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ON STEIN TRANSFORMATION IN SEMIDEFINITE LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. In the setting of semidefinite linear complementarity problems on $\mathbf{S}^n$, we focus on the Stein Transformation $S_A(X) := X - AXA^T$, and show that S_A is (strictly) monotone if and only if $\nu_r(UAU^T \circ UAU^T)\,(<) \leq 1$, for all orthogonal matrices U where $\circ$ is the Hadamard product and ν_r is the real numerical radius. In particular, we show that if $\rho(A) < 1$ and $\nu_r(UAU^T \circ UAU^T) \leq 1$, then SDLCP$(S_A, Q)$ has a unique solution for all $Q \in \mathbf{S}^n$. In an attempt to characterize the **GUS**-property of a nonmonotone S_A, we give an instance of a nonnormal 2×2 matrix A such that SDLCP(S_A, Q) has a unique solution for Q either a diagonal or a symmetric positive or negative semidefinite matrix. We show that this particular S_A has the $\mathbf{P_2'}$-property.

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1. Introduction

Given a continuous function f from a real Hilbert space H into itself and a closed convex set K in H, the *variational inequality problem* VI(f, K) is to find a vector x^* in H such that

$$x^* \in K \quad \text{and} \quad \langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in K. \tag{1}$$

This problem has been extensively studied in the literature. In the infinite dimensional setting, it appears in the study of partial differential equations, mechanics, etc. [17]. In the finite dimensional setting, it appears in optimization, economics, traffic equilibrium problems etc. [13].

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Now suppose K is a cone, i.e., $tK \subseteq K$ for all $t \geq 0$. Then by putting $x = 0$ and $x = 2x^*$, the condition

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in K \tag{2}$$

leads to

$$\langle f(x^*), x^* \rangle = 0. \tag{3}$$

If we define the dual cone K^* of K by

$$K^* := \{y \in H : \langle x, y \rangle \geq 0 \quad \forall x \in K\}, \tag{4}$$

then, (2) and (3) together imply that $f(x^*) \in K^*$.

So when K is a closed convex cone, (1) becomes the problem of finding an $x^* \in H$ such that

$$x^* \in K, \quad f(x^*) \in K^*, \quad \text{and} \quad \langle x^*, f(x^*) \rangle = 0. \tag{5}$$

This is a *cone complementarity problem.* This problem and its special cases often arise in optimization (Karush-Kuhn-Tucker conditions), game theory (bimatrix games), mechanics (contact problem, structural engineering), economics (equilibrium in a competitive economy) etc. For a detailed description of these applications, we refer to [16], [6], [5], [20], [21]. In this paper, we focus on the so-called *semidefinite linear complementarity problem* (SDLCP) introduced by Gowda and Song [8]: Let $\mathbf{S}^n$ denote the space of all real symmetric $n \times n$ matrices, and $\mathbf{S}^n_+$ be the set of symmetric positive semidefinite matrices in $\mathbf{S}^n$. With the inner product defined by $\langle Z, W \rangle := tr(ZW)$, $\forall Z, W \in \mathbf{S}^n$, the space $\mathbf{S}^n$ becomes a Hilbert space. Clearly, $\mathbf{S}^n_+$ is a closed convex cone in $\mathbf{S}^n$. Given a linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ and a matrix $Q \in \mathbf{S}^n$, the *semidefinite linear complementarity problem*, denoted by SDLCP(L, Q), is the problem of finding a matrix $X \in \mathbf{S}^n$ such that

$$X \in \mathbf{S}^n_+, \; Y := L(X) + Q \in \mathbf{S}^n_+, \quad \text{and} \quad \langle X, Y \rangle = 0. \tag{6}$$

Examples of the semidefinite linear complementarity problem are: the standard linear complementarity problem [4], the block SDLCP [27], and the geometric SDLCP of Kojima, Shindoh, and Hara [18]. For detalils on how to reformulate these as the SDLCP of Gowda and Song (6), we refer to the Ph.D. Thesis of Song (Section 1.3 [22]). We give the description of the standard linear complementarity problem here that is needed in the paper. Consider the Euclidean space R^n with the cone of nonnegative vectors R^n_+. Given a matrix $M \in R^{n \times n}$ and a vector $q \in R^n$, the *linear complementarity problem* LCP(M, q) is to find a vector in R^n such that

$$x \in R^n_+, \quad y := Mx + q \in R^n_+, \quad \text{and} \quad \langle x, y \rangle = 0, \tag{7}$$

where $\langle x, y \rangle$ is the usual inner product in R^n. This problem has been studied in great detail, see [4], [6], [5], [20]. In this setting, we have the following result:

(a) (3.3.1, 3.3.7. [4]) LCP(M, q) has a unique solution for any given $q \in R^n$ if and only if M is a $\mathbf{P}$-matrix (that is, all its principal minors are positive).

