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PERTURBATION RESULTS FOR HYPERBOLIC EVOLUTION SYSTEMS IN HILBERT SPACES

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Abstract. The purpose of this paper is to derive a perturbation theory of evolution systems of the hyperbolic second order hyperbolic equations. We give an example of a partial functional equation as an application of the preceding result in case of the mixed problems for hyperbolic equations of second order with unbounded principal operators.

1. Introduction

The purpose of this paper is to derive a perturbation theory of the following perturbed inhomogeneous second order hyperbolic equation:

$$\begin{cases} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{1.1}$$

Phillips [11] started the study of properties of C_0-semigroups which are conserved under bounded perturbations, and perturbations of infinitesimal generators of analytic semigroups by a bounded operator is due to Kato [7]. Recently, Belarbi and Benchohra [1] proved the existence of solutions for a perturbed impulsive hyperbolic differential inclusion with variable times under the mixed generalized Lipschitz and Carathéodory's conditions.

Kato [8] was first to succeed in constructing the fundamental solution of temporally inhomogeneous second hyperbolic equation:

$$\begin{cases} u''(t) + A(t)u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{LE}$$

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in a Hilbert space H. For more general results see any of a number of source, including [7] and Tanabe [12]. Applications to initial value problem of hyperbolic equations have been referred to Goldstein [4] and Yosida [14], in addition [12]. Typical models can be found in the works of materials with biology, engineering, population models, etc. (see e.t., [2, 13] and the bibliography therein). As the second order nonlinear functional evolutions, Kalsatos and Markov in [6] have analyzed some questions on existence of solutions for functional differential inclusions of second order in time, and in [3] proved them in the case where a damping term is added. In [5] they have studied the wellposedness of solutions and the regularity properties of solutions for the mixed problems for semilinear hyperbolic equations of second order with unbounded principal operators.

In this paper, in order to give a construction of an evolution system of $A(t) + B(t)$, we will assume general conditions that $A(t)$, for each $t \in [0, T]$, is self adjoint and bounded and $A(t)v$ for each $v \in V$ is strongly continuous differentiable on $[0, T]$.

Let V be a Hilbert space forming a Gelfand triple $V \subset H \subset V^*$ with pivot space H. Recall that

$$\mathcal{A}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -u_1 \\ A(t)u_0 \end{pmatrix},$$
$$\mathcal{B}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B(t) & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ B(t)u_0 \end{pmatrix}$$

for any $\binom{u_0}{u_1} \in X = (V \times H)^T$ (or $\widetilde{X} = (H \times V^*)^T$), our problem can be applied to second order time dependent equations by writing them as first order systems. Consequently, we deal with constructing of the fundamental solution of (LE) explained the arguments in given in [1, 7]. In addition to assumptions of $A(t)$, Tanabe [12] dealt with a singular perturbation of evolution systems in a Banach space X with conditions that $B(t)$ is strongly continuous and there exists a real number λ_0 satisfying $\lambda_0 \in \rho(A(t))$ for all $t \in [0, T]$, such that

$$A(t)B(t)(A(t) - \lambda_0)^{-1} \in \mathcal{L}(X), \tag{1.2}$$

where $\mathcal{L}(X)$ denotes the set of all bounded linear operators from X into itself. But in Section 4, we will give a perturbation approach under the more general conditions that X is a Hilbert space and $B(t)v$ for each $v \in V$ is strongly continuous differentiable on $[0, T]$ instead of (1.2) even in special cases of second order equations. In the last section we give an example of a partial functional equation as an application of the preceding result in a mixed problem for hyperbolic case that

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_j}(a_{ij}(t, x)\frac{\partial u}{\partial x_i}), \quad B(t) = \sum_{i=1}^{n} b_i(t, x)\frac{\partial u}{\partial x_i} + c(t, x)u,$$

where the matrix $(a_{ij}(x, t))$ is uniformly positive definite.

2. Construction of fundamental solutions

Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Let V be embedded in H as a dense subspace with inner product and norm by $((\cdot,\cdot))$ and $||\cdot||$, respectively. By considering $H = H^*$, we may write $V \subset H \subset V^*$ where V^* denotes the dual space of V; its inner product and norm will be denoted by $(\cdot,\cdot)_*$ and $||\cdot||_*$, respectively. For $l \in V^*$ we denote (l,v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l,v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for the brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V.$$

