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THE NORMALIZED LAPLACIAN ESTRADA INDEX OF GRAPHS†

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ABSTRACT. Suppose G is a simple graph. The $\ell-$eigenvalues $\delta_1, \delta_2, \ldots, \delta_n$ of G are the eigenvalues of its normalized Laplacian ℓ. The normalized Laplacian Estrada index of the graph G is defined as $\ell EE = \ell EE(G) = \Sigma_{i=1}^{n} e^{\delta_i}$. In this paper the basic properties of ℓEE are investigated. Moreover, some lower and upper bounds for the normalized Laplacian Estrada index in terms of the number of vertices, edges and the Randic index are obtained. In addition, some relations between ℓEE and graph energy $E_\ell(G)$ are presented.

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1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices and m edges. The eigenvalues of the adjacency matrix $A(G)$ are called the eigenvalues of G and form the spectrum of G. Suppose $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of G such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If G has exactly s distinct eigenvalues $\delta_1, \ldots, \delta_s$ and the multiplicity of δ_i is t_i, $1 \leq i \leq s$, then we use the following compact form

$$Spec(G) = \begin{pmatrix} \delta_1 & \delta_2 & \cdots & \delta_s \\ t_1 & t_2 & & t_s \end{pmatrix}$$

for the spectrum of G.

The Estrada index of the graph G is defined as $EE = EE(G) = \Sigma_{i=1}^{n} e^{\lambda_i}$. This graph invariant was introduced by Ernesto Estrada, which has noteworthy

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chemical applications, see [12, 13, 14] for details. We encourage the interested readers to consult [2, 11, 16] for the mathematical properties of Estrada index.

The Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$, where $A(G)$ and $D(G)$ are the adjacency and diagonal matrices of G, respectively. If $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ are the Laplacian eigenvalues of G, then the Laplacian Estrada index, $L-$Estrada index for short, of G is defined as the sum of the terms e^{μ_i}, $1 \leq i \leq n$. This quantity is denoted by $LEE(G)$. There exists a vast literature that studies the $L-$Estrada index of graphs. We refer the readers to [15, 19, 24] for more information.

The normalized Laplacian matrix $\ell(G) = [\ell_{i,j}]_{n\times n}$ is defined as:

$$\ell_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } \deg(v_i) \neq 0 \\ -\frac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j. \\ 0 & \text{otherwise} \end{cases}$$

The normalized Laplacian eigenvalues or $\ell-$spectrum of G are denoted by $0 = \delta_1 \leq \delta_2 \leq \cdots \leq \delta_n$. The multiplicity of $\delta_1 = 0$ is equal to the number of connected components of G. Define $\varphi(G, \delta) = det(\delta I_n - \ell(G))$, where I_n is the unit matrix of order n. This polynomial is called the normalized Laplacian characteristic polynomial. The basic properties of the normalized Laplacian eigenvalues can be found in [8, 9]. The normalized Laplacian eigenvalues of an $n-$vertex connected graph G satisfying the following elementary conditions: $\Sigma_{i=1}^{n}\delta_i = n$ and $\Sigma_{i=1}^{n}\delta_i^2 = n + 2R_{-1}(G)$, where $R_{-1}(G)$ is Randic index of G, see [6, 8, 9] for details.

We now define the *normalized Laplacian Estrada index*, simply called *$\ell-$Estrada index*, of G by the following equation:

$$\ell EE = \ell EE(G) = \sum_{i=1}^{n} e^{\delta_i}.$$

From the power-series expansion of e^x, we have:

$$\ell EE = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=1}^{n} \delta_i^k,$$

where we assumed that $0^0 = 1$.

We now introduce some notation that will be used throughout this paper. The *complete graph* on n vertices is denoted by K_n. The line graph $l(G)$ of a graph G is another graph $l(G)$ that represents the adjacencies between edges of G. In a graph theoretical language $V(l(G)) = E(G)$ and two edges of G are adjacent in $l(G)$ if they have a common vertex. Suppose $\overline{G}$ denotes the *complement* of G. For two graphs G and H, $G \cup H$ is the disjoint union of G and H. The *join* $G + H$ is the graph obtained from $G \cup H$ by connecting all vertices from $V(G)$ with all vertices from $V(H)$. If $G_1, G_2, \ldots, G_k$ are graphs with mutually disjoint vertex sets, then we denote $G_1 + G_2 + \cdots + G_k$ by $\Sigma_{j=1}^{k} G_j$. In the case that $G_1 = G_2 = \ldots = G_k = G$, we denote $\sum_{j=1}^{k} G_j$ by $G^{(k)}$.

The following results are crucial throughout this paper.

Lemma 1.1 (See [9] for details)**.** *Let G be a graph of order $n \geq 2$ that contains no isolated vertices. We have*
i) If G is connected with m edges and diameter D, then $\delta_2(G) \geq \frac{1}{2mD} > 0$.
ii) $\delta_n(G) \geq \frac{n}{n-1}$ with equality if and only if G is the complete graph on n vertices.