(b) (3.3.4. [4]) M is a $\mathbf{P}$-matrix if and only if

$$z_i(Mz)_i \leq 0 \quad \forall i \quad \Longrightarrow \quad z = 0,$$

where y_i denotes the ith element of the vector y.

As mentioned above, the standard linear complementarity problem is a special case of the semidefinite linear complementarity problem in the following way [22]:

Let a matrix $M \in R^{n\times n}$ and a vector $q \in R^n$ be given. Define a linear transformation $\mathbf{M} : \mathbf{S}^n \to \mathbf{S}^n$ by $\mathbf{M}(X) := Diag(Mdiag(X))$, where $diag(X)$ is a vector whose entries are the diagonal entries of the matrix X, and $Diag(u)$ is a diagonal matrix whose diagonal is the vector u. Corresponding to LCP(M, q) in (7), one can consider SDLCP$(\mathbf{M}, Diag(q))$, which is to find $X \in \mathbf{S}^n$ such that $X \in \mathbf{S}^n_+$, $Y := \mathbf{M}(X) + Diag(q) \in \mathbf{S}^n_+$, and $\langle X, Y\rangle = 0$. If X is a solution of SDLCP$(\mathbf{M}, Diag(q))$, then $diag(X)$ solves LCP(M, q). Conversely, if x solves LCP(M, q), then $Diag(x)$ solves SDLCP$(\mathbf{M}, Diag(q))$. In this sense, these two complementarity problems are equivalent.

We want to note that the cone of nonnegative vectors R^n_+ in R^n is polyhedral. That is, R^n_+ is the intersection of a finite number of sets defined by linear inequalities. However, the cone of symmetric positive semidefinite matrices $\mathbf{S}^n_+$ in $\mathbf{S}^n$ is not polyhedral. We want to note that because of the nonpolyhedrality of the cone $\mathbf{S}^n_+$ and the noncommutativity of the matrix product, the results available for the standard linear complementarity problem do not simply carry over to the SDLCPs. To ellaborate on this, for example, consider the so-called the **P** and **GUS**-properties introduced by Gowda and Song [8]: A linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ has the

(c) **P**-property if $[XL(X) = L(X)X \text{ negative semidefinite}] \quad \Longrightarrow \quad X = 0,$

(d) **Globally Uniquely Solvable (GUS)**-property if for all $Q \in \mathbf{S}^n$, the semidefinite linear complementarity problem (6) SDLCP(L, Q) has a unique solution.

Note that (c) is an analogus definition to (b) and in the LCP case, $\mathbf{P} \Longleftrightarrow \mathbf{GUS}$ (see (a)). Does the equivalence still hold for SDLCPs? Authors [7], [1], [12] specialized these properties to the Lyapunov transformation $L_A(X) := AX + XA^T$, the Stein transformation $S_A(X) := X - AXA^T$, the two-sided multiplication transformation $M_A(X) := AXA^T$, and studied interconnections with the (global) asymptotic stability of the continuous and discrete linear dynamical systems (see Introduction of [12])

$$\frac{dx}{dt} = -Ax(t), \quad \text{and} \quad x(k+1) = Ax(k). \tag{8}$$

Gowda and Song showed that (Thm 5 [8]) the Lyapunov transformation $L_A(X) := AX + XA^T$ has the **P**-property if and only if A is positive stable (i.e., every eigenvalue of A has a positive real part) and hence the continuous system in (8) is globally asymptotically stable [19]; and has (Thm 9 [8]) the **GUS**-property if and only if A is positive stable and positive semidefinite. Therefore, the **P** and **GUS** properties are *not* equivalent in SDLCP setting.

Bhimasankaram et al. [1] showed that for the two-sided multiplication transformation $M_A(X) := AXA^T$, **GUS** and **P** are both equivalent to A being positive definite or negative definite. Zhang [28], in particular, looked at a problem of solving the matrix equation $AXA^T + BYB^T = C$ seeking a solution $X \succeq 0$ for given matrices A, B and C and provided necessary and sufficient conditions.