Let $a(t;u,v)$ be quadratic form defined on $V \times V$ and let us also make the following assumptions:

i) $a(t;u,v)$ is bounded and uniformly Lipschitz continuous and $\frac{da(t;u,v)}{dt}$ is strong continuous with respect to t, i.e., there are some positive constants c_0, c_1 such that

$$|a(t;u,v)| \leq c_0||u||\,||v||,$$
$$|a(t;u,v) - a(s;u,v)| \leq c_1|t-s|\,||u||\,||v||,$$
$$|d/dt\ a(t;u,v)| = |\dot{a}(t;u,v)| \leq c_1||u||\,||v||;$$

ii) $a(t;u,v)$ is symmetric, i.e., $a(t;u,v) = \overline{a(t;v,u)}$;

iii) $a(t;u,v)$ satisfies the Gårding's inequality, i.e.,

$$Re\, a(t;u,u) \geq \delta||u||^2, \quad \delta > 0.$$

Let us define $A(t)$ the operator determined by $a(t;u,v)$, i.e., we set

$$a(t;u,v) = (A(t)u,v), \quad u,v \in V.$$

Then it is easily seen that $A(t)$ is an isomorphism V onto V^* and for $u \in V$, we have

$$\delta||u|| \leq ||A(t)u||_* \leq c_0||u||. \tag{2.1}$$

The restriction of $A(t)$ to

$$D(A_H(t)) = \{u \in V; \quad A(t)u \in H\}$$

is denoted by $A_H(t)$. Then it is well known that $D(A_H(t))$ is dense in H by Lax-Milgram theorem and it is easy to see that

$$\delta||u|| \leq |A_H(t)u| \leq c_0||u||_{D(A_H(t))}.$$

It is obvious that $A(t)$ is an extension of the operator $A_H(t)$. Here and in what follows we consider that $D(A(t)) = V$ is independent of t in terms of (2.1).

Put $X = (V \times H)^T$, $\widetilde{X} = (H \times V^*)^T$. We define inner product of X and $\widetilde{X}$ by

$$\left(\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)\right)_X = ((u_0, v_0)) + (u_1, v_1),$$

and

$$\left(\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}\right)_{\widetilde{X}} = (f_0, g_0) + (f_1, g_1)_*,$$

respectively. We introduce a new inner product $((\ ,\ ))_t$ and norm $||\cdot||_t$ into X as

$$\left(\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)\right)_t = a(t; u_0, v_0) + (u_1, v_1)$$

and

$$\left\|\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right\|_t = \{a(t; u_0, u_0) + (u_1, u_1)\}^{\frac{1}{2}}$$

for $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in X$, respectively. Let us introduce a new norm in V^* as follows. For $f_1, g_1 \in V^*$, putting

$$(f_1, g_1)_{*,t} = a(t; A(t)^{-1} f_1, A(t)^{-1} g_1) = (f_1, A(t)^{-1} g_1),$$

it satisfies the inner product properties and its norm is given by

$$||f_1||_{*,t} = (f_1, f_1)_{*,t}^{1/2} = a(t; A(t)^{-1} f_1, A(t)^{-1} f_1)^{1/2} = (f_1, A(t)^{-1} f_1)^{1/2}.$$

It is easily seen that the norm $||\cdot||_{*,t}$ is equivalent to $||\cdot||_*$, i.e, we have

$$\frac{\delta}{\sqrt{c_0}} ||\cdot||_{*,t} \le ||\cdot||_* \le \frac{c_0}{\sqrt{\delta}} ||\cdot||_{*,t}.$$

We also introduce a inner product $(\ ,\ )_t$ and norm $|\cdot|_t$ into $\widetilde{X}$ as

$$\begin{aligned}\left(\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}\right)_t &= (f_0, g_0) + a(t; A(t)^{-1} f_1, A(t)^{-1} g_1) \\ &= (f_0, g_0) + (f_1, A(t)^{-1} g_1)_{*,t}\end{aligned}$$

and

$$\left|\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}\right|_t = \left(|f_0|^2 + ||f_1||_{*,t}^2\right)^{1/2}.$$

The Hilbert spaces defined by the inner products mentioned above denote by X_t and $\widetilde{X}_t$, respectively.

Let $\mathcal{A}_X(t)$ be an operator defined by

$$D(\mathcal{A}_X(t)) = (D(A_H(t)) \times V)^T,$$

$$\mathcal{A}_X(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A_H(t) & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -u_1 \\ A_H(t) u_0 \end{pmatrix} \in (V \times H)^T = X.$$

In virtue of Lax-Milgram theorem we can also consider as

$$D(\mathcal{A}(t)) = (V \times H)^T = X,$$

$$\mathcal{A}(t)\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} -g_1 \\ A(t)g_0 \end{pmatrix} \in (H \times V^*)^T = \widetilde{X}.$$

Now we consider the initial-value problem of the inhomogeneous second hyperbolic equation(LE). Let $\mathbf{x}(t) = \binom{u_0(t)}{u_1(t)}$ where $u_1(t) = \frac{d}{dt}u_0(t)$, and let $F(t) = \binom{0}{f(t)}$. Then the equation (LE) can be rewritten by

$$\begin{cases} \mathbf{x}'(t) + \mathcal{A}(t)\mathbf{x}(t) = F(t) \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{2.2}$$

where $\mathbf{x}_0 = \binom{u_0}{u_1}$. We have known that $A_H(t)$ and $A(t)$ generate analytic semigroups in H and V^*, respectively, so the equation (LE) is considered in the space both H and V^*.

