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ON THE MONOPHONIC NUMBER OF A GRAPH

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ABSTRACT. For a connected graph $G = (V, E)$ of order at least two, a set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G, denoted by $m(G)$. Certain general properties satisfied by the monophonic sets are studied. Graphs G of order p with $m(G) = 2$ or p or $p - 1$ are characterized. For every pair a, b of positive integers with $2 \leq a \leq b$, there is a connected graph G with $m(G) = a$ and $g(G) = b$, where $g(G)$ is the geodetic number of G. Also we study how the monophonic number of a graph is affected when pendant edges are added to the graph.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [5]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. It is known that d is a metric on the vertex set V of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1,6] and further studied in [2,4]. The *detour distance*

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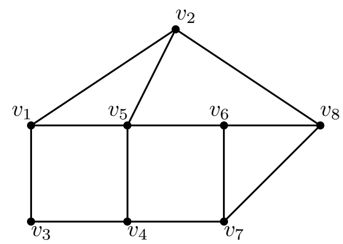

Figure 1. A graph G with $rad_m G = 3$ and $diam_m G = 5$

$D(u, v)$ between two vertices u and v in G is the length of a longest $u - v$ path in G. An $u - v$ path of length $D(u, v)$ is called an $u - v$ *detour*. It is known that D is a metric on the vertex set V of G. The concept of detour distance was introduced and studied in [3].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m G$ of G is $rad_m G = \min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m G$ of G is $diam_m G = \max\{e_m(v) : v \in V(G)\}$. A vertex u in G is *monophonic eccentric vertex* of a vertex v in G if $e_m(u) = d_m(u, v)$. For the graph G given in Figure 1, $d(v_1, v_4) = 2$, $D(v_1, v_4) = 6$ and $d_m(v_1, v_4) = 4$. Thus the monophonic distance is different from both the distance and the detour distance. The usual distance d and the detour distance D are metrics on the vertex set V of a connected graph G, whereas the monophonic distance d_m is not a metric on V. For the graph G given in Figure 1, $d_m(v_4, v_6) = 5$, $d_m(v_4, v_5) = 1$ and $d_m(v_5, v_6) = 1$. Hence $d_m(v_4, v_6) > d_m(v_4, v_5) + d_m(v_5, v_6)$ and so the triangle inequality is not satisfied. It is clear that for vertices u and v in a connected graph G of order p, $0 \leq d(u, v) \leq d_m(u, v) \leq D(u, v) \leq p - 1$. The monophonic distance was introduced and studied in [7]. For the graph G given in Figure 1, the monophonic distance between vertices and the monophonic eccentricities of vertices are given in Table 1. Thus $rad_m G = 3$ and $diam_m G = 5$.

The following theorems will be used in the sequel.

Theorem 1.1 ([6]). *Each extreme vertex of a connected graph G belongs to every geodetic set of G.*

Theorem 1.2 ([6]). *For any tree T with k endvertices, $g(T) = k$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Monophonic number of a graph

Definition 2.1. A set S of vertices of a graph G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path in G for some $x, y \in S$. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$.

Table 1. Monophonic eccentricities of the graph G given in Figure 1

$d_m(v_i, v_j)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$e_m(v)$
v_1	0	1	1	4	1	4	3	4	4
v_2	1	0	4	3	1	5	4	1	5
v_3	1	4	0	1	2	4	4	4	4
v_4	4	3	1	0	1	5	1	4	5
v_5	1	1	2	1	0	1	3	3	3
v_6	4	5	4	5	1	0	1	1	5
v_7	3	4	4	1	3	1	0	1	4
v_8	4	1	4	4	3	1	1	0	4

Example 2.2. For the graph G given in Figure 2, $S_1 = \{x, w\}$ and $S_2 = \{u, w\}$ are the minimum monophonic sets of G and so $m(G) = 2$.