Gowda and Parthasarathy (Theorem 11, Remark 4 [7]) showed that the Stein transformation $S_A(X) := X - AXA^T$ has the **P**-property if and only if $\rho(A) < 1$ and hence the discrete system in (8) is globally asymptotically stable [23] . However, the characterization of the **GUS**-property of the Stein transformation is still open. The known results so far are in 2003, Gowda, Song, and Ravindran (Thm 3 [11]) showed that if S_A is strictly monotone (that is, $\langle X, S_A(X)\rangle > 0$ for all $0 \neq X \in \mathbf{S}^n$), then S_A has the **GUS**-property; If A is normal, the converse also holds. Moreover, in 2013, J. Tao [26] examined conditions which gives the so-called **Cone-GUS**-property (that is, SDLCP(S_A, Q) has a unique solution for all $Q \in \mathbf{S}^n_+$.

Therefore, the aims of this paper are to give a characterization of S_A being strictly monotone in terms of the real numerical radius of A and hence providing a sufficient condition for the **GUS**-property of S_A; and to examine a particular instance of A so that SDLCP(S_A, Q) has a unique solution for all Q diagonal or symmetric *negative* semidefinite. **The main results of the paper** are as follows: In section 2, we show that the Stein Transformation S_A is (strictly) monotone if and only if $\nu_r(UAU^T \circ UAU^T)\,(<)\, \leq 1$ for all U orthogonal, where $\circ$ denotes the Hadamard product and ν_r denotes the real numerical radius defined in this paper (see Section 2). In particular, if $\rho(A) < 1$ and $\nu_r(UAU^T \circ UAU^T) \leq 1$, then SDLCP$(S_A, Q)$ has a unique solution for all $Q \in \mathbf{S}^n$. As a by-product, we get a result that if $tr(AA^T) > n$, then S_A is not monotone. In section 3, we look at a particular case of the 2×2 matrix A (motivated by [10]):

$$A = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix},$$

and show that SDLCP(S_A, Q) has a unique solution if Q is either a diagonal or a symmetric *negative* semidefinite matrix. Moreover, we show that not only S_A but also every principal subtransformations of $\widehat{S_A}$ defined by $\widehat{S_A}(X) = P^T S_A(PXP^T)P$, $P \in R^{n\times n}$ invertible $(X \in \mathbf{S}^n)$ has the **Cone-Gus**-properties.

We list below some necessary definitions:

(a) A linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ is called *monotone* if $\langle L(X), X\rangle \geq 0 \quad \forall X \in \mathbf{S}^n$; *strictly monotone* if $\langle L(X), X\rangle > 0$ for all nonzero $X \in \mathbf{S}^n$.

(b) A linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ is said to have the
- **Cone-Gus**-*property* if for all Q in the cone of symmetric positive semedefinite matrices, SDLCP(L, Q) has a unique solution.
- $\mathbf{P_2'}$-*property* if $X \succeq 0$, $XL(X)X \preceq 0 \Longrightarrow X = 0$.

- **Cross Commutative property** if for any $Q \in \mathbf{S}^n$ and for any two solutions X_1 and X_2 of SDLCP(L, Q), $X_1 Y_2 = Y_2 X_1$ and $X_2 Y_1 = Y_1 X_2$, where $Y_i = L(X_i) + Q$ $(i = 1, 2)$.

(c) A matrix $M \in R^{n \times n}$ is called
- *positive semidefinite* if $\langle Mx, x \rangle \geq 0$ for all $x \in R^n$. If M is symmetric positive semidefinite, we use the notation $M \succeq 0$. The notation $M \preceq 0$ means $-M \succeq 0$. Note that a nonsymmetric matrix M is positive semidefinite if and only if the symmetric matrix $M + M^T$ is positive semidefinite.
- *positive definite* if $\langle Mx, x \rangle > 0$ for all nonzero $x \in R^n$. If M is symmetric positive definite, we use the notation $M \succ 0$.
- **P**-*matrix* if all its principal minors are positive.
- *orthogonal* if $MM^T = M^T M = I$, where I is the $n \times n$ identity matrix.
- *normal* if $MM^T = M^T M$.