As seen in [5, Theorems 2.1 and 2.2], we obtain the following results.

Proposition 2.1. *The linear operators $\mathcal{A}_X(t)$ and $\mathcal{A}(t)$ mentioned above are the infinitesimal generators of C_0-groups of unitary operators in X_t and $\widetilde{X}_t$, respectively. Moreover, $\mathcal{A}_X(t)$ and $\mathcal{A}(t)$ are stable on X and $\widetilde{X}$, respectively.*

For every $\binom{u_0}{u_1} \in D(\mathcal{A}_X(t)) = D(A_H(t)) \times V$, we have

$$\frac{d}{dt}\mathcal{A}_X(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} -u_1 \\ A_H(t)u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ d/dt\, A_H(t)u_0 \end{pmatrix}. \tag{2.3}$$

From which and $d/dt\ (A_H(t)u, v) = \dot{a}(t; u, v)$, it follows that $d/dt\,\mathcal{A}_X(t)\binom{u_0}{u_1}$ is strong continuous with respect to t. Thus the following Proposition is from Corollary in Section 4.4 of [12] (or Theorem 2.3 of [11]).

Proposition 2.2. *Let $\mathcal{A}(t)$ (or $\mathcal{A}_X(t)$) be the operators mentioned above. Then there exist fundamental solution $\mathcal{U}(t,s)$ (or $\mathcal{U}_X(t,s)$) satisfying*

(a) $\mathcal{U}(t,s)$ *is strongly continuously in s and t, and* $||\mathcal{U}(t,s)||_{\mathcal{L}(\widetilde{X})} \le Me^{\beta(t-s)}$,

(b) $\mathcal{U}(s,s) = I$, *and* $\mathcal{U}(t,s) = \mathcal{U}(t,r)\mathcal{U}(r,s)$ *for* $s \le r \le t$,

(c) $\partial/\partial t\,\mathcal{U}(t,s)v = -A(t)\mathcal{U}(t,s)v$,

(d) $\partial/\partial s\,\mathcal{U}(t,s)v = \mathcal{U}(t,s)A(s)v$ *in* $\widetilde{X}$ (*or in* X, *respectively*).

Let $\mathbf{x}(t) = \binom{u(t)}{u'(t)}$ and $F(t) = \binom{0}{f(t)}$. We can show that a solution $\mathbf{x}(t)$ of (LE) is represented by

$$\mathbf{x}(t) = \mathcal{U}(t,0)\mathbf{x}(0) + \int_0^t \mathcal{U}(t,s)F(s)ds \tag{2.4}$$

using the fundamental solution $\mathcal{U}(t,s)$ constructed in Proposition B. Indeed, we have

$$(\partial/\partial s)\mathcal{U}(t,s)\mathbf{x}(s) = \mathcal{U}(t,s)\mathbf{x}'(s) + \mathcal{U}(t,s)\mathcal{A}(s)\mathbf{x}(s) = \mathcal{U}(t,s)F(s),$$

which, being integrated from 0 to t, yields (2.3). Let $T > 0$. Define

$$W_T = \{u : u \in L^2(0,T,D(A_H)),\ \dot{u} \in L^2(0,T,V),\ \ddot{u} \in L^2(0,T,H)\},$$
$$||u||_{W_T} = ||u||_{L^2(0,T,D(A_H))} + ||\dot{u}||_{L^2(0,T,V)} + ||\ddot{u}||_{L^2(0,T,H)},$$

and

$$\widetilde{W}_T = \{u : u \in L^2(0,T,V),\ \dot{u} \in L^2(0,T,H),\ \ddot{u} \in L^2(0,T,V^*)\},$$
$$||u||_{\widetilde{W}_T} = ||u||_{L^2(0,T,V)} + ||\dot{u}||_{L^2(0,T,H)} + ||\ddot{u}||_{L^2(0,T,V^*)},$$

where $\dot{u}$ denote the derivative of u in the generalized sense. Since

$$\mathcal{A}(t)^{-1} = \begin{pmatrix} 0 & A(t)^{-1} \\ -I & 0 \end{pmatrix} :\ \widetilde{X} \to X$$

is a bounded operator. It holds $\mathcal{A}(t)\mathcal{U}t,s)\mathcal{A}(t)^{-1} : \widetilde{X} \to \widetilde{X}$ is bounded and strong continuous jointly in s,t. Therefore, there is constant $M > 0$ such that

$$(2.5) \qquad ||\mathcal{U}(t,s)||_{\mathcal{L}(\widetilde{X})} \le M, \quad ||\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}||_{\mathcal{L}(\widetilde{X})} \le M.$$