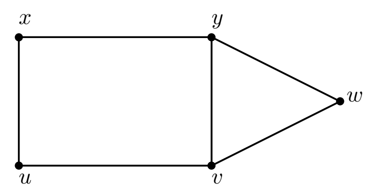

Figure 2. A graph G with $m(G) = 2$

A vertex v in a graph G is a *monophonic vertex* if v belongs to every minimum monophonic set of G. If G has a unique minimum monophonic set S, then every vertex in S is a monophonic vertex. In the next theorem, we show that there are certain vertices in a nontrivial connected graph G that are monophonic vertices of G.

Theorem 2.3. *Each extreme vertex of a connected graph G belongs to every monophonic set of G. Moreover, if the set S of all extreme vertices of G is a monophonic set, then S is the unique minimum monophonic set of G.*

Proof. Let u be an extreme vertex and let S be a monophonic set of G. Suppose that $u \notin S$. Then u is an internal vertex of an $x-y$ monophonic path, say P, for some x, $y \in S$. Let v and w be the neighbors of u on P. Then v and w are not adjacent and so u is not an extreme vertex, which is a contradiction. Therefore u belongs to every monophonic set of G. The second part of the theorem is clear. □

Corollary 2.4. *For the complete graph $K_p (p \geq 2)$, $m(K_p) = p$.*

Theorem 2.5. *Let G be a connected graph with a cutvertex v and let S be a monophonic set of G. Then every component of $G - v$ contains an element of S.*

Proof. Suppose that there is a component B of $G - v$ such that B contains no vertex of S. Let u be any vertex in B. Since S is a monophonic set, there exists a pair of vertices x and y in S such that u lies in some $x-y$ monophonic path $P: x=u_0, u_1, u_2, ..., u, ..., u_n=y$ in G with $u \neq x, y$. Since v is a cutvertex of G, the $x-u$ subpath P_1 of P and the $u-y$ subpath P_2 of P both contain v, it follows that P is not a path, which is a contradiction. □

Theorem 2.6. *No cutvertex of a connected graph G belongs to any minimum monophonic set of G.*

Proof. Let v be a cutvertex of G and let S be a minimum monophonic set of G. Then by Theorem 2.5, every component of $G-v$ contains an element of S. Let U and W be two distinct components of $G-v$ and let $u \in U$ and $w \in W$. Then v is an internal vertex of an $u-w$ monophonic path. Let $S^{'} = S - \{v\}$. It is clear that every vertex that lies on an $u-v$ monophonic path also lies on an $u-w$ monophonic path. Hence it follows that $S^{'}$ is a monophonic set of G, which is a contradiction to S a minimum monophonic set of G. □

Corollary 2.7. *If T is a tree with k endvertices, then $m(T)=k$.*

Proof. This follows from Theorem 2.3 and Theorem 2.6. □

We denote the vertex connectivity of a connected graph G by $\kappa(G)$ or κ.

Theorem 2.8. *If G is a non-complete connected graph such that it has a minimum cutset consisting of κ vertices, then $m(G) \leq p-\kappa$.*

Proof. Since G is a non-complete connected graph, it is clear that $1 \leq \kappa \leq p-2$. Let $U=\{u_1, u_2, u_3, ..., u_\kappa\}$ be a minimum cutset of G. Let $G_1, G_2, ..., G_r$, $(r \geq 2)$ be the components of $G-U$ and let $S=V-U$. Then every vertex u_i $(1 \leq i \leq \kappa)$ is adjacent to at least one vertex of G_j, for each j $(1 \leq j \leq r)$. It is clear that S is a monophonic set of G and so $m(G) \leq |S| = p-\kappa$. □

Remark 2.1. The bound in Theorem 2.8 is sharp. For the cycle C_4, $m(C_4)=2$. Also $\kappa=2$ and $p-\kappa=2$. Thus $m(G)=p-\kappa$.

The following theorem is clear.

Theorem 2.9. *For any connected graph G, $2 \leq m(G) \leq p$.*

The bounds in the above theorem are sharp. For the complete graph $K_p(p \geq 2)$, $m(K_p)=p$. The set of two endvertices of a path $P_n(n \geq 2)$ is its unique minimum monophonic set so that $m(P_n)=2$.