Lemma 1.2 ([21] Theorems 2.2 and 2.3)**.** *Let G be a graph of order n with no isolated vertices. Suppose G has minimum vertex degree equal to d_{min} and maximum vertex degree equal to d_{max}. Then $\frac{n}{2d_{max}} \leq R_{-1}(G) \leq \frac{n}{2d_{min}}$. Equality occurs in both bounds if and only if G is a regular graph.*

Lemma 1.3. *Let G be an $n-$vertex graph. Then $\delta_2 = \cdots = \delta_n$ if and only if $G \cong \overline{K}_n$ or $G \cong K_n$.*

Proof. We know that $\delta_1 = 0$. Suppose that $\delta_2 = \cdots = \delta_n$. If G is connected on $n \geq 3$ vertices, then by [7, Corollary 2.6.4] G has exactly two distinct $\ell-$eigenvalues if and only if G is the complete graph. If G is not connected, then $\delta_2 = 0$ and if $\delta_i = 0$ and $\delta_{i+1} \neq 0$ then by [9, Lemma 1.7 (iv)], G has exactly i connected components. So, all Laplacian eigenvalues are equal to zero, which obviously implies that $G \cong \overline{K}_n$. □

2. Examples

In this section, the normalized Laplacian Estrada index of some well-known graphs are computed.

Example 2.1. In this example the normalized Laplacian Estrada index of complete and cocktail-party graphs are computed. We begin with the complete graph. The normalized Laplacian spectrum of K_n and cocktail-party graph $CP_{\frac{n}{2}}$ are computed as follows:

$$\ell Spec(K_n) = \begin{pmatrix} 0 & \frac{n}{n-1} \\ 1 & n-1 \end{pmatrix} \text{ and } \ell Spec(CP_{\frac{n}{2}}) = \begin{pmatrix} 0 & 1 & \frac{n}{n-2} \\ 1 & \frac{n}{2} & \frac{n}{2}-1 \end{pmatrix}.$$

So, $\ell EE(K_n) = 1 + (n-1)e^{\frac{n}{n-1}}$ and $\ell EE(CP_{\frac{n}{2}}) = 1 + \frac{n}{2}e + (\frac{n}{2}-1)e^{\frac{n}{n-2}}$.

Example 2.2. The normalized Laplacian spectrum of the cycle C_n consists of $1 - cos\frac{2\pi i}{n}$, where $0 \leq i \leq n-1$. So,

$$\ell EE(C_n) = \sum_{i=0}^{n-1} e^{1-cos\frac{2\pi i}{n}} = ne\left(\frac{1}{n}\sum_{i=0}^{n-1} e^{-cos\frac{2\pi i}{n}}\right) \approx \frac{ne}{2\pi}\int_0^{2\pi} e^{-cosx}dx.$$

Suppose $Z_0 = \displaystyle\int_0^{2\pi} e^{-cosx}dx \approx 7.954926524$. Then $\ell EE(C_n) \approx \frac{ne}{2\pi}Z_0$.

Example 2.3. The normalized Laplacian spectrum of $n-$vertex path P_n consists of $1 - cos\frac{\pi i}{n-1}$, where $0 \leq i \leq n-1$. Thus,

$$\begin{aligned}
\ell EE(P_n) &= \sum_{i=0}^{n-1} e^{1-\cos\frac{\pi i}{n-1}} \\
&= \frac{e}{2} e \sum_{i=1}^{n-1} e^{1-\cos\frac{\pi i}{n-1}} + \frac{e}{2} \sum_{i=0}^{n-2} e^{1-\cos\frac{\pi i}{n-1}} + \frac{e}{2}(e^{-1}+e) \\
&= \frac{1+e^2}{2} + \frac{(n-1)e}{2}\left(\frac{1}{n-1}\sum_{i=1}^{n-1} e^{-\cos\frac{\pi i}{n-1}} + \frac{1}{n-1}\sum_{i=0}^{n-2} e^{-\cos\frac{\pi i}{n-1}}\right) \\
&\approx \frac{1+e^2}{2} + \frac{(n-1)e}{2}\left(\frac{1}{\pi}\int_0^{\pi} e^{-\cos x}dx + \frac{1}{\pi}\int_0^{\pi} e^{-\cos x}dx\right) \\
&= \frac{1+e^2}{2} + \frac{(n-1)e}{\pi}\left(\int_0^{\pi} e^{-\cos x}dx\right).
\end{aligned}$$

Therefore, $\ell EE(P_n) \approx 0.753004179 + 3.441523869n$, for large n.

Consider the Petersen graph P on 10 vertices. Then the normalized Laplacian spectrum of P is $\ell Spec(P) = \begin{pmatrix} 0 & \frac{2}{3} & \frac{5}{3} \\ 1 & 5 & 4 \end{pmatrix}$. Hence, $\ell EE(P) = 1+5e^{\frac{2}{3}}+4e^{\frac{5}{3}}$.

Example 2.4. Take the star graph and add a new edge to each of its n vertices to get a star-like graph T_{2t+1} with $n = 2t+1$ vertices. By [8], the $\ell-$eigenvalues of a star-like graph are as follows:

$$\ell Spec(T_{2t+1}) = \begin{pmatrix} 0 & 1 & 1-\frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2} & 2 \\ 1 & 1 & t-1 & t-1 & 1 \end{pmatrix}.$$

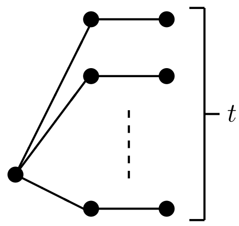

FIGURE 1. The Star-Like Graph T_{2t+1}.

Therefore, $\ell EE(G) = 1 + e + e^2 + e(n-3)\cosh\frac{1}{\sqrt{2}}$.