By the assumption i) of $a(s;u,v)$, it holds that for every $u, v \in V$,

$$|d/ds\ (A(s)u,v)| = |\dot{a}(s;u,v)| \le c_1||u||\,||v||,$$

that is, we have that for every $u \in V$, $s \mapsto d/dsA(s)u$ is strongly continuous in V^* and so, $||d/dsA(s)||_{\mathcal{L}(V,V^*)}$ is bounded on $[0,T]$. Hence, noting that for every $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X$, it follows from (2.3) that $d/ds\,\mathcal{A}(s)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ is strong continuous with respect to t in $\widetilde{X}$ and so, $||d/ds\mathcal{A}(s)||_{\mathcal{L}(X,\widetilde{X})}$ is bounded on $[0,T]$. Therefore, we may assume that

$$(2.6) \qquad ||\frac{d}{ds}\mathcal{A}(s)\mathcal{A}(s)^{-1}||_{\mathcal{L}(\widetilde{X})} \le M.$$

Now we give useful regularity results and the energy inequalities for our problem (LE) (see [9, 11]).

Proposition 2.3. *Assume that* $f \in C([0,T];V^*) \cap W^{1,2}(0,T;V^*)(T > 0)$ *and the initial data* $(u_0,u_1) \in V \times H$. *Then the solution* u *of (LE) exists and is unique in*

$$u \in \widetilde{W}_T \cap C([0,T];V) \cap C^1([0,T);H).$$

Furthermore, the following energy inequality holds: there exists a constant C_T *depending on* T *such that*

$$||u||_{\widetilde{W}_T} \le C_T(||u_0|| + |u_1| + ||f(0)||_* + ||f||_{W^{1,2}(0,T;V^*)}).$$

If $f \in C([0,T];H) \cap W^{1,2}(0,T;H)$ *and* $(u_0,u_1) \in D(A_H) \times V$, *then the solution* u *of (LE) exists and is unique in*

$$u \in W_T \cap C([0,T];D(A_H)) \cap C^1([0,T);V),$$

satisfying

$$||u||_{W_T} \le C_T(||u_0||_{D(A_H)} + ||u_1|| + |f(0)| + ||f||_{W^{1,2}(0,T;H)}).$$

3. Perturbation for fundamental solutions

Consider the following perturbed inhomogeneous second order hyperbolic equation:

$$\text{(PE)} \qquad \begin{cases} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

where $A(t)$ satisfies the conditions in Section 2. From now on, both $A_H(t)$ and $A(t)$ are denoted simply by $A(t)$ without the risk of confusing. Let $B(t)$ be defined on $[0, T]$ as a strongly continuously differentiable satisfying

$$(3.1) \qquad B(t)u \in C^1([0, T); H), \quad |B(t)u| \le B|u| \quad \text{for all} \quad u \in H$$

for some constant $B > 0$. For $\binom{u_0}{u_1} \in (V \times H)^T = X$, let $\mathcal{B}(t)$ be an operator defined by

$$\mathcal{B}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -B(t) & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -B(t)u_0 \end{pmatrix} \in X.$$

Then we have that $\mathcal{B}(t) : H_0^1(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ is a bounded operator and strongly continuous differentiable with respect to t.

Theorem 3.1. *Assume that $\{\mathcal{A}(t) : 0 \le t \le T\}$ satisfies the conditions in Section 2. Assume also that $B(t)$ is defined on $[0, T]$ as a strongly continuously differentiable with values in $B(X)$. Then there exists a fundamental solution $\mathcal{W}(t, s)$ of (PE) satisfy the following results: for each $\mathbf{x} \in D(\mathcal{A}(t)) = (D(A(t)) \times V)^T$,*

(a) *$\mathcal{W}(t, s)$ is strongly continuously in s and t, and $||\mathcal{W}(t, s)|| \le Me^{\beta(t-s)}$,*
(b) *$\mathcal{W}(s, s) = I$, and $\mathcal{W}(t, s) = \mathcal{W}(t, r)\mathcal{W}(r, s)$ for $s \le r \le t$,*
(c) *$\partial/\partial t\mathcal{W}(t, s)\mathbf{x} = -(\mathcal{A}(t) + \mathcal{B}(t))\mathcal{W}(t, s)\mathbf{x}$,*
(d) *$\partial/\partial s\mathcal{W}(t, s)\mathbf{x} = \mathcal{W}(t, s)(\mathcal{A}(t) + \mathcal{B}(t))\mathbf{x}$.*

Proof. Let us denote $\mathcal{U}(t, s)$ the evolution fundamental system of

$$\mathbf{x}'(t) + \mathcal{A}(t)\mathbf{x}(t) = F(t)$$

whose existence is proved by Propositions 2.2 and 2.3. For the sake of simplicity in sense of (2.5), we assume that there are constants M_0, M_1 such that

$$(3.2) \qquad ||\mathcal{U}(t, s)|| \le M_0, \quad ||\mathcal{A}(t)\mathcal{U}(t, s)\mathcal{A}(s)^{-1}|| \le M_1.$$

