

BOUNDEDNESS IN NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper, we investigate bounds for solutions of nonlinear perturbed differential systems.

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1. Introduction

The behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for showing the qualitative behavior of the solutions of perturbed nonlinear system : Lyapunov's second method, the use of integral inequalities , and the method of variation of constants formula. The method incorporating integral inequalities takes an important place among the methods developed for the qualitative analysis of solutions to linear and nonlinear system of differential equations. In the presence the method of integral inequalities is as efficient as the direct Lyapunov's method.

The notion of h -stability (hS) was introduced by Pinto [15,16] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h -systems. Using this notion, Choi and Ryu [3,5] investigated bounds of solutions for nonlinear perturbed systems and nonlinear functional differential systems. Also, Goo et al. [8] studied the boundedness of solutions for nonlinear perturbed systems.

In this paper, we obtain some results on boundedness of solutions of nonlinear perturbed differential systems under suitable conditions on perturbed term. To do this we need some integral inequalities.

2. Preliminaries

We are interested in the relations of the unperturbed system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad (1)$$

and the solutions of the perturbed system

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s)) ds, \quad y(t_0) = y_0, \quad (2)$$

where x, y, f and g are elements of \mathbb{R}^n , an n -dimensional real Euclidean space.

We assume that $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$, and that f is continuously differentiable with respect to the components of x on $\mathbb{R}^+ \times \mathbb{R}^n$, $f(t, 0) = 0$ for all $t \in \mathbb{R}^+$. The symbol $|\cdot|$ will be used to denote arbitrary vector norm in \mathbb{R}^n .

Let $x(t, t_0, x_0)$ denote the unique solutions of (1) and (2), satisfying the initial conditions $x(t_0, t_0, x_0) = x_0$, and $y(t_0, t_0, y_0) = y_0$, existing on $[t_0, \infty)$, respectively. Then we can consider the associated variational systems around the zero solution of (1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \quad v(t_0) = v_0 \quad (3)$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \quad z(t_0) = z_0. \quad (4)$$

Here, $f_x(t, x)$ is the matrix whose element in the i th row, j th column is the partial derivative of the i th component of f with respect to the j th component of x . The fundamental matrix $\Phi(t, t_0, x_0)$ of (4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (3).

We recall some notions of h -stability [15].

Definition 2.1. The system (1) (the zero solution $x = 0$ of (1)) is called an h -system if there exist a constant $c \geq 1$, and a positive continuous function h on \mathbb{R}^+ such that

$$|x(t)| \leq c|x_0| h(t) h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 2.2. The system (1) (the zero solution $x = 0$ of (1)) is called h -stable (hS) if there exists $\delta > 0$ such that (1) is an h -system for $|x_0| \leq \delta$ and h is bounded.

Let \mathcal{M} denote the set of all $n \times n$ continuous matrices $A(t)$ defined on \mathbb{R}^+ and \mathcal{N} be the subset of \mathcal{M} consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞ -similarity in \mathcal{M} was introduced by Conti [6].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is t_∞ -similar to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over \mathbb{R}^+ , i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{5}$$

for some $S(t) \in \mathcal{N}$.

We give some related properties that we need in the sequel.

Lemma 2.1 ([16]). *The linear system*

$$x' = A(t)x, \quad x(t_0) = x_0, \tag{6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h -system (h -stable, respectively) if and only if there exist $c \geq 1$ and a positive continuous (bounded, respectively) function h defined on \mathbb{R}^+ such that

$$|\phi(t, t_0)| \leq c h(t) h(t_0)^{-1} \tag{7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (6).

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.2. *Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (2), respectively. If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) \int_{t_0}^s g(\tau, y(\tau)) d\tau ds.$$

Theorem 2.3 ([3]). *If the zero solution of (1) is hS , then the zero solution of (3) is hS .*

Theorem 2.4 ([4]). *Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (3) is hS , then the solution $z = 0$ of (4) is hS .*

Lemma 2.5 ([13]). *Let $u, f, g \in C(\mathbb{R}^+)$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s) \left\{ \int_0^s g(\tau)u(\tau)d\tau \right\} ds, \quad t \in \mathbb{R}^+,$$

holds, where u_0 is a nonnegative constant. Then,

$$u(t) \leq u_0 \left(1 + \int_0^t f(s) \exp\left(\int_0^s (f(\tau) + g(\tau))d\tau \right) ds \right), \quad t \in \mathbb{R}^+,$$