(d) The Hadamard product (or Schur product) of two matrices A and B is the entrywise product of A and B.

2. Characterization of the Monotonicity of the Stein Transformation

Recall [14] that the *Numerical Radius* of an $n \times n$ matrix A is defined as $\nu(A) = \max\{|x^T Ax| : \|x\| = 1, x \in C^n\}$, where $\|x\|$ denotes the Euclidean norm of the vector x. Here we define the *real* version of the Numerical Radius for a real matrix $A \in R^{n \times n}$ as

$$\nu_r(A) := \max\{|x^T Ax| : \|x\| = 1, x \in R^n\}$$

and relate it with the monotonicity of the Stein Transformation S_A.

Example For $r > 0$, let

$$A = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix} \quad \in \quad R^{2 \times 2}.$$

Since $x^T Ax = 0$ for all x, $\nu(A) = \nu_r(A) = 0$.

Note that for a square matrix A, $\rho(A) \leq \nu(A)$ since each square matrix A has an eigenvector in C^n, but this inequality is not necessarily true for $\nu_r(A)$. See in the above example that $\sigma(A) = \{\pm ir\}$ and therefore $\rho(A) = r > \nu_r(A)$, where $\rho(A)$ denotes the spectral radius of A and $\sigma(A)$ is the spectrum of A (i.e., the set of all eigenvalues of A). We also note that for a square matrix A, $\nu_r(A) = \rho\left(\frac{A + A^T}{2}\right)$ because $|x^T Ax| = |x^T A^T x|$ and $\frac{1}{2}(A + A^T)$ is a symmetric matrix.

We characterize the monotonicity of S_A in terms of the real numerical radius.

Theorem 2.1. *For $A \in R^{n \times n}$, the Stein Transformation $S_A : \mathbf{S}^n \to \mathbf{S}^n$, $S_A(X)$ $:= X - AXA^T$ is (strictly) monotone if and only if for all orthogonal matrices U, $\nu_r(UAU^T \circ UAU^T)$ $(<)$ ≤ 1.*

Proof. Suppose S_A is monotone. Then $\langle S_A(D), D\rangle \geq 0$, where D is a diagonal matrix with the diagonal equals a unit vector d. Note that

$$0 \leq \langle S_A(D), D\rangle = \langle D, D - ADA^T\rangle = 1 - tr(ADA^T D) = 1 - \langle d, (A \circ A)d\rangle,$$

and hence $\nu_r(A \circ A) \leq 1$. Since S_A is monotone, S_{UAU^T} is also monotone for all orthogonal matrices U. Therefore, $\nu_r(UAU^T \circ UAU^T) \leq 1$. For the converse, let $B = UAU^T$ and suppose $\nu_r(B \circ B) \leq 1$ for all orthogonal matrices U. Take $X \in \mathbf{S}^n$, then $X = UDU^T$ where D is a diagonal matrix with the diagonal d. Upon replacing X by UDU^T and carrying out the calculation we get

$$\langle S_A(X), X\rangle = tr(UD^2U^T) - tr(BDB^T D) = \|d\|^2 \left(1 - \langle \frac{d}{\|d\|}, (B \circ B)\frac{d}{\|d\|}\rangle\right)$$

as $tr(BDB^T D) = \langle d, (B \circ B)d\rangle$. By our assumption $\nu_r(B \circ B) \leq 1$, therefore $\langle S_A(X), X\rangle \geq 0$ for all $X \in \mathbf{S}^n$. Hence S_A is monotone.
The proof for strict monotonicty is similar. □

Remark 2.1. (a) If we let $B = UAU^T$, then $\nu_r(B \circ B)$ can be computed as $\nu_r(B \circ B) = \rho\left(\frac{(B\circ B)+(B\circ B)^T}{2}\right) = \|(B \circ B) + (B \circ B)^T\|_{op}$, where $\|\cdot\|_{op}$ denotes the operator norm induced by the Euclidean vector norm.