Put

$$(3.3) \qquad \mathcal{W}_0(t, s) = \mathcal{U}(t, s), \quad \mathcal{W}_m(t, s) = -\int_s^t \mathcal{U}(t, \tau)\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau, s)d\tau,$$

$$(3.4) \qquad \mathcal{W}(t, s) = \sum_{m=0}^{\infty} \mathcal{W}_m(t, s),$$

for $m = 1, 2 \ldots$. Then we have

$$\mathcal{W}(t,s) = \mathcal{U}(t,s) - \int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}(\tau,s)d\tau, \tag{3.5}$$

and the series on the right hand side of (3.4) is strongly convergent uniformly in $0 \le s \le t \le T$. Indeed, by (3.4)

$$\begin{aligned}\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}(\tau,s)d\tau &= \int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\sum_{m=0}^{\infty}\mathcal{W}_m(\tau,s)d\tau \\ &= \sum_{m=0}^{\infty}\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}_m(\tau,s)d\tau \\ &= -\sum_{m=0}^{\infty}\mathcal{W}_{m+1}(t,s) \\ &= -\sum_{m=0}^{\infty}\mathcal{W}_m(t,s) + \mathcal{U}(t,s),\end{aligned}$$

which yields (3.5). From (3.1), (3.2), it follows, by mathematical induction, that

$$\begin{aligned}&||\mathcal{U}(t,s)|| \le M_0, \\ &||\mathcal{W}_m(t,s)|| \le || - \int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)d\tau|| \le M_0^{m+1}B^m\frac{(t-s)^m}{m!}.\end{aligned}$$

Hence $\sum_0^\infty \mathcal{W}_m(t,s)$ is uniformly convergence.

First, we will show that $\partial/\partial t\, \mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}$ exists and is strongly continuous on $B(X)$ for all $m = 1, 2, \ldots$. From (d) of Proposition 2.2 and Proposition 2.3, we have

$$\mathcal{U}(t,s) = \frac{\partial}{\partial s}\mathcal{U}(t,s)\mathcal{A}(s)^{-1} \tag{3.6}$$

and

$$\begin{aligned}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1} &= -\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}d\tau \\ &= -\int_s^t \frac{\partial}{\partial\tau}\mathcal{U}(t,\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}d\tau \\ &= -\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}_{m-1}(t,s)\mathcal{A}(s)^{-1} \\ &\quad + \mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s,s)\mathcal{A}(s)^{-1} \\ &\quad + \int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1})d\tau.\end{aligned} \tag{3.7}$$

Here,

$$\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}) \tag{3.8}$$

$$= \mathcal{A}(\tau)^{-1}(-\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau) + \dot{\mathcal{B}}(\tau))\mathcal{W}_{m-1}(\tau, s)\mathcal{A}(s)^{-1}$$
$$+ \mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\frac{\partial}{\partial\tau}\mathcal{W}_{m-1}(\tau, s)\mathcal{A}(s)^{-1}.$$

Now we shall show that the right side of (3.7) is differentiable with respect to t and therefore $\mathcal{W}(t, s)\mathcal{A}(s)^{-1}$ is differentiable. Noting that

$$\frac{\partial}{\partial t}\mathcal{U}(t, s) = -\mathcal{A}\mathcal{U}(t, s),$$

consider that

(3.9)

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathcal{W}_m(t, s)\mathcal{A}(s)^{-1} \\
= &-\frac{\partial}{\partial t}(\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}_{m-1}(t, s)\mathcal{A}(s)^{-1}) \\
&- \mathcal{A}(t)\mathcal{U}(t, s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s, s)\mathcal{A}(s)^{-1} \\
&+ \frac{\partial}{\partial t}(\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}_{m-1}(\tau, s)\mathcal{A}(s)^{-1}) \\
&- \int_s^t \mathcal{A}(t)\mathcal{U}(t, \tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau, s)\mathcal{A}(s)^{-1})d\tau \\
= &-\mathcal{A}(t)\mathcal{U}(t, s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s, s)\mathcal{A}(s)^{-1} \\
&- \int_s^t \mathcal{A}(t)\mathcal{U}(t, \tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau, s)\mathcal{A}(s)^{-1})d\tau \\
= &-\mathcal{A}(t)\mathcal{U}(t, s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s, s)\mathcal{A}(s)^{-1} \\
&- \int_s^t \mathcal{A}(t)\mathcal{U}(t, \tau)\mathcal{A}(\tau)^{-1}\{-\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau) + \dot{\mathcal{B}}(\tau)\}\mathcal{W}_{m-1}(\tau, s)\mathcal{A}(s)^{-1}d\tau \\
&- \int_s^t \mathcal{A}(t)\mathcal{U}(t, \tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\frac{\partial}{\partial\tau}\mathcal{W}_{m-1}(\tau, s)\mathcal{A}(s)^{-1}d\tau.
\end{aligned}$$