Example 2.5. Suppose G is a $m-$petal graph on $n = 2m+1$ vertices, $V(G) = \{v_0, v_1, \ldots, v_{2m}\}$ and $E(G) = \{v_0v_i, v_{2i-1}v_{2i}\}$, for $i > 1$.
By [9], G has $\ell-$eigenvalues 0, $\frac{1}{2}$ with multiplicity $m-1$, and $\frac{3}{2}$ with multiplicity $m+1$. Hence, $\ell EE(G) = 1+(m-1)e^{\frac{1}{2}}+(m+1)e^{\frac{3}{2}}$. We now generalize this graph as follows: Fix $s, m \geq 2$ and let $H = \{u\}+(sK_m)$, see Figure 2 for an illustration.

By [8], The $\ell-$eigenvalues of H are 0, $\frac{1}{m}$ with multiplicity $s-1$ and $\frac{m+1}{m}$ with multiplicity $s(m-1)+1$. Then, $\ell EE(H) = 1+(s-1)e^{\frac{1}{m}}+(s(m-1)+1)e^{\frac{m+1}{m}}$.

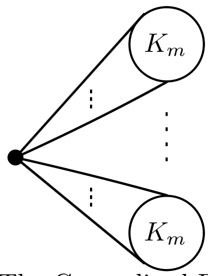

FIGURE 2. The Generalized Petal Graph.

Example 2.6. Let G be the graph constructed as follows. Fix $m \geq 1$. Take the vertex set to be $\{u_1, u_2, u_3\} \cup V_1 \cup V_2 \cup V_3$, where each V_i is a set of m vertices. Then G has exactly $3(m+1)$ vertices. Define the edge set of G by

$$E(G) =\{u_1x : x \in V_1 \cup V_2\} \cup \{u_2x : x \in V_1 \cup V_3\} \cup \{u_3x : x \in V_2 \cup V_3\} \cup\{u_1u_2, u_2u_3, u_1u_3\} \cup \bigcup_{i=1}^{3}\{xy : x, y \in v_i, x \neq y\},$$

see Figure 3. By [7], the $\ell-$eigenvalues of G are 0, $\frac{3}{2(m+1)}$ with multiplicity 2 and $\frac{m+2}{m+1}$ with multiplicity $3m$. Hence, $\ell EE(G) = 1 + 2e^{\frac{3}{2(m+1)}} + 3me^{\frac{m+2}{m+1}}$.

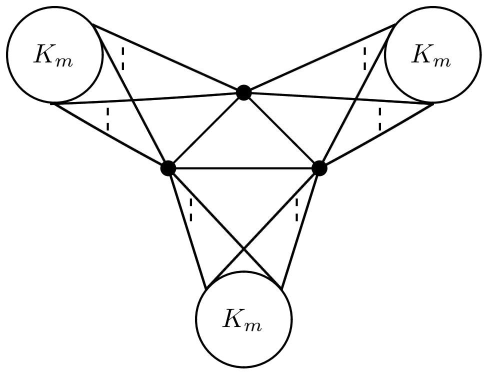

FIGURE 3. The Generalized Triangle-Petal Graph.

Example 2.7. The hypercube graph Q_n is a regular graph with 2^n vertices, which correspond to the subsets of an $n-$element set. Two vertices A and B are joined by an edge if and only if A can be obtained from B by adding or removing a single element. The $\ell-$eigenvalues of the hypercube Q_n are $\frac{2i}{n}$ with multiplicity $\binom{n}{i}$, for $0 \leq i \leq n$. So, $\ell EE(Q_n) = \sum_{i=0}^{n} \binom{n}{i} e^{\frac{2i}{n}} = (e^{\frac{2}{n}}+1)^n$.

Example 2.8. The wheel graph on $n+1$ vertices is defined by $W_n = C_n + K_1$. Thus, the normalized Laplacian spectrum is

$$Spec(W_n) = \left\{0, \frac{4}{3}, 1 - \frac{2}{3}cos\frac{2\pi}{n}, 1 - \frac{2}{3}cos\frac{4\pi}{n}, \ldots, 1 - \frac{2}{3}cos\frac{2(n-1)\pi}{n}\right\}.$$

Thus,

$$\begin{aligned}
\ell EE(W_n) &= 1 + e^{\frac{4}{3}} + \sum_{i=1}^{n-1} e^{1-\frac{2}{3}cos\frac{2\pi i}{n}} \\
&= 1 + e^{\frac{4}{3}} + \frac{e}{2}(\sum_{i=1}^{n} e^{-\frac{2}{3}cos\frac{2\pi i}{n}} + \sum_{i=0}^{n-1} e^{-\frac{2}{3}cos\frac{2\pi i}{n}} - 2e^{-\frac{2}{3}}) \\
&= 1 + e^{\frac{4}{3}} - e^{\frac{1}{3}} + \frac{ne}{2}(\frac{1}{n}\sum_{i=1}^{n} e^{-\frac{2}{3}cos\frac{2\pi i}{n}} + \frac{1}{n}\sum_{i=0}^{n-1} e^{-\frac{2}{3}cos\frac{2\pi i}{n}}) \\
&\approx 1 + e^{\frac{4}{3}} - e^{\frac{1}{3}} + \frac{ne}{2}(\frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{2}{3}cosx}dx + \frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{2}{3}cosx}dx) \\
&= 1 + e^{\frac{4}{3}} - e^{\frac{1}{3}} + \frac{ne}{2\pi}(\int_0^{2\pi} e^{-\frac{2}{3}cosx}dx).
\end{aligned}$$

Define $N_0 = \int_0^{2\pi} e^{-\frac{2}{3}cosx}dx \approx 7.000950642$. Since, $e^{\frac{4}{3}} \approx 3.79367$, $e^{\frac{1}{3}} \approx 1.39561$, $\ell EE(W_n) \approx 3.398055468 + \frac{ne}{2\pi}N_0 = 3.398055468 + 3.028807202n$, for large n.