Theorem 2.10. *For any integer k such that $2 \leq k \leq p$ there is a connected graph G of order p such that $m(G)=k$.*

Proof. For $k=p$, the theorem follows from Corollary 2.4 by taking $G=K_p$. For $2 \leq k \leq p-1$, the tree G given in Figure 3 has p vertices and it follows from Corollary 2.7 that $m(G)=k$. □

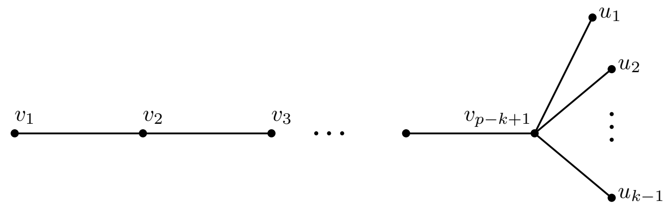

Figure 3. The graph G in Theorem 2.10 with $m(G) = k$

Now we proceed to characterize graphs G for which the bounds in Theorem 2.9 are attained.

Theorem 2.11. *For any connected graph G of order p, $m(G) = p$ if and only if G is complete.*

Proof. Let $m(G) = p$. Suppose that G is not a complete graph. Then there exist two vertices u and v such that u and v are not adjacent in G. Since G is connected, there is a monophonic path from u to v, say P, with length at least 2. Let x be a vertex of P such that $x \neq u, v$. Then $S = V - \{x\}$ is a monophonic set of G and hence $m(G) \leq p - 1$, which is a contradiction. The converse follows from Corollary 2.4. □

Definition 2.12. Let x be any vertex in G. A vertex y in G is said to be an x - *monophonic superior vertex* if for any vertex z with $d_m(x, y) < d_m(x, z)$, z lies on an $x - y$ monophonic path.

Example 2.13. For any vertex x in the cycle $C_p (p \geq 4), V(C_p) - N[x]$ is the set of all x - monophonic superior vertices.

We give below a property related with monophonic eccentric vertex of x and x - monophonic superior vertex in a graph G.

Theorem 2.14. *Let x be any vertex in G. Then every monophonic eccentric vertex of x is an x - monophonic superior vertex.*

Proof. Let y be a monophonic eccentric vertex of x so that $e_m(x) = d_m(x, y)$. If y is not an x - monophonic superior vertex, then there exists a vertex z in G such that $d_m(x, y) < d_m(x, z)$ and z does not lie on any $x - y$ monophonic path and hence $e_m(x) \geq d_m(x, z) > d_m(x, y)$, which is a contradiction. □

Note 2.15. The converse of Theorem 2.14 is not true. For the cycle C_6 : $v_1, v_2, v_3, v_4, v_5, v_6, v_1$, the vertex v_4 is a v_1 - monophonic superior vertex and it is not a monophonic eccentric vertex of v_1.

Theorem 2.16. *Let G be a connected graph. Then $m(G) = 2$ if and only if there exist two vertices x and y such that y is an x- monophonic superior vertex and every vertex of G is on an $x - y$ monophonic path.*

Proof. Let $m(G) = 2$ and let $S = \{x, y\}$ be a minimum monophonic set of G. If y is not an x - monophonic superior vertex, then there is a vertex z in G with

$d_m(x, y) < d_m(x, z)$ and z does not lie on any $x - y$ monophonic path. Thus S is not a monophonic set of G, which is a contradiction. The converse is clear from the definition. □

Theorem 2.17. *Let G be a connected graph of order $p \geq 3$. Then $m(G) = p - 1$ if and only if $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$.*