From (2.6), (3.1) we know that $-\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)+\dot{\mathcal{B}}(\tau)$ is uniformly bounded, and so there exists a constant M_2 such that

$$||\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau) + \dot{\mathcal{B}}(\tau)|| \leq M_2. \tag{3.10}$$

If $m = 1$ in (3.9), then

$$\begin{aligned}
&||\frac{\partial}{\partial t}\mathcal{W}_1(t, s)\mathcal{A}(s)^{-1}|| \\
\leq\ & M_1B||\mathcal{A}(s)^{-1}|| + \int_s^t M_1M_2M_0||\mathcal{A}(s)^{-1}||d\tau + \int_s^t M_1B||\frac{\partial}{\partial\tau}\mathcal{U}(\tau, s)\mathcal{A}(s)^{-1}||d\tau \\
\leq\ & M_1B||\mathcal{A}(s)^{-1}|| + M_1M_2M_0||\mathcal{A}(s)^{-1}||(t-s) + M_1^2B(t-s).
\end{aligned}$$

If $m \geq 2$, then $\mathcal{W}_{m-1}(s,s) = 0$ by (3.3) and hence

$$
\begin{aligned}
&||\frac{\partial}{\partial t}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}|| \\
\leq &\int_s^t M_1M_2M_0^mB^{m-1}\frac{(\tau-s)^{m-1}}{(m-1)!}||\mathcal{A}(s)^{-1}||d\tau \\
&+\int_s^t M_1B||\frac{\partial}{\partial \tau}\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}||d\tau \\
\leq &M_1M_2M_0^mB^{m-1}||\mathcal{A}(s)^{-1}||\frac{(t-s)^m}{m!} + M_1B\int_s^t ||\frac{\partial}{\partial \tau}\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}||d\tau.
\end{aligned}
$$

By mathematical induction, it satisfies the following that

$$
\begin{aligned}
&||\frac{\partial}{\partial t}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}|| \\
\leq &M_1^mB^m||\mathcal{A}(s)^{-1}||\frac{(t-s)^{m-1}}{(m-1)!} \\
&+M_1M_2M_0B^{m-1}\sum_{i=0}^{m-1}M_0^{m-1-i}M_1^i||\mathcal{A}(s)^{-1}||\frac{(t-s)^m}{m!}+M_1^{m+1}B^m\frac{(t-s)^m}{m!}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&||\frac{\partial}{\partial t}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}|| \\
\leq &M_1^mB^m m||\mathcal{A}(s)^{-1}||\frac{(t-s)^{m-1}}{m!} \\
&+M_1M_2M_0B^{m-1}m\{\max\{M_0,M_1\}\}^{m-1}||\mathcal{A}(s)^{-1}||\frac{(t-s)^m}{m!} \\
&+M_1^{m+1}B^m\frac{(t-s)^m}{m!}
\end{aligned}
$$

for all m, so that $\sum_{m=0}^\infty ||\partial/\partial t\, \mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}||$ is uniformly convergence. Thus

$$
\frac{\partial}{\partial t}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} = \frac{\partial}{\partial t}\sum_{m=0}^\infty \mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}
$$

exists and is strongly continuous. Noting that

$$
\mathcal{W}(t,s) = \mathcal{U}(t,s) - \int_s^t \mathcal{U}(t,s)\mathcal{B}(\tau)\mathcal{W}(\tau,s)d\tau
$$

and $\mathcal{U}(t,s) = \partial/\partial s\mathcal{U}_(t,s)\mathcal{A}(s)^{-1}$, it holds

(3.11)

$$
\begin{aligned}
&\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \\
&= \mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \int_s^t \frac{\partial}{\partial \tau}\mathcal{U}(t,\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1}d\tau
\end{aligned}
$$

$$= \mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} + \mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} + \int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau.$$

from which it follows

$$\begin{aligned}
(3.12)\qquad & \frac{\partial}{\partial t}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \\
&= -\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \frac{\partial}{\partial t}\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \\
&\quad - \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} + \frac{\partial}{\partial t}(\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1}) \\
&\quad - \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau \\
&= -\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(t)^{-1} - \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} \\
&\quad - \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau.
\end{aligned}$$

Put $\widetilde{\mathcal{A}(t)} = \mathcal{A}(t) + \mathcal{B}(t)$, Then from (3.11) we obtain that

$$\begin{aligned}
(3.13)\qquad & \widetilde{\mathcal{A}(t)}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \\
&= \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1} + \mathcal{B}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \\
&\quad - \mathcal{B}(t)\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} + \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} \\
&\quad + \mathcal{B}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} \\
&\quad + \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau \\
&\quad + \mathcal{B}(t)\int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau.
\end{aligned}$$