Example 2.9. A Möbius ladder L_n of order $2n$ is a graph obtained by introducing a twist in a 3−regular prism graph of order n that is isomorphic to the circulant graph, see Figure 4.

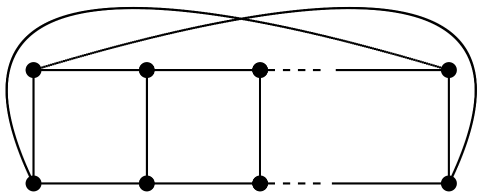

FIGURE 4. The Möbuis Ladder Graph.

In this example the normalized Laplacian Estrada index of a Möbius graph is computed. By [10], the normalized Laplacian eigenvalues of L_n are $\delta_i = 1 - \frac{(-1)^i}{3} - \frac{2}{3}cos\frac{\pi i}{n}$, where $0 \le i \le 2n-1$. So,

$$\ell EE(L_n) = e^{\frac{2}{3}} \sum_{\substack{i=0 \\ i\ even}}^{2n-2} e^{-\frac{2}{3}cos\frac{\pi i}{n}} + e^{\frac{4}{3}} \sum_{\substack{i=0 \\ i\ odd}}^{2n-1} e^{-\frac{2}{3}cos\frac{\pi i}{n}}$$

$$= e^{\frac{2}{3}} \sum_{i=0}^{n-1} e^{-\frac{2}{3}cos\frac{2\pi i}{n}} + e^{\frac{4}{3}} \sum_{i=0}^{n-1} e^{-\frac{2}{3}cos\frac{(2i+1)\pi}{n}}$$
$$\approx \frac{ne^{\frac{2}{3}}}{2\pi} \int_0^{2\pi} e^{-\frac{2}{3}cosx}dx + \frac{ne^{\frac{4}{3}}}{2\pi} \int_0^{2\pi} e^{-\frac{2}{3}cosx}dx$$
$$= \frac{n}{2\pi}(e^{\frac{2}{3}} + e^{\frac{4}{3}})N_0.$$

Note that,

$$\sum_{i=0}^{n-1} e^{-\frac{2}{3}cos\frac{(2i+1)\pi}{n}} = \sum_{i=0}^{2n-1} e^{-\frac{2}{3}cos\frac{\pi i}{n}} - \sum_{i=0}^{n-1} e^{-\frac{2}{3}cos\frac{2\pi i}{n}} \approx \frac{n}{2\pi} \int_0^{2\pi} e^{-\frac{2}{3}cosx}dx.$$

Since, $e^{\frac{4}{3}} \approx 3.79367$, $e^{\frac{2}{3}} \approx 1.94773$. Then $\ell EE(L_n) \approx 6.397276157n$, for large n.

3. The $\ell-$Estrada Index of Graphs

This section is concerned with the use of algebraic techniques in the study of the normalized Estrada index of graphs. We begin with the following simple theorem:

Theorem 3.1. *Let G be a connected graph with n vertices. Then $\ell EE(G) > ne$.*

Proof. By Arithmetic-Geometric mean inequality [18], we have:

$$\frac{1}{n}\ell EE(G) \geq \sqrt[n]{\prod_i e^{\delta_i}} = \sqrt[n]{e^{\sum_i \delta_i}} = \sqrt[n]{e^n} = e,$$

with equality if and only if for all $1 \leq i, j \leq n$, $e^{\delta_i} = e^{\delta_j}$ if and only if $\delta_i = \delta_j$. This implies that all δ_i 's are zero. This contradicts the fact that G is connected. □

Theorem 3.2. *Let G be a graph with n vertices and c connected components. Then, $\ell EE(G) \geq c + (n-c)e^{\frac{n}{n-c}}$. Equality holds if and only if G is a union of copies of cK_s, for some fixed integers s.*

Proof. Using a similar method as in [3, Theorem 3], we obtain $\delta_1 = \cdots = \delta_c = 0$ and $\delta_{c+1} + \cdots + \delta_n = n$. Therefore,

$$\ell EE(G) = c + \sum_{i=c+1}^{n} e^{\delta_i} \geq c + (n-c)e^{\frac{\delta_{c+1}+\cdots+\delta_n}{n-c}},$$

where the last inequality is obtained by applying the Arithmetic-Geometric mean inequality. Suppose $G = cK_s$, $s \geq 2$. Then, $n = cs$, and the normalized

Laplacian spectrum of G is as follows: $\begin{pmatrix} 0 & \frac{s}{s-1} \\ c & c(s-1) \end{pmatrix}$. Further,

$$\ell EE(G) = c + \sum_{i=c+1}^{n} e^{\frac{s}{s-1}} = c + (n-c)e^{\frac{n}{n-c}}.$$

This shows that the equality holds for G. Conversely, let equality hold for G. Then all of non-zero normalized Laplacian eigenvalues of G must be mutually equal. Then, the normalized Laplacian spectrum of the graph H is 0, δ with multiplicity $s-1$, where $\delta > 0$ and s is a positive integer. Therefore, $H = K_s$, and then $G = cK_s$, as desired. □