Proof. Let $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$. Then G has exactly one cutvertex and all other vertices are extreme and hence by Theorems 2.3 and 2.6, $m(G) = p - 1$. Conversely, let $m(G) = p - 1$. Let S be a monophonic set such that $|S| = p - 1$. Let $v \notin S$. We show that v is a cutvertex of G. Otherwise, $G - v$ has just one component. By Theorem 2.3, v is not an extreme vertex of G. Hence there exist vertices $x, y \in N(v)$ such that x and y are not adjacent in $G - v$. Let P be an $x - y$ monophonic path in $G - v$ of length at least 2. Choose a vertex z on P such that $z \neq x, y$. Note that $z \neq v$. Then it is clear that $S_1 = V - \{v, z\}$ is a monophonic set of G so that $m(G) \leq p - 2$, which is a contradiction. Hence v is a cutvertex of G and by Theorem 2.6, v is the only cutvertex of G.

Now, let $G_1, G_2, ..., G_r$ be the components of $G - v$. First, we show that each G_i is complete. Suppose that some component, say G_1, is not complete. Then there exist two vertices x and y in G_1 such that x and y are not adjacent. Choose a vertex z in an $x - y$ geodesic such that $z \neq x, y$. Then $S_2 = V - \{v, z\}$ is a monophonic set of G so that $m(G) \leq p - 2$, which is a contradiction. Now, it remains to show that v is adjacent to every vertex of G_i for each i $(1 \leq i \leq r)$. Otherwise, there exists a component, say G_i, such that v is not adjacent to at least one vertex of G_i. Hence there is a vertex u in G_i such that u is not extreme in G. Then $S_3 = V - \{v, u\}$ is a monophonic set of G so that $m(G) \leq p - 2$, which is a contradiction. Hence $G = K_1 + \cup m_j K_j$, where $K_1 = \{v\}$ and $\sum m_j \geq 2$. □

3. Bounds for the monophonic number of a graph

In the following theorem we give an improved upper bound for the monophonic number of a graph in terms of its order and monophonic diameter. For convenience, we denote the monoponic diameter $diam_m G$ by d_m itself.

Theorem 3.1. *If G is a non-trivial connected graph of order p and monophonic diameter d_m, then $m(G) \leq p - d_m + 1$.*

Proof. Let u and v be vertices of G such that $d_m(u, v) = d_m$ and let $P : u = v_0, v_1, ..., v_{d_m} = v$ be a $u - v$ monophonic path of length d_m. Let $S = V - \{v_1, v_2, ..., v_{d_m - 1}\}$. Then it is clear that S is a monophonic set of G so that $m(G) \leq |S| = p - d_m + 1$. □

For the complete graph $K_p (p \geq 2)$, $d_m = 1$ and $m(K_p) = p$ so that the bound in Theorem 3.1 is sharp.

A *caterpillar* is a tree for which the removal of all the endvertices gives a path.

Theorem 3.2. *For every non-trivial tree T of order p and monophonic diameter d_m, $m(T) = p - d_m + 1$ if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : u = v_0, v_1, ..., v_{d_m}$ be a monophonic diametral path. Let k be the number of endvertices of T and l be the number of internal vertices of T other than $v_1, v_2, ..., v_{d_m-1}$. Then $d_m - 1 + l + k = p$. By Corollary 2.7, $m(T) = k$ and so $m(T) = p-d_m-l+1$. Hence $m(T) = p-d_m+1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic diametral path P, if and only if T is a caterpillar. □

For any connected graph G, $rad_mG \leq diam_mG$. It is shown in [7] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the monophonic number can be prescribed when $rad_mG < diam_mG$.

Theorem 3.3. *For positive integers r, d and $k \geq 4$ with $r < d$, there exists a connected graphs G such that $rad_mG = r$, $diam_mG = d$ and $m(G) = k$.*

Proof. We prove this theorem by considering two cases.

Case 1. $r = 1$. Then $d \geq 2$. Let $C_{d+2} : v_1, v_2, ..., v_{d+2}, v_1$ be a cycle of order $d+2$. Let G be the graph obtained by adding $k-2$ new vertices $u_1, u_2, ..., u_{k-2}$ to C_{d+2} and joining each of the vertices $u_1, u_2, ..., u_{k-2}, v_3, v_4, ..., v_{d+1}$ to the vertex v_1. The graph G is shown in Figure 4. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = 1, e_m(v_2) = d$. Then $rad_mG = 1$ and $diam_mG = d$. Let $S = \{u_1, u_2, ..., u_{k-2}, v_2, v_{d+2}\}$ be the set of all extreme vertices of G. Since S is a monophonic set of G, it follows from Theorem 2.3 that $m(G) = k$.