Therefore from which and (3.12) it follows that

$$\begin{aligned}
& \frac{\partial}{\partial t}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} + \widetilde{\mathcal{A}(t)}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \\
&= \mathcal{B}(t)\{\mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \mathcal{W}(t,s)\mathcal{A}(s)^{-1} - \mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \\
&\quad + \mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} + \int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau\}.
\end{aligned}$$

By (3.11) the right side of (3.13) equals zero. Thus, it is evident that $\mathcal{W}(t,s)\mathbf{x}$ is differentiable in s and t and satisfies

$$\frac{\partial}{\partial t}W(t,s)\mathbf{x} = -(\mathcal{A}(t) + \mathcal{B}(t))W(t,s)\mathbf{x},$$
$$\frac{\partial}{\partial s}W(t,s)\mathbf{x} = W(t,s)(\mathcal{A}(t) + \mathcal{B}(t))\mathbf{x}$$

for each $\mathbf{x} \in D(\mathcal{A}(t)) = (D(A(t)) \times V)^T$ (or $\mathbf{x} \in (V \times H)^T = X$). Hence such an operator valued function $\mathcal{W}(t,s)$ is a fundamental solution of $\partial/\partial t\, \mathbf{x}(t) + (\mathcal{A}(t) + \mathcal{B}(t))\mathbf{x}(t) = 0$. □

Remark 3.2. Let us assume also that $B(t)$ is defined on $[0,T]$ as a strongly continuously differentiable with values in $B(\widetilde{X})$. Then for each $\mathbf{x} \in (V \times H)^T = X$, there exists a fundamental solution $\mathcal{W}(t,s)$ of (PE) satisfying (a), (b), (c), and (d) in Theorem 3.1 in $\widetilde{X}$.

4. Mixed problem of hyperbolic equations

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We define the following spaces:

$$H^1(\Omega) = \left\{ u : u,\ \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ \ i = 1, 2, \ldots, n \right\},$$

$$H^2(\Omega) = \left\{ u : u,\ \frac{\partial u}{\partial x_i},\ \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega),\ \ i, j = 1, 2, \ldots, n \right\},$$

$$H_0^1(\Omega) = \{ u : u \in H^1(\Omega),\ \ u|_{\partial\Omega} = 0 \} = \text{the closure of } C_0^\infty(\Omega) \text{ in } H^1(\Omega)$$

where $\partial/\partial x_i\, u$ and $\partial^2/\partial x_i \partial x_j\, u$ are the derivative of u in the distribution sense. The norm of $H_0^1(\Omega)$ is defined by

$$||u|| = \left\{ \int_\Omega \sum_{i=1}^n (\frac{\partial u(x)}{\partial x_i})^2 dx \right\}^{1/2}.$$

Hence $H_0^1(\Omega)$ is a Hilbert space. Let $H^{-1}(\Omega)$=$H_0^1(\Omega)^*$ be a dual space of $H_0^1(\Omega)$. For any $l \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$, the notation (l, v) denotes the value l at v. In what follows, we consider the regularity for given equations in the space $L^2(\Omega)$ in place of H in Section 2. Then the space instead of V is $H_0^1(\Omega) = \{ u \in H^1(\Omega); u = 0 \text{ on } \partial\Omega \}$.

Consider the mixed problem for the hyperbolic equation:

$$\text{(4.1)}\quad \begin{cases} \frac{\partial^2}{\partial t^2} u(t,x) - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}) + \sum_{i=1}^n b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u & \\ = f(t), & 0 \le t < \infty, \quad x \in \Omega, \\ u(t,x) = 0, & 0 \le t < \infty, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad \frac{\partial}{\partial t} u(0,x) = u_1(x), & x \in \Omega. \end{cases}$$

We deal with the Dirichlet condition's case as follows. The matrix $(a_{ij}(x,t))$ is uniformly positive definite, i.e., there exists a positive constant δ such that

$$\sum_{i,j=1}^n a_{ij}(x,t)\xi_i \bar{\xi_j} \ge \delta |\xi|^2$$

for all $x \in \Omega$, $t \in [0,T]$ and for all real vectors ξ. Let

$$a_{ij}, \quad \frac{\partial}{\partial x_j} a_{ij}, \frac{\partial}{\partial t} a_{ij}, \quad \frac{\partial^2}{\partial t \partial x_j} a_{ij}, \quad \frac{\partial}{\partial t} b_i, \quad c \ge 0, \quad \frac{\partial}{\partial t} c$$

be all continuous and bounded on $\Omega \times [0, T]$, and

$$a_{ij}, \quad \frac{\partial}{\partial x_j} a_{ij}, \quad c$$

satisfy uniformly Lipschitz's condition with respect to t.