Theorem 3.3. *If G is a connected $r-$regular graph with n vertices, then $\ell EE(G) \geq 1 + (n-1)e^{\frac{n}{n-1}}$, with equality if and only if $G \cong K_n$.*

Proof. The $\ell-$spectrum of G is 0, $1 - \frac{\lambda_i}{r}$, for $2 \leq i \leq n$. Then $\ell EE(G) = 1 + e\sum_{i=2}^{n} e^{\frac{-\lambda_i}{r}}$. By arithmetic-geometric mean inequality, we get

$$\begin{aligned}(\ell EE(G) - 1)e^{-1} = \sum_{i=2}^{n} e^{\frac{-\lambda_i}{r}} &\geq (n-1)(\prod_{i=2}^{n} e^{\frac{-\lambda_i}{r}})^{\frac{1}{n-1}} \\ &= (n-1)e^{\frac{-1}{r(n-1)}\sum_{i=2}^{n}\lambda_i} \\ &= (n-1)e^{\frac{1}{n-1}},\end{aligned}$$

where the last equality follows from $\Sigma_{i=2}^{n}\lambda_i = -r$. Therefore, $\ell EE(G) \geq 1 + (n-1)e^{\frac{n}{n-1}}$ with equality if and only if $\lambda_2 = \cdots = \lambda_n$. By assumption, this is equivalent to $G \cong K_n$. □

Theorem 3.4. *If G is an $r-$regular bipartite graph, then $\ell EE(G) < eEE(G)^{\frac{1}{r}}$.*

Proof. The $\ell-$eigenvalues of G are 0, $1 - \frac{\lambda_i}{r}$ for $2 \leq i \leq n$. Thus,

$$\ell EE(G) = \sum_{i=1}^{n}(e^{r-\lambda_i})^{\frac{1}{r}} = e\sum_{i=1}^{n}(e^{-\lambda_i})^{\frac{1}{r}} \leq e(\sum_{i=1}^{n} e^{-\lambda_i})^{\frac{1}{r}} = eEE(G)^{\frac{1}{r}}$$

where the last equality follows from $\Sigma_{i=1}^{n}e^{\lambda_i} = \Sigma_{i=1}^{n}e^{-\lambda_i}$. Since G is bipartite, the eigenvalues of G are symmetric around zero. The equality is attained if and only if $\lambda_1 = \cdots = \lambda_n$ and this is equivalent to $G \cong \overline{K}_n$, which is impossible. □

Theorem 3.5. *Let G be a connected with $n \geq 2$ vertices, m edges and diameter D. Then $\ell EE(G) \geq 1 + e^{\frac{1}{2mD}} + e^{\frac{2n}{(n-1)} - \frac{1}{2mD}} + (n-3)e^{\frac{n}{n-1}}$.*

Proof. Since G is connected, $\delta_1 = 0$ and $\delta_2, \delta_n > 0$. Then,

$$\ell EE(G) = e^{\delta_1} + e^{\delta_2} + \cdots + e^{\delta_n}$$

$$\geq 1+e^{\delta_2}+e^{\delta_n}+(n-3)(\prod_{i=3}^{n-1}e^{\delta_i})^{\frac{1}{n-3}}$$
$$=1+e^{\delta_2}+e^{\delta_n}+(n-3)e^{\frac{n-\delta_2-\delta_n}{n-3}}.$$

Define $f(x,y)=1+e^x+e^y+(n-3)e^{\frac{n-x-y}{n-3}}$, $x,y>0$. Then we have:

$$f_x=e^x-e^{\frac{n-x-y}{n-3}},$$
$$f_y=e^y-e^{\frac{n-x-y}{n-3}},$$
$$f_{xx}=e^x+\frac{1}{n-3}e^{\frac{n-x-y}{n-3}},$$
$$f_{yy}=e^y+\frac{1}{n-3}e^{\frac{n-x-y}{n-3}},$$
$$f_{xy}=f_{yx}=\frac{1}{n-3}e^{\frac{n-x-y}{n-3}}.$$

Moreover, if $f_x=f_y=0$ then $(n-2)x+y=n$ and so $x+y=\frac{2n}{n-1}$. If $x+y=\frac{2n}{n-1}$, then $f_{xx}>0$ and

$$f_{xx}f_{yy}-f_{xy}^2=e^{\frac{2n}{n-1}}+\frac{1}{n-3}e^{\frac{n}{n-1}}[e^x+e^{\frac{2n}{n-1}}]>0.$$

From the above, we conclude that $f(x,y)$ has a minimum at $x+y=\frac{2n}{n-1}$ and that the minimum value is $1+e^x+e^{\frac{2n}{n-1}-x}+(n-3)e^{\frac{n}{n-1}}$. Hence f is an increasing function for $x>0$. By Lemma 1.1(i), $\delta_2(G)\geq\frac{1}{2mD}>0$. Thus,

$$\ell EE(G)\geq 1+e^{\frac{1}{2mD}}+e^{\frac{2n}{(n-1)}-\frac{1}{2mD}}+(n-3)e^{\frac{n}{n-1}},$$

proving the result. □

Theorem 3.6. *If G is an $r-$regular graph with n vertices, then*

$$\ell EE(l(G))\leq LEE(G)^{\frac{1}{2(r-1)}}+\frac{n(r-2)}{2}e^{\frac{r}{r-1}},$$

with equality if and only if $G\cong\overline{K}_n$. In particular, for $r-$regular graphs, $\ell EE(l(G))\leq\sqrt{LEE(G)}$ if and only if $r=2$.