Case 2. $r \geq 2$. Let $C : v_1, v_2, ..., v_{r+2}, v_1$ be a cycle of order $r + 2$ and let $W = K_1 + C_{d+2}$ be the wheel with $V(C_{d+2}) = \{u_1, u_2, ..., u_{d+2}\}$, $K_1 = \{v_1\}$ and all other vertices distinct. Now, add $k-3$ new vertices $w_1, w_2, ..., w_{k-3}$ and join each $w_i (1 \leq i \leq k-3)$ to the vertex v_1 and obtain the graph G of Figure 5. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_1) = r$ and $e_m(u_1) = d$. Thus $rad_mG = r$ and $diam_mG = d$. Let $S = \{w_1, w_2, ..., w_{k-3}\}$ be the set of all extreme vertices of G. By Theorem 2.3, every monophonic set of G contains S. It is clear that S is not a monophonic set of G. Let $T = S \bigcup \{u_1, u_3, v_3\}$. It is easily verified that T is a minimum monophonic set of G and so $m(G) = k$. □

Problem 3.4. *For any three positive integers r, d and $k \geq 4$ with $r = d$, does there exist a connected graph G with $rad_m = r$, $diam_m = d$ and $m(G) = k$?*

Theorem 3.5. *For each triple d, k, p of integers with $2 \leq k \leq p-d+1$ and $d \geq 2$, there is a connected graph G of order p such that $diam_mG = d$ and $m(G) = k$.*

