with $0 < t_0 < \pi$, there are $Z \in \mathcal{U}$, $f_Z \in \mathcal{V}$ and $\epsilon > 0$ such that:

- $Z^t(p) = Y^t(p)$ for any $t \in \mathbb{R}$;
- $P^{t_0}_Z(p) = P^{t_0}_Y(p)$;
- $Z|_{\mathcal{T}^c} = Y|_{\mathcal{T}^c}$;

- $$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P_Y^{\pi}(p) \circ \varphi_p(x), & x \in B_{\epsilon_0}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x), & x \notin B_{4\epsilon_0}(p) \cap \varphi_p^{-1}(N_p), \end{cases}$$

  where $\epsilon_0 > 0$ is small.

Notice that $P_Z^{\pi}(p)$ still has an eigenvalue σ with modulus 1. Firstly, assume that $\sigma = 1$, fix the associated non-zero eigenvector v such that $\|v\| = \epsilon_0/2$ and define $\mathcal{I}_v = \{sv : 0 \le s \le 1\}$.

Since $Z \in \mathcal{S}_\mu^1(M)$, for any $\epsilon > 0$, there is $\delta > 0$ such that any (δ, T)-pseudo-orbit is ϵ-shadowed by some orbit y of Z^t for $T > 0$. Fix $0 < \epsilon < \frac{\epsilon_0}{4}$. The idea now is to construct a (δ, T)-pseudo-orbit of Z^t, adapting the strategy followed by Lee and Sakai in [9, Proposition A]. Let us present the highlights of that proof.

Let $x_0 = p$ and $t_0 = 0$. Since p is a parabolic closed orbit, we construct a finite sequence $\{(x_i, t_i)\}_{i=0}^I$, where $I \in \mathbb{N}$, $t_i > 0$ and $x_i \in \varphi_p^{-1}(\mathcal{I}_v)$ for $1 \le i \le I$ such that:

- $x_I = \varphi_p^{-1}(v)$;
- $dist(Z^t(f_Z(x_i)), Z^t(x_{i+1})) < \delta$ for $|t| \le T$ and $0 \le i \le I-1$;
- $Z^{t_i}(x_i) = f_Z(x_i)$ for $1 \le i \le I$.

So, taking $S_n = \sum_{i=0}^n t_i$ for $0 \le n \le I$, the map $\psi : \mathbb{R} \to M$ defined by

$$\psi(t) = \begin{cases} Z^t(x_0), & t < 0 \\ Z^{t-S_n}(x_{n+1}), & S_n \le t < S_{n+1},\ 0 \le n \le I-2 \\ Z^{t-S_{I-1}}(x_I), & t \ge S_{I-1}, \end{cases}$$

is a (δ, T)-pseudo-orbit of Z^t. So, since $Z \in \mathcal{U}$, there is a reparametrization $\alpha \in Rep(\epsilon)$ and a point $y \in B_\epsilon(p) \cap \varphi_p^{-1}(N_{p,\epsilon})$ that ϵ-shadows ψ. So,

$$dist(Z^{\alpha(t)}(y), \psi(t)) < \epsilon$$

for any $t \in \mathbb{R}$. Note that, since $\sigma = 1$,

$$dist(x_0, x_I) = dist\big(p, \varphi_p^{-1}(v)\big) = dist\big(p, f_Z(\varphi_p^{-1}(v))\big) = \|v\| = \frac{\epsilon_0}{2} > 2\epsilon.$$

However, since Z has the shadowing property,

$$dist(x_0, x_I) \le dist\big(x_0, Z^{\alpha(S_{I-1})}(y)\big) + dist\big(Z^{\alpha(S_{I-1})}(y), \psi(S_{I-1})\big) < 2\epsilon,$$

which is a contradiction.

Now, if $|\sigma| = 1$ but $\sigma \ne 1$, we point out that, by the Franks Lemma ([3, Lemma 3.2]), we can find $W \in \mathcal{U}$ such that $P_W^{\pi}(p)$ is a rational rotation. Then, there is $T \ne 0$ such that $P_W^{T+\pi}(p)$ has 1 as an eigenvalue. Therefore, reproducing the previous argument, we conclude that any closed orbit of $X \in int(\mathcal{S}_\mu^1(M))$ is hyperbolic. □

Lemma 3.2. *If $X \in int(\mathcal{S}_\mu^1(M))$, then $Sing(X) = \emptyset$.*

Proof. Fix $X \in int(\mathcal{S}^1_\mu(M))$ and let $\mathcal{U}$ be a C^1-neighborhood of X in $\mathcal{S}^1_\mu(M)$, small enough such that the dichotomy of Theorem 2.3 holds.