Proof. By [4, Theorem 3.8], the eigenvalues of $l(G)$ are -2 with multiplicity $\frac{n(r-2)}{2}$, and $\lambda_i(G)+r-2$ for $1\leq i\leq n$. Since the line graph of G is $(2r-2)-$regular, and $\mu_i(l(G))=2r-2-\lambda_i(l(G))$ for $1\leq i\leq n$, the normalized Laplacian eigenvalues of $l(G)$ are $\frac{r}{r-1}$with multiplicity $\frac{n(r-2)}{2}$, and $\frac{\mu_i}{2r-2}$ for $1\leq i\leq n$. Thus, we have:

$$\ell EE(l(G))=\sum_{i=1}^{n}e^{\frac{\mu_i}{2r-2}}+\frac{n(r-2)}{2}e^{\frac{r}{r-1}}$$
$$\leq(\sum_{i=1}^{n}e^{\mu_i})^{\frac{1}{2r-2}}+\frac{n(r-2)}{2}e^{\frac{r}{r-1}}$$

$$= LEE(G)^{\frac{1}{2r-2}} + \frac{n(r-2)}{2}e^{\frac{r}{r-1}}.$$

From [24, Lemma 1.2] it follows that the above equality holds if and only if G is an empty graph. □

Corollary 3.7. *Let $l(G) = l^1(G)$ and $l^{k+1}(G) = l(l^k(G))$. If G is $r-$regular then $\ell EE(l^{k+1}(G)) = \ell EE(l^k(G))^{\frac{1}{2r_k-2}} + \frac{n_k(r_k-2)}{2}e^{\frac{r_k}{r_k-1}}$, where $l^k(G)$ is r_k-regular with n_k vertices, $r_k = (r-2)2^k + 2$ and $n_k = \frac{n}{2^k}\prod_{i=0}^{k-1}(2^i r - 2^{i-1} + 2)$.*

Corollary 3.8. *If G is $2-$regular and bipartite, then $\ell EE(l(G)) \leq \sqrt{EE(G)}$.*

A *fullerene graph* of order n is a cubic $3-$connected planar graph with exactly 12 pentagonal faces and $\frac{n}{2} - 10$ hexagonal faces.

Corollary 3.9. *If F_n is an $n-$vertex fullerene graph, then*

$$\ell EE(l(F_n)) \leq LEE(F_n)^{\frac{1}{4}} + 2.24n.$$

Consider G is $r-$regular graph with $n-$vertex and m-edges, and the eigenvalues of G are $r = \lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$. A *para-line graph* of G, denoted by $C(G)$, is defined as a line graph of the subdivision graph $S(G)$ (i.e., $S(G)$ is the graph obtained from G by inserting a vertex to every edge of G.) of G. The para-line graph has also been called the clique-inserted graph. Note that para-line graph is $r-$regular and the number of vertices of $C(G)$ equals nr. The eigenvalues of the para-line graph $C(G)$ of G are $\frac{r+2\pm\sqrt{r^2+4(\lambda_i(G)+1)}}{2}$ for $1 \leq i \leq n$, -2, with multiplicity $m - n$, and 0, with multiplicity $m - n$, see [22, 23] for details.

Theorem 3.10. *Let G be a $r-$regular graph with n vertices and m edges. Then*

$$\ell EE(C(G)) > 1+\frac{n(r-2)e}{2}+(\frac{n(r-2)}{2}+1)e^{\frac{r+2}{r}}+2(n-1)e^{\frac{r+2}{2r}}+(n-1)(r^2+6)-4r.$$

Proof. By above discussion, the normalized Laplacian eigenvalues of the para-line graph $C(G)$ of G are

$$\begin{pmatrix} 0 & 1 & \frac{r+2}{r} & \frac{r+2-\sqrt{r^2+4(\lambda_i(G)+1)}}{2r} & \frac{r+2+\sqrt{r^2+4(\lambda_i(G)+1)}}{2r} \\ 1 & \frac{n(r-2)}{2} & \frac{n(r-2)}{2}+1 & 2 \leq i \leq n & 2 \leq i \leq n \end{pmatrix}.$$

By definition,

$$\ell EE(C(G)) = 1 + \frac{n(r-2)e}{2} + (\frac{n(r-2)}{2} + 1)e^{\frac{r+2}{r}} + \sum_{i=2}^{n} e^{\frac{r+2\pm\sqrt{r^2+4(\lambda_i(G)+1)}}{2r}}.$$

In the other hand,

$$\sum_{i=2}^{n} e^{\frac{r+2\pm\sqrt{r^2+4(\lambda_i(G)+1)}}{2r}} = 2(n-1)e^{\frac{r+2}{2r}} + \sum_{i=2}^{n} e^{\pm\sqrt{r^2+4(\lambda_i(G)+1)}}$$

$$= 2(n-1)e^{\frac{r+2}{2r}} + 2\sum_{i=2}^{n} cosh(\sqrt{r^2+4(\lambda_i(G)+1)})$$
$$> 2(n-1)e^{\frac{r+2}{2r}} + \sum_{i=2}^{n}(r^2+4\lambda_i(G)+6)$$
$$= 2(n-1)e^{\frac{r+2}{2r}} + (n-1)(r^2+6) + 4\sum_{i=2}^{n}\lambda_i(G)$$
$$= 2(n-1)e^{\frac{r+2}{2r}} + (n-1)(r^2+6) - 4r,$$

where the last equality follows from $\sum_{i=2}^{n}\lambda_i(G) = -r$. Therefore,

$$\ell EE(C(G)) > 1+\frac{n(r-2)e}{2}+(\frac{n(r-2)}{2}+1)e^{\frac{r+2}{r}}+2(n-1)e^{\frac{r+2}{2r}}+(n-1)(r^2+6)-4r.$$