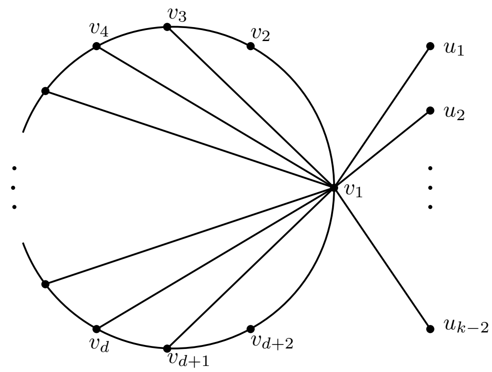

Figure 4. The graph G in Case 1 of Theorem 3.3

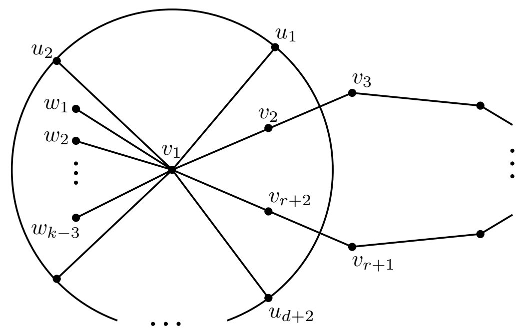

Figure 5. The graph G in Case 2 of Theorem 3.3

Proof. Let $P_{d+1} : u_1, u_2, ..., u_{d+1}$ be a path of length d. Add $p-d-1$ new vertices, $v_1, v_2, ..., v_{k-2}, w_1, w_2, ..., w_{p-d-k+1}$ to P_{d+1} and join each $w_i(1 \leq i \leq p-d-k+1)$ to u_1, u_2 and u_3, and also join each $v_j(1 \leq j \leq k-2)$ to u_2, thereby producing the graph G of Figure 6. Then G has order p and monophonic diameter d. If $p-d-k+1 \leq 1$, then $S = \{v_1, v_2, ..., v_{k-2}, u_1, u_{d+1}\}$ is the set of all extreme vertices of G. Since S is a monophonic set of G, it follows from Theorem 2.3 that $m(G) = k$. So, let $p-d-k+1 \geq 2$. If $d = 2$, then $S_1 = \{v_1, v_2, ..., v_{k-2}\}$ is the set of all extreme vertices of G. It is clear that neither S_1 nor $S_1 \cup \{x\}$, where $x \notin S_1$, is a monophonic set of G. Since $S_2 = S_1 \cup \{u_1, u_3\}$ is a monophonic set of G, it follows from Theorem 2.3 that $m(G) = k$. If $d \geq 3$, then $S_3 = \{v_1, v_2, ..., v_{k-2}, u_{d+1}\}$ is the set of all extreme vertices of G. Now, S_3 is not a monophonic set of G. Since $S_4 = S_3 \cup \{u_1\}$ is a monophonic set of G, it follows from Theorem 2.3 that $m(G) = k$. □

Theorem 3.6. *For any connected graph G of order p, $2 \leq m(G) \leq g(G) \leq p$.*

Proof. Since every geodesic is a monophonic path, it follows that every geodetic set is a monophonic set, and hence $m(G) \leq g(G)$. The other inequalities are trivial. □

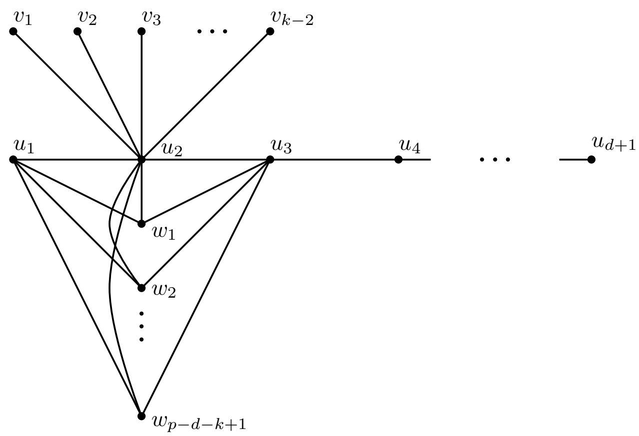

Figure 6. The graph G in Theorem 3.5 with $diam_m G = d$ and $m(G) = k$

Remark 3.1. The bounds in Theorem 3.6 are sharp. For the complete graph K_p, $m(K_p) = g(K_p) = p$. For a non-trivial path P_n, $m(P_n) = g(P_n) = 2$. Also, if G is a non-trivial tree, or an even cycle, or a complete bipartite graph, then $m(G) = g(G)$. All the inequalities in Theorem 3.6 are strict. For the graph G given in Figure 7, $S = \{v_6, v_7, v_3\}$ is a minimum monophonic set of G so that $m(G) = 3$ and no 3-elements subset of the vertex set is a geodetic set of G. Since $S \cup \{v_1\}$ is a geodetic set of G, it follows that $g(G) = 4$. Thus we have $2 < m(G) < g(G) < p$.

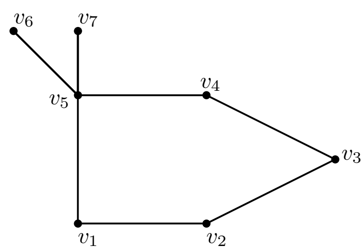

Figure 7. A graph G in Remark 3.1 with $2 < m(G) < g(G) < p$

In view of this remark, we have the following problem.

Problem 3.7. *Characterize graphs G for which $m(G) = g(G)$.*

Theorem 3.8. *For every pair a, b of positive integers with $2 \le a \le b$, there is a connected graph G with $m(G) = a$ and $g(G) = b$.*

Proof. For $2 \le a = b$, any tree with a endvertices has the desired properties, by Theorem 1.2 and Corollary 2.7. So, assume that $2 \le a < b$. Let $P_i : x_i, w_i, y_i$ $(1 \le i \le b-a)$ be $b-a$ copies of a path of length 2 and $P : v_1, v_2, v_3, v_4$ a path of length 3. Let G be the graph obtained by joining each $x_i (1 \le i \le b-a)$ in P_i and v_2 in P, joining each $y_i (1 \le i \le b-a)$ in P_i and v_4 in P; and adding $a-1$ new vertices $u_1, u_2, ..., u_{a-1}$ and joining each $u_i (1 \leq i \leq a-1)$ to v_4. The graph G is shown in Figure 8. Let $S = \{v_1, u_1, ..., u_{a-1}\}$ be the set of all extreme vertices of G. It is easily verified that S is a monophonic set of G and so by Theorem 2.3, $m(G) = |S| = a$.