By contradiction, assume that $Sing(X) \neq \emptyset$ and fix $p \in Sing(X)$. By Lemma 2.5, there is $Y \in \mathcal{U}$ such that $p \in Sing(Y)$ is linear hyperbolic, and so of saddle-type. Hence, by Proposition 2.4, P^t_Y does not admit any dominated splitting over $M \backslash Sing(Y)$. However, since any closed orbit of Y is hyperbolic (Lemma 3.1), it is straightforward to see that, reproducing the techniques used in the proof of [8, Lemma 3.1], P^t_Y admits a dominated splitting over $M \backslash Sing(Y)$. Therefore, $Sing(X) = \emptyset$. □

Now, we prove that any divergence-free vector field in the C^1-interior of the set $\mathcal{E}^1_\mu(M)$ has all the closed orbits hyperbolic. For this, we adapt the strategy described in [11], by Moriyasu, Sakai and Sun, to the divergence-free setting.

Lemma 3.3. *If $X \in int(\mathcal{E}^1_\mu(M))$, then any closed orbit of X is hyperbolic.*

Proof. Fix $X \in int(\mathcal{E}^1_\mu(M))$ and a C^1-neighborhood $\mathcal{U}$ of X in $\mathcal{E}^1_\mu(M)$. Let p be a point in a closed orbit γ of X with period π and U_p a small neighborhood of p on M. By contradiction, assume that there is an eigenvalue σ_0 of $P^\pi_X(p)$ such that $|\sigma_0| = 1$. Applying Zuppa's Theorem (Theorem 2.6), there is $Y \in \mathcal{U}$ such that $Y \in \mathfrak{X}^\infty_\mu(M)$, $Y^\pi(p) = p$ and $P^\pi_Y(p)$ has an eigenvalue σ such that $|\sigma| = 1$, as explained in Remark 3.1.

Let φ and f_Y be as in the proof of Lemma 3.1 and fix a C^1-neighborhood $\mathcal{V}$ of f_Y. By the Franks Lemma ([3, Lemma 3.2]), taking $\mathcal{T}$ a small flowbox of $Y^{[0,t_0]}(p)$, with $0 < t_0 < \pi$, there are $Z \in \mathcal{U}$ and $f_Z \in \mathcal{V}$ such that:

- $Z^t(p) = Y^t(p)$ for any $t \in \mathbb{R}$;
- $P^{t_0}_Z(p) = P^{t_0}_Y(p)$;
- $Z|_{\mathcal{T}^c} = Y|_{\mathcal{T}^c}$;
- $$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P^\pi_Y(p) \circ \varphi_p(x), & x \in B_{\epsilon/4}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x), & x \notin B_\epsilon(p) \cap \varphi_p^{-1}(N_p). \end{cases}$$

Observe that $P^\pi_Z(p)$ still has an eigenvalue σ with modulus 1.

Since $Z \in \mathcal{E}^1_\mu(M)$, for a sufficiently small $\epsilon > 0$, there exists $0 < \delta < \epsilon$ such that, if $x, y \in M$ satisfy $dist(Z^t(x), Z^{\alpha(t)}(y)) \leq \delta$ for every $t \in \mathbb{R}$ and for some continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = Z^s(x)$, where $|s| \leq \epsilon$. So, take $0 < \delta' < \delta$ such that if $x, y \in M$ satisfy $dist(x, y) < \delta'$ then $dist(Z^t(x), Z^t(y)) < \delta$ for any $0 \leq t \leq \pi$.

As shown in the proof of Lemma 3.1, it is enough to assume that the eigenvalue σ is equal to 1. Fix a non-zero eigenvector v associated to σ such that $\|v\| < \delta'$. Now, choose $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p) \backslash \{p\}$ and observe that

$$f_Z(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P^\pi_Y(p) \circ \varphi_p(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P^\pi_Y(p)(v) = \varphi_p^{-1}(v).$$

Thus, $dist(p, \varphi_p^{-1}(v)) = dist(p, f_Z(\varphi_p^{-1}(v))) = \|v\| < \delta'$ and, by the choice of δ', we have that $dist(Z^t(p), Z^t(\varphi_p^{-1}(v))) < \delta$ for every $0 \leq t \leq \pi$. Then, there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$, with $\alpha(0) = 0$, such that

$$dist(Z^t(p), Z^{\alpha(t)}(\varphi_p^{-1}(v))) < \delta$$

for every $t \in \mathbb{R}$. Since $Z \in \mathcal{E}^1_\mu(M)$, we have that $\varphi_p^{-1}(v) = Z^s(p)$ for $|s| \leq \epsilon$. This is a contradiction, because $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p)\backslash\{p\}$. Hence, any closed orbit of X in $int(\mathcal{E}^1_\mu(M))$ is hyperbolic. □

We remark that, in [6, Lemma 1], Bowen and Walters prove that if $p \in M$ is a singularity of an expansive vector field, then there is $\epsilon > 0$ such that $B_\epsilon(p) = \{p\}$. Therefore, since M is a connected manifold, M must be regular. So, in particular, if $X \in int(\mathcal{E}^1_\mu(M))$, then $Sing(X) = \emptyset$.

Hence, by Theorem 2.1, $int(\mathcal{E}^1_\mu(M)) \subset \mathcal{A}^1_\mu(M)$, which concludes the proof of Theorem 1.

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