□

Corollary 3.11. *Let $C^0(G) = G$, $C^k(G) = C(C^{k-1}(G))$, $k \geq 1$. Then*

$$\ell EE(C^k(G)) > 1+\frac{n'_k(r-2)e}{2}+(\frac{n'_k(r-2)}{2}+1)e^{\frac{r+2}{r}}+2(n'_k-1)e^{\frac{r+2}{2r}}+(n'_k-1)(r^2+6)-4r,$$

where $C^k(G)$ is $r-$regular with $n'_k = nr^k$, vertices for $k \geq 0$.

Theorem 3.12. *Let G be an $r-$regular graph. Then*

$$\ell EE(\overline{G}) \leq 1 + e^{\frac{n-r}{n-r-1}} \sqrt[n-r-1]{EE(G)-1}.$$

Equality holds if and only if G is an empty graph.

Proof. By [10, Theorem 2.6], if the spectrum of G contains $r = \lambda_1, \lambda_2, \ldots, \lambda_n$, then the spectrum of $\overline{G}$ is $n-r-1$ and $-1-\lambda_i$, where $2 \leq i \leq n$. Since $\mu_i = r-\lambda_i$ and complement of G is $(n-r-1)-$regular, the normalized Laplacian eigenvalues of $\overline{G}$ are 0 and $\frac{n-\mu_i}{n-r-1}$, where $2 \leq i \leq n$. Thus,

$$\ell EE(\overline{G}) = 1 + \sum_{i=2}^{n}(e^{n-\mu_i})^{\frac{1}{n-r-1}}$$
$$= 1 + e^{\frac{n}{n-r-1}} \sum_{i=2}^{n}(e^{-\mu_i})^{\frac{1}{n-r-1}}$$
$$\leq 1 + e^{\frac{n-r}{n-r-1}} (\sum_{i=2}^{n} e^{\lambda_i})^{\frac{1}{n-r-1}}$$
$$= 1 + e^{\frac{n-r}{n-r-1}} \sqrt[n-r-1]{EE(G)-1}.$$

Clearly, equality holds if and only if G is an empty graph. □

Theorem 3.13. *Let G_1 and G_2 be $r-$ and $s-$regular graphs on n and m vertices, respectively. Suppose $0 = \delta_1(G_1) \leq \delta_2(G_1) \leq \cdots \leq \delta_n(G_1) \leq 2$ are the $\ell-$eigenvalues of G_1 and $0 = \delta_1(G_2) \leq \delta_2(G_2) \leq \cdots \leq \delta_n(G_2) \leq 2$ are the $\ell-$eigenvalues of G_2. Then*

$$\ell EE(G_1 + G_2) \leq 1 + e^{\frac{m}{m+r}} (\ell EE(G_1) - 1)^{\frac{r}{m+r}} + e^{\frac{n}{n+s}} (\ell EE(G_2) - 1)^{\frac{s}{n+s}} + e^{\frac{m}{m+r}+\frac{s}{n+s}},$$

with equality if and only if $G_1 \cong \overline{K}_n$ and $G_2 \cong \overline{K}_m$.

Proof. From [6, Theorem 12], the normalized Laplacian eigenvalues of $G_1 + G_2$ are as follows:

$$\begin{pmatrix} 0 & \frac{m+r\delta_i(G_1)}{m+r} & \frac{n+s\delta_j(G_2)}{n+s} & \frac{m}{m+r} + \frac{n}{n+s} \\ 1 & 2 \leq i \leq n & 2 \leq j \leq m & 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \ell EE(G_1 + G_2) &= 1 + e^{\frac{m}{m+r}} \sum_{i=2}^{n} e^{\frac{r\delta_i(G_1)}{m+r}} + e^{\frac{n}{n+s}} \sum_{i=2}^{m} e^{\frac{s\delta_j(G_2)}{n+s}} + e^{\frac{m}{m+r}+\frac{n}{n+s}} \\ &\leq 1 + e^{\frac{m}{m+r}} (\sum_{i=2}^{n} e^{\delta_i(G_1)})^{\frac{r}{m+r}} + e^{\frac{n}{n+s}} (\sum_{i=2}^{m} e^{\delta_j(G_2)})^{\frac{s}{n+s}} + e^{\frac{m}{m+r}+\frac{n}{n+s}} \\ &= 1 + e^{\frac{m}{m+r}} (\ell EE(G_1) - 1)^{\frac{r}{m+r}} + e^{\frac{n}{n+s}} (\ell EE(G_2) - 1)^{\frac{s}{n+s}} + e^{\frac{m}{m+r}+\frac{n}{n+s}}, \end{aligned}$$

where the last equality follows from $\delta_1(G_1) = 0$ and $\delta_1(G_2) = 0$. The equality is attained if and only if $\delta_i(G_1) = 0$, $2 \leq i \leq n$, and $\delta_j(G_2) = 0$, $2 \leq j \leq m$. So, $G_1 \cong \overline{K}_n$ and $G_2 \cong \overline{K}_m$. This completes the proof. □