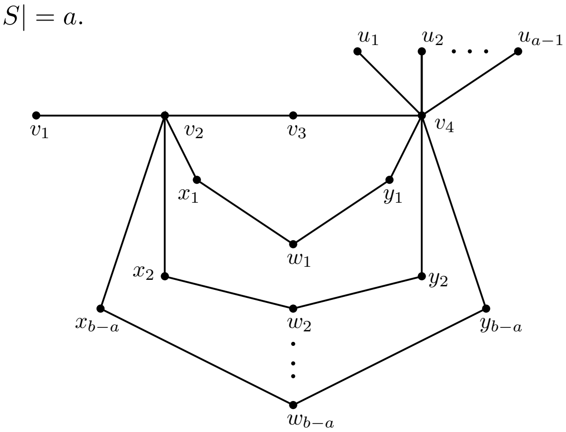

Figure 8. The graph G in Theorem 3.8 with $m(G) = a$ and $g(G) = b$

Next, we show that $g(G) = b$. By Theorem 1.1, every geodetic set of G contains S. Clearly, S is not a geodetic set of G. It is easily verified that at least one of the vertex of each $P_i (1 \leq i \leq b-a)$ must belong to every geodetic set of G. Since $T = S \cup \{w_1, w_2, ..., w_{b-a}\}$ is a geodetic set of G, it follows from Theorem 1.1 that T is a minimum geodetic set of G and so $g(G) = b$. □

4. Monophonic number of a graph by adding some pendant edges

Theorem 4.1. *If G' is a graph obtained by adding l pendant edges to a connected graph G, then $m(G) \leq m(G') \leq m(G) + l$.*

Proof. Let G' be the connected graph obtained from G by adding l pendant edges $u_i v_i (1 \leq i \leq l)$, where each $u_i (1 \leq i \leq l)$ is a vertex of G and each $v_i (1 \leq i \leq l)$ is not a vertex of G. Let S be a minimum monophonic set of G. Then $S \cup \{v_1, v_2, ..., v_l\}$ is a monophonic set of G' and so $m(G') \leq m(G) + l$.

Now, we claim that $m(G) \leq m(G')$. Suppose that $m(G) > m(G')$. Then let S' be a monophonic set of G' with $|S'| < m(G)$. Since each $v_i (1 \leq i \leq l)$ is an extreme vertex of G', it follows from Theorem 2.3 that $\{v_1, v_2, ..., v_l\} \subseteq S'$. Let $S = (S' - \{v_1, v_2, ..., v_l\}) \cup \{u_1, u_2, ..., u_l\}$. Then S is a subset of $V(G)$ and $|S| = |S'| < m(G)$. Now, we show that S is a monophonic set of G. Let $w \in V(G) - S$. Since S' is a monophonic set of G', w lies on an $x - y$ monophonic path P in G' for some vertices $x, y \in S'$. If neither x nor y is $v_i (1 \leq i \leq l)$, then $x, y \in S$. If exactly one of x, y is $v_i (1 \leq i \leq l)$, say $x = v_i$. Then w lies on the $u_i - y$ monophonic path in G obtained from P by removing v_i. If both $x, y \in \{v_1, v_2, ..., v_l\}$, then let $x = v_i$ and $y = v_j$ where $i \neq j$. Hence w lies on the $u_i - u_j$ monophonic path in G obtained from P by removing v_i and

v_j. Thus S is a monophonic set of G. Hence $m(G) \leq |S| < m(G)$, which is a contradiction. □