(b) Chen and Qi in 2006 [3] introduced the Cartesian P-property which is equivalent to the strict monotonicty in some special case (p179 [3]) and showed that (Corollary 1 [3]) a linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ is strictly monotone if and only if for any $0 \neq X \in \mathbf{S}^n$ and any U orthogonal, there exists an index $i \in \{1, \cdots, n\}$ such that $[UXL(X)U^T]_{ii} > 0$, where $[M]_{ij}$ is the (i, j)-component of the matrix M. We also gave sufficent and necessary conditions for monotocity in Theorem 2 above when L is restricted to S_A. We think our result is easier in view of computations involved because in Chen and Qi, it involves both random nonzero symmetric matrices X and orthogonal matrices U to verify the condition. However, our result involves only orthogonal matrices. There are numerical methods to radomly generate orthogonal matrices, for example, see [24], and the real numerical radius ν_r can again be computed numerically as the operator norm as mentioned in Part(a) above.

We now state a simple check condition for a *non*monotonicity of S_A below.

Corollary 2.2. *If $tr(AA^T) > n$ for $A \in R^{n\times n}$, then S_A is not monotone.*

Proof. We shall show that if S_A is monotone, then $tr(AA^T) \leq n$.
First we observe that for a diagonal matrix D,

$$\langle S_A(D), D\rangle = tr((I - A)D(I + A)^T D) = \langle (I - A) \circ (I + A)d,\ d\rangle.$$

Therefore, if S_A is monotone, $(I - A) \circ (I + A)$ is positive semidefinite, as well as $(I - B) \circ (I + B)$ (for S_A monotone implies S_B monotone) for all $B = UAU^T$, where U is an arbitrary orthogonal matrix.
Suppose S_A is monotone. Then $(I-A)\circ(I+A) = I-A\circ A$ is positive semidefinite

which means $\langle e, (I - A \circ A)e \rangle \geq 0$, where e is the vector of all 1's. This reduces to $\langle e, e \rangle - \langle e, (A \circ A)e \rangle \geq 0$. Since $\langle e, e \rangle = n$ and $\langle e, (A \circ A)e \rangle = tr(AA^T)$, we get the desired result. □

We state below a sufficient condition for S_A to be **GUS** in terms of the matrix A.

Theorem 2.3. *For $A \in R^{n\times n}$, if $\rho(A) < 1$, $\nu_r(UAU^T \circ UAU^T) \leq 1$ for all orthogonal matrices U, then S_A has the* **GUS**-*property.*

Proof. The condition $\rho(A) < 1$ is equivalent to S_A having the **P**-property which is also equivalent to the existence of the solution to SDLCP(S_A, Q) for all $Q \in \mathbf{S}^n$. (Thm 11, Remark 4 [7]). By Theorem 2, S_A is monotone. Suppose there are two solutions X_1 and X_2 to SDLCP(S_A, Q). Let $Y_i = S_A(X_i) + Q$, $i = 1, 2$. Thus, $0 \leq \langle S_A(X_1 - X_2), X_1 - X_2 \rangle = \langle Y_1 - Y_2, X_1 - X_2 \rangle \leq 0$ since $tr(X_1Y_1) = 0 = tr(X_2Y_2)$ and $X_i, Y_i \succeq 0$ for $i = 1, 2$. This leads to $tr(X_2Y_1) + tr(X_1Y_2) = 0$. Since the involved matrices are all positive semidefinite, each trace is nonnegative. Hence $tr(X_2Y_1) = 0 = tr(X_1Y_2)$, resulting $X_2Y_1 = 0 = X_1Y_2$. Hence S_A has the **Cross Commutative**-property. It is known (Thm 7 [8]) that **P** + **Cross Commutativity** $\Longleftrightarrow$ **GUS**, hence the proof is complete. □

Remark 2.2. As it is seen in the above proof, monotonicity implies the Cross Commutative-property. Whether monotonicity is equivalent to the Cross Commutative-property is an open problem. This would complete the characterization of the **GUS**-property of S_A.

3. On a special $S_A : S^2 \to S^2$

In what follows, let $R_C(X) := Diag(Cx_d) + X_0$, where $C \in R^{n\times n}$ is a matrix, x_d is the vector composed of the main diagonal of the matrix X, and X_0 is the matrix obtained after replacing all the diagonal elements of X with zeros. If we let $A = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$, and $C = I - A \circ A$, then

$$R_C(X) = S_A(X) = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 - \gamma^2 x_1 \end{bmatrix} \text{ where } X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}. \tag{9}$$

Note that this A is not normal. Motivated by the question raised in (p12, Problem 2. [10]), we studied this particular instance of S_A. So far, the following is known:

(a) S_A has the **GUS**-property when $\gamma^2 \leq 2$ (p12 [10]).
(b) S_A is **Not GUS** when $\gamma^2 > 4$ (p12 [10]).
(c) S_A has the **Cone-Gus**-property when $\gamma^2 \leq 4$. This is obtained by applying J. Tao's result on this particular S_A. In Corollary 4.1 [25], it states that $S_A : S^2 \to S^2$ has the Cone-Gus property if and only if $\rho(A) < 1$ and $I \pm A$ are positive semidefinite matrices.