Lemma 2.6 ([5]). *Let $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. If ,for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)u(s)ds + \int_{t_0}^t \lambda_1(s) \left\{ \int_{t_0}^s \lambda_2(\tau)w(u(\tau))d\tau \right\} ds, \quad t \geq t_0 \geq 0,$$

then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t \lambda_2(s)ds \right] \exp \left(\int_{t_0}^t \lambda_1(s)ds \right), \quad t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t \lambda_2(s)ds \in \text{dom}W^{-1} \right\}.$$

Lemma 2.7 ([11]). *Let $u, \lambda_1, \lambda_2, \lambda_3 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u , $u \leq w(u)$. Suppose that for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^t \lambda_1(s)w(u(s))ds + \int_{t_0}^t \lambda_2(s) \left(\int_{t_0}^s \lambda_3(\tau)u(\tau)d\tau \right) ds, \quad 0 \leq t_0 \leq t.$$

Then

$$u(t) \leq W^{-1} \left[W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau) ds \right], \quad t_0 \leq t < b_1, \quad (8)$$

where $W(u) = \int_{u_0}^u \frac{ds}{w(s)}$, $u > 0$, $u_0 > 0$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau)d\tau) ds \in \text{dom}W^{-1} \right\}.$$

3. Main results

In this section, we investigate bounds for the nonlinear differential systems. Also, we examine the bounded property for the perturbed system of (1)

$$y' = f(t, y) + \int_{t_0}^t g(s, y(s))ds, \quad y(t_0) = y_0, \quad (9)$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$.

The generalization of a function h 's condition and the strong condition of a function g in Theorem 3.1 [10] are the following result.

Theorem 3.1. *Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1) is hS with a positive continuous function h , and g in (9) satisfies*

$$\left| \int_{t_0}^s g(\tau, y(\tau))d\tau \right| \leq a(s) \left(|y(s)| + h(s) \int_{t_0}^s k(\tau)|y(\tau)|d\tau \right), \quad t \geq t_0 \geq 0,$$

where $a, k \in C(\mathbb{R}^+)$, $\int_{t_0}^\infty a(s)ds < \infty$, and $\int_{t_0}^s k(s)ds < \infty$. Then, the solution $y = 0$ of (9) is hS.

Proof. Using the nonlinear variation of Alekseev[1], any solution $y(t) = y(t, t_0, y_0)$ of (9) passing through (t_0, y_0) is given by

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^t \Phi(t, s, y(s)) \int_{t_0}^s g(\tau, y(\tau)) d\tau ds. \tag{10}$$

By Theorem 2.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (4) is hS. By Lemma 2.1 and (10), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left| \int_{t_0}^s g(\tau, y(\tau)) d\tau \right| ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) h(s)^{-1} a(s) (|y(s)| + h(s) \int_{t_0}^s k(\tau) |y(\tau)| d\tau) ds \\ &\leq c_1 |y_0| h(t) h(t_0)^{-1} + \int_{t_0}^t c_2 h(t) a(s) h(s)^{-1} |y(s)| ds \\ &\quad + \int_{t_0}^t c_2 h(t) a(s) \int_{t_0}^s k(\tau) h(\tau) h(\tau)^{-1} |y(\tau)| d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.5, we obtain

$$\begin{aligned} |y(t)| &\leq c_1 |y_0| h(t) h(t_0)^{-1} \left(1 + c_2 \int_{t_0}^t a(s) \exp\left(\int_{t_0}^s (c_2 a(\tau) + k(\tau)h(\tau)) d\tau\right) ds \right) \\ &\leq c |y_0| h(t) h(t_0)^{-1}, c = c_1 \left(1 + c_2 \int_{t_0}^t a(s) \exp\left(\int_{t_0}^s (c_2 a(\tau) + k(\tau)h(\tau)) d\tau\right) ds \right). \end{aligned}$$

It follows that $y = 0$ of (9) is hS. Hence, the proof is complete. □

Remark 3.1. In the linear case, we can obtain that if the zero solution $x = 0$ of (6) is hS, then the perturbed system

$$y' = A(t)y + \int_{t_0}^t g(s, y(s))ds, y(t_0) = y_0,$$

is also hS under the same hypotheses in Theorem 3.1 except the condition of t_∞ -similarity.