Remark 4.1. The bounds for $m(G')$ in Theorem 4.1 are sharp. Consider a tree T with number of endvertices $k \geq 3$. Let $S = \{v_1, v_2, ..., v_k\}$ be the set of all endvertices of T. Then by Corollary 2.7, $m(G) = k$. If we add a pendant edge to an endvertex of T, then we obtain another tree T' with k endvertices. Hence $m(T) = m(T')$. On the otherhand, if we add l pendant edges to a cutvertex of T, then we obtain another tree T'' with $k + l$ endvertices. Then by Corollary 2.7, $m(T') = m(T) + l$.

Now, we proceed to characterize graphs G for which $m(G) = m(G')$, where G' is obtained from G by adding l pendant edges.

Theorem 4.2. *Let G' be a graph obtained from a connected graph G by adding l pendant edges $u_iv_i (1 \leq i \leq l)$, where $u_i \in V(G)$ and $v_i \notin V(G)$. Then $m(G) = m(G')$ if and only if $l \leq m(G)$ and $\{u_1, u_2, ..., u_l\}$ is a subset of some minimum monophonic set of G.*

Proof. Let $l \leq m(G)$ and let $\{u_1, u_2, ..., u_l\}$ be a subset of some minimum monophonic set S of G. Let $S' = (S - \{u_1, u_2, ..., u_l\}) \bigcup \{v_1, v_2, ..., v_l\}$. Then $|S'| = |S|$. We show that S' is a monophonic set of G'. Let $z \in V(G') - S'$. If $z = u_i$ $(1 \leq i \leq l)$, then z lies on every $v_i - w$ monophonic path in G', where $w \in S'$, since u_i is the only vertex adjacent to v_i. So we may assume that $z \neq u_i (1 \leq i \leq l)$. Since z is a vertex of G and S is a monophonic set of G, it follows that z lies on some $x - y$ monophonic path P in G for some $x, y \in S$. Then by an argument similar to the one used in the proof of Theorem 4.1, we can show that S' is a monophonic set of G'. Hence $m(G') \leq |S'| = |S| = m(G)$. Now, the result follows from Theorem 4.1.

Conversely, let $m(G) = m(G')$. Suppose that $l > m(G)$. Since each $v_i (1 \leq i \leq l)$ is an endvertex of G', by Theorem 2.3, $m(G') \geq l$. Hence $m(G') > m(G)$, which is a contradiction. Thus $l \leq m(G')$. Now, let S' be a minimum monophonic set of G'. Since each u_i $(1 \leq i \leq l)$ is a cutvertex of G', it follows from Theorem 2.6 that $u_i \notin S'$ for $1 \leq i \leq l$. Since each $v_i (1 \leq i \leq l)$ is an endvertex of G', it follows from Theorem 2.3 that $v_i \in S'$ for $1 \leq i \leq l$. Let $S = (S' - \{v_1, v_2, ..., v_l\}) \bigcup \{u_1, u_2, ..., u_l\}$. Then S is a subset of $V(G)$ and $|S| = |S'|$. Then, as in the proof of Theorem 4.1, S is a monophonic set of G. Since $|S| = |S'| = m(G') = m(G)$, it follows that S is a minimum monophonic set of G that contains $\{u_1, u_2, ..., u_l\}$. □

Theorem 4.3. *For each triple a, b and l of integers with $2 \leq a \leq b$, $1 \leq l \leq b$, and $a + l - b \geq 0$, there exists a connected graph G with $m(G) = a$ and $m(G') = b$, where G' is a graph obtained by adding l pendant edges to G.*

Proof. Let G be a tree with number of endvertices a. Let G' be a graph obtained by adding $b - a$ pendant edges to a cutvertex of G and also adding $l + a - b$

pendant edges each with different endvertices of G. Then $G^{'}$ is another tree with b endvertices. By Corollary 2.7, $m(G) = a$ and $m(G^{'}) = b$. □

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