In [10], Gowda and Song raised the question:

Is S_A **GUS** when $2 < \gamma^2 \leq 4$?

We were able to show that this S_A has the **GUS**-property regardless of the value of γ if the given Q is either diagonal or symmetric *negative* semidefinite. We start with a lemma which shows that every positive definite solution of an SDLCP(L, Q) is locally unique if L has the **P**-property.

Lemma 3.1. *Suppose a linear transformation $L : \mathbf{S}^n \to \mathbf{S}^n$ has the $\mathbf{P}$-property. Then SDLCP(L, Q) cannot have two distinct positive definite solutions.*

Proof. Suppose X_1 and X_2 are two distinct positive definite solutions to a given SDLCP(L, Q). Then we get $X_1(L(X_1) + Q) = 0 = X_2(L(X_2) + Q)$. Since both X_1 and X_2 are invertible, this would mean $L(X_1) = -Q = L(X_2)$. Hence $(X_1 - X_2)L(X_1 - X_2) = 0$. By the **P**-property of L, $X_1 = X_2$ contradicting our assumption. □

Now, let's consider our special S_A given in (9). Since all eigenvalues are zeros for any γ, note that our S_A has the **P**-property for all γ.

Theorem 3.2. *Let*

$$A = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix},$$

where γ a real number. Then

(a) *SDLCP(S_A, Q) has a unique solution for all Q diagonal and the solution is diagonal.*

(b) *SDLCP(S_A, Q) has a unique solution for all $Q \preceq 0$ and the solution is $S_A^{-1}(-Q)$.*

Proof. For part(a), let $Q = D$, where D is a diagonal matrix. By Proposition 3(ii) of [10], the solution set of SDLCP(S_A, D) is equal to the solution set of LCP(C, d), where $C = I - A \circ A$, under obvious modifications (here, d denotes the vector $diag(D)$). Since C is a **P**-matrix (that is, the matrix with all its principal minors positive) for all $\gamma \in R$, the assertion follows.

For part(b), if given Q is diagonal, then there is a unique solution by part(a). Assume Q is a nondiagonal symmetric negative semidefinite matrix. Since $-Q \succeq 0$, we claim that there is a unique $X \succ 0$ such that $S_A(X) = -Q$. This is because, for

$$M = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}, \quad S_A^{-1}(M) = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 + \gamma^2 m_1 \end{bmatrix},$$

and S_A^{-1} is a linear transformation from $\mathbf{S}^n$ to $\mathbf{S}^n$ that maps a nondiagonal positive semidefinite matrix into a positive definite matrix; and $S_A^{-1}(\mathbf{S}^n_+) \subseteq \mathbf{S}^n_+$. Therefore, $S_A^{-1}(-Q) := X \succ 0$. Note X solves SDLCP(S_A, Q). Now suppose there is $X_1 \succeq 0$ such that $Y_1 := S_A(X_1) + Q \succeq 0$ and $X_1 Y_1 = 0$. Then $S_A(X_1) = Y_1 - Q$ and

$$X_1 = S_A^{-1}(Y_1 - Q) = S_A^{-1}(Y_1) + S_A^{-1}(-Q) = S_A^{-1}(Y_1) + X \succ 0$$

because $Y_1 \succeq 0$. Then both X_1 and X are positive definite solutions and by Lemma 7, $X_1 = X$. This completes the proof. □

In addition to this, in what follows, we show that not only this S_A, but also its variants $\widehat{S_A}(X) := P^T S_A(PXP^T)P, \quad P \in R^{n\times n}$ invertible $\quad (X \in \mathbf{S}^n)$, all have the **Cone-Gus**-properties when $\gamma^2 < 4$. We achieve this by showing S_A has the so-called $\mathbf{P_2}'$-property and then applying results of J. Tao [26]. The $\mathbf{P_2}'$-property was introduced by Chandrashekaran et. al [2] in 2010, and J. Tao showed that (Thm 3.3 [26] interpreted for $V = \mathbf{S}^n$) L has the $\mathbf{P_2}'$-property if and only if $\widehat{L}$ and all its principal subtransformations have the **Cone-Gus** properties where $\widehat{L}(X) := P^T L(PXP^T)P$, $P \in R^{n\times n}$ invertible $(X \in \mathbf{S}^n)$.