Remark 3.2. Letting $k(t) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.3 in [9].

The weak condition of a function h and the strong condition of a function g in Theorem 3.3 [8] are the following result.

Theorem 3.2. Let $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u , $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the

solution $x = 0$ of (1) is hS with a positive continuous function h , and g in (9) satisfies

$$\left| \int_{t_0}^s g(\tau, y(\tau))d\tau \right| \leq a(s) \left(|y(s)| + h(s) \int_{t_0}^s k(\tau)w(|y(\tau)|)d\tau \right),$$

where $\int_{t_0}^\infty a(s)ds < \infty$, $\int_{t_0}^\infty b(s)ds < \infty$, and $\int_{t_0}^\infty k(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (9) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + \int_{t_0}^t k(s)h(s)ds \right] \exp \left(\int_{t_0}^t c_2a(s)ds \right), t_0 \leq t < b_1$$

where $c = c_1|y_0|h(t_0)^{-1}$ and W, W^{-1} are the same functions as in Lemma 2.6 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + \int_{t_0}^t k(s)h(s)ds \in \text{dom}W^{-1} \right\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (9), respectively. By Theorem 2.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (4) is hS. Using Lemma 2.1 and (10), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left| \int_{t_0}^s g(\tau, y(\tau))d\tau \right| ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)a(s) \frac{|y(s)|}{h(s)} ds \\ &\quad + \int_{t_0}^t c_2h(t)a(s) \int_{t_0}^s k(\tau)h(\tau)w\left(\frac{|y(\tau)|}{h(\tau)}\right)d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Now an application of Lemma 2.6 yields

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + \int_{t_0}^t k(s)h(s)ds \right] \exp \left(\int_{t_0}^t c_2a(s)ds \right), t_0 \leq t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded, and the theorem is proved. □

The generalization of a function h 's condition and a slight modification of a function g 's condition in Theorem 3.4[11] are the following result.

Theorem 3.3. Let $a, b, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u , $u \leq w(u)$ and $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞ -similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1) is hS with the positive continuous function h , and g in (9) satisfies

$$\left| \int_{t_0}^s g(\tau, y(\tau))d\tau \right| \leq a(s) \left(w(|y(s)|) + h(s) \int_{t_0}^s k(\tau)|y(\tau)|d\tau \right),$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$, $\int_{t_0}^{\infty} b(s)ds < \infty$, and $\int_{t_0}^{\infty} k(s)ds < \infty$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (9) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t a(s) \left(1 + \int_{t_0}^s k(\tau)h(\tau)d\tau \right) ds \right],$$

where W, W^{-1} are the same functions as in Lemma 2.6 and

$$b_1 = \sup \left\{ t \geq t_0 : W(c) + c_2 \int_{t_0}^t a(s) \left(1 + \int_{t_0}^s k(\tau)h(\tau)d\tau \right) ds \in \text{dom}W^{-1} \right\}.$$

Proof. It is known that the solution of (9) is represented by the integral equation (10). By Theorem 2.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (3) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (4) is hS. Using Lemma 2.1 and (10), we have

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^t |\Phi(t, s, y(s))| \left| \int_{t_0}^s g(\tau, y(\tau))d\tau \right| ds \\ &\leq c_1|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^t c_2h(t)a(s)w\left(\frac{|y(s)|}{h(s)}\right)ds \\ &\quad + \int_{t_0}^t c_2h(t)a(s) \int_{t_0}^s k(\tau)h(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau ds. \end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Now an application of Lemma 2.7 yields

$$|y(t)| \leq h(t)W^{-1} \left[W(c) + c_2 \int_{t_0}^t a(s) \left(1 + \int_{t_0}^s k(\tau)h(\tau)d\tau \right) ds \right],$$

where $c = c_1|y_0|h(t_0)^{-1}$. The above estimation implies the boundedness of $y(t)$, and the proof is complete. \square

Remark 3.3. Letting $k(t) = 0$ in Theorem 3.5 and adding the increasing condition of the function h , we obtain the same result as that of Theorem 3.2 in [8].

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