For each $t \in [0, T]$ and $u, v \in H_0^1(\Omega)$, let us consider the following sesquilinear form:

$$a(t; u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(t, x) \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} dx.$$

Then there exist constants $c_0,\ c_1 > 0$ such that

$$|a(t, u, v)| \le c_0 ||u|| ||v||,$$

$$\Big|\frac{d}{dt} a(t, u, v)\Big| = \Big|\int_\Omega \sum_{i,j=1}^{n} \dot{a}_{ij}(t, x) \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} dx\Big| \le c_1 ||u|| \cdot ||v||$$

and it holds Gårding's inequality;

$$a(t; u, u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \overline{\frac{\partial u}{\partial x_j}} dx \ge \delta \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^2 dx = \delta ||u||^2.$$

Define the operator $A(t)$ by

$$(A(t)u, v) = a(t; u, v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} dx,$$

$$D(A(t)) = \{u : u \in H^2(\Omega) \cap H_0^1(\Omega)\} = \{u : u \in H^2(\Omega),\ \ u|_{\partial\Omega} = 0\}.$$

The operator $A(t)$ in $L^2(\Omega)$ is defined as the following that for any $v \in H_0^1(\Omega)$ there exists $f \in L^2(\Omega)$ such that

$$a(t; u, v) = (f, v)$$

then, for any $u \in D(A(t))$, $A(t)u = f$ and $A(t)$ is a positive definite self-adjoint operator. Let u be fixed if we consider the functional $H_0^1(\Omega) \ni v \to a(t; u, v)$, this function is a continuous linear. For any $l \in H^{-1}(\Omega)$, it follow that $(l, v) = a(t; u, v)$. We denote that for any $u, v \in H_0^1(\Omega)$

$$a(t; u, v) = (\widetilde{A}(t)u, v),$$

that is, $\widetilde{A}(t)u = l$. The operator $\widetilde{A}(t)$ is one to one mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. The relation of operators $A(t)$ and $\widetilde{A}(t)$ satisfy the following that for any $u \in D(A(t))$,

$$D(A(t)) = \{u \in H_0^1(\Omega),\ \ \widetilde{A}(t)u \in L^2(\Omega)\}, \quad A(t)u = \widetilde{A}(t)u.$$

From now on, both $A(t)$ and $\widetilde{A}(t)$ are denoted simply by A. Put

$$D(B(t)) = H_0^1(\Omega), \quad B(t)u = \sum_{i,j=1}^{n} b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u,$$

and for $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\mathcal{A}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -u_1 \\ A(t)u_0 \end{pmatrix},$$

$$\mathcal{B}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B(t) & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ B(t)u_0 \end{pmatrix}.$$

Then $\mathcal{B}(t)$ is a bounded operator from $X = (H_0^1(\Omega) \times L^2(\Omega))^T$ to itself and strongly continuous differentiable with respect to t. Since

$$|B(t)u_0| \leq \max\{(|b_1|, |b_2|, \ldots, |b_n|, |c|\}(\sum |\frac{\partial u_0}{\partial x_i}|^2 + |u_0|^2)^{\frac{1}{2}} \leq c||u_0||_{H_0^1(\Omega)},$$

we have

$$\left\|\mathcal{B}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_X \leq c \left|\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right|_X (\mathcal{B}(t) \in B(X)).$$

Then we treat (4.1) as the initial value problem for the abstract second order equations

$$\left\{ \begin{array}{l} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_0, \qquad u'(0) = u_1. \end{array} \right. \tag{4.2}$$

Now we can apply the results of Theorem 4.1 and Remark 3.2 as follows.

Theorem 4.1. *Assume that $\{\mathcal{A}(t) : 0 \leq t \leq T\}$ is defined as mentioned above and $B(t)$ is defined on $[0, T]$ as a strongly continuously differentiable with values in $\mathcal{L}(L^2(\Omega))$. Let us assume that $f \in C([0, T]; H^{-1}(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega))$ $(T > 0)$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, there exists a fundamental solution $\mathcal{W}(t, s)$ of* (4.2) *satisfying* (a), (b), (c), *and* (d) *in Theorem 3.1 and the solution u of* (4.1) *exists and is unique in*

$$u \in \widetilde{W}_T \cap C([0, T]; H_0^1(\Omega)) \cap C^1([0, T); L^2(\Omega)), \quad T > 0$$

where

$$\widetilde{W}_T = L^2(0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)) \cap W^{2,2}(0, T; H^{-1}(\Omega)).$$

Furthermore, the following energy inequality holds: there exists a constant C_T depending on T such that

$$||u||_{\widetilde{W}_T} \leq C_T(1 + ||u_0|| + |u_1| + ||f(0)||_* + ||f||_{W^{1,2}(0,T;H^{-1}(\Omega))}).$$

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