Theorem 3.3. *The transformation S_A given in (9) has the $\mathbf{P_2}'$-property for $\gamma^2 < 4$.*

Proof. Let $A = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$ with $\gamma^2 < 4$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \succeq 0$.
Suppose $XS_A(X)X \preceq 0$. If X is invertible (i.e., $X \succ 0$), then

$$XS_A(X)X = Y \preceq 0 \quad \Longrightarrow \quad S_A(X) = X^{-1}YX^{-1} \preceq 0$$

since X^{-1} is symmetric. Let $Q := S_A(X) \preceq 0$. Then by Theorem 3(b), SDLCP(S_A, Q) has a unique solution and the solution is $S_A^{-1}(-Q)$ which equals to $S_A^{-1}(S_A(-X))$ by linearity of S_A. But this equals $-X$ which is a solution to SDLCP(S_A, Q), and so $0 \succ -X \succeq 0$ which is a contradiction. Therefore, X can't be invertible.
Now suppose det$X = 0$, i.e., $x_1x_3 = x_2^2$. Carrying out the algebra,

$$XS_A(X)X = \begin{bmatrix} 2x_1x_2^2 + x_2^2x_3 + x_1^3 - x_1x_2^2\gamma^2 & 0 \\ 0 & x_1x_2^2 + 2x_2^2x_3 + x_3^3 - x_1x_3^2\gamma^2 \end{bmatrix}.$$

Then $XS_A(X)X \preceq 0$ leads to $[XS_A(X)X]_{11} \leq 0$ and $[XS_A(X)X]_{22} \leq 0$. Upon putting $x_2^2 = x_1x_3$ in $[XS_A(X)X]_{22}$, we get

$$0 \geq [XS_A(X)X]_{22} = x_3(x_1^2+x_3^2+(2-\gamma^2)x_1x_3) = x_3((x_1-x_3)^2+(4-\gamma^2)x_1x_3) \geq 0,$$

because $X \succeq 0$ and $\gamma^2 < 4$. Hence the last item in the above equations vanishes, therefore, we have cases
(i) $x_3 = 0$, or (ii) $(x_1 - x_3)^2 + (4 - \gamma^2)x_1x_3 = 0$. For the case (i), if $x_3 = 0$, then $x_2 = 0$. Then $[XS_A(X)X]_{11}$ leads to $0 \leq x_1^3 \leq 0$. Hence $x_1 = 0$, and therefore $X = 0$. In case (ii), we get $0 \leq (x_1 - x_3)^2 = -(4 - \gamma^2)x_1x_3 \leq 0$, which leads to $[x_1 = x_3]$ and $[x_1 = 0$ or $x_3 = 0]$ since $\gamma^2 \neq 4$. We get $X = 0$ in each case. Therefore, S_A has the $\mathbf{P_2'}$-property. □

Hence, for any real invertible matrix P, $L(X) := P^T S_A(PXP^T)P$ has only the trivial solution for any given $Q \succeq 0$ which would make $X \succeq 0$, $L(X)+Q \succeq 0$, and $tr(X(L(X)+Q)) = 0$, when $\gamma^2 < 4$. From the above proof, it is more or less clear that S_A is not $\mathbf{P_2'}$ if $\gamma^2 > 4$. When $\gamma^2 = 4$, the following example shows that S_A is not $\mathbf{P_2'}$ as well.
Example Let

$$X = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

Then $XS_A(X)X = 0 \preceq 0$, yet $X \neq 0$. Hence S_A is not $\mathbf{P_2'}$.

4. Conclusion and Acknowledgements

In this paper, we took a baby step to characterize the **GUS**-property of the Stein transformation which hasn't been succeeded for the last decade. We hope the presentation of this paper would bring interests of many other talented mathematicians to work on this newly generated problem. We are indebted to Professor **Muddappa S. Gowda** of University of Maryland Baltimore County for the idea of the real numerical radius and helpful suggestions. We are grateful to Professor **Jiyuan Tao** of Loyola University of Maryland for pointing out right resources for the paper.

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