Apply Theorems 3.12 and 3.13 to evaluate the $\ell-$Estrada indices of the complete bipartite graphs, star graphs, $CP_n + 2K_1$ and $K_{n-2} + 2K_1$. Start with the complete bipartite graph $K_{n,m}$. We have:

$$\begin{aligned} \ell EE(K_{n,m}) &= \ell EE(\overline{K}_n + \overline{K}_m) = e^2 + (n + m - 2)e + 1, \\ \ell EE(S_n) &= \ell EE(K_1 + \overline{K}_{n-1}) = e^2 + (n - 2)e + 1, \\ \ell EE(CP_n + 2K_1) &= e^{\frac{n+2}{n}} + (n - 2)e^{\frac{n+1}{n}} + ne + 1, \\ \ell EE(K_{n-2} + 2K_1) &= e^{\frac{n+1}{n-1}} + (n - 3)e^{\frac{n}{n-1}} + ne + 1. \end{aligned}$$

Corollary 3.14. *If G_j, $1 \leq j \leq k$, is an $r-$regular $n-$vertex graph, then*

$$\ell EE(\sum_{j=1}^{k} G_j) \leq 1 + e^{\frac{n(k-1)}{n(k-1)+r}} (\ell EE(G_k) - 1)^{\frac{r}{n(k-1)+r}} + e^{\frac{n}{n(k-1)+r}} (\ell EE(\sum_{j=1}^{k-1} G_j) - 1)^{\frac{n(k-2)+r}{n(k-1)+r}} + e^{\frac{nk}{n(k-1)+r}}.$$

Corollary 3.15. *If G is an $r-$regular graph $n-$vertex graph, then*

$$\begin{aligned}\ell EE(G^{(k)}) &\leq 1 + e^{\frac{n(k-1)}{n(k-1)+r}}(\ell EE(G)-1)^{\frac{r}{n(k-1)+r}} \\ &\quad + e^{\frac{n}{n(k-1)+r}}(\ell EE((k-1)G)-1)^{\frac{n(k-2)+r}{n(k-1)+r}} + e^{\frac{nk}{n(k-1)+r}}.\end{aligned}$$

Theorem 3.16. *If G is connected graph with n vertices, then*

$$\sqrt{n(n-1)e^2 + 4R_{-1}(G) + 5n} < \ell EE(G) < e^n + R_{-1}(G) + \frac{n}{2}(3-n) - 1.$$

Proof. Using a similar method as [15, Proposition 7], we have:

$$\begin{aligned}\ell EE(G) &= \sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{(\delta_i)^k}{k!} \\ &\leq \frac{5n}{2} + R_{-1}(G) + \sum_{k\geq 3}\frac{1}{k!}(\sum_{i=1}^{n}\delta_i)^k \\ &= e^n + R_{-1}(G) + \frac{n}{2}(3-n) - 1,\end{aligned}$$

resulting in the upper bound. If $\sum_{i=1}^{n}{\delta_i}^k = (\sum_{i=1}^{n}\delta_i)^k$, then $\delta_i = 0$, where $2 \leq i \leq n$. Thus, $G \cong \overline{K}_n$. Obviously, the right equality is impossible. On the other hand, $\ell EE(G)^2 = \sum_{i=1}^{n} e^{2\delta_i} + 2\sum_{1\leq i<j\leq n} e^{\delta_i}e^{\delta_j}$ and so, by the arithmetic-geometric inequality

$$\begin{aligned}2\sum_{1\leq i<j\leq n} e^{\delta_i}e^{\delta_j} &\geq n(n-1)(\prod_{1\leq i<j\leq n} e^{\delta_i}e^{\delta_j})^{\frac{2}{n(n-1)}} \\ &= n(n-1)[(\prod_{i=1}^{n} e^{\delta_i})^{n-1}]^{\frac{2}{n(n-1)}} \\ &= n(n-1)e^2.\end{aligned}$$

By means of a power-series expansion, we get

$$\begin{aligned}\sum_{i=1}^{n} e^{2\delta_i} &= \sum_{i=1}^{n}\sum_{k\geq 0}\frac{(2\delta_i)^k}{k!} \\ &= 3n + 2(n + 2R_{-1}(G)) + \sum_{k\geq 3}\frac{(2\delta_i)^k}{k!} \\ &\geq 4R_{-1}(G) + 5n.\end{aligned}$$

Therefore, $\ell EE(G)^2 = n(n-1)e^2 + 4R_{-1}(G) + 5n$. This implies the lower bound. If $\sum_{k\geq 3}\frac{(2\delta_i)^k}{k!} = 0$, then $\delta_i = 0$ for $2 \leq i \leq n$. Thus, $G \cong \overline{K}_n$. The left equality is clearly impossible, proving the result. □

Theorem 3.17. *If G is a connected graph with $n > 2$ vertices, then*

$$\ell EE(G) > 2 + \sqrt{n(n-1)e^2 - 6n + 4}.$$

Proof. Using a similar method as in [19, Proposition 3.3], one can observe that for $k \geq 2$, $\sum_{i=1}^{n}(2\delta_i)^k \geq 4\sum_{i=1}^{n}\delta_i^k$ with equality for all $k \geq 2$ if and only if $\delta_1 = \cdots = \delta_n = 0$, i.e., $G \cong \overline{K}_n$. Then

$$\begin{aligned}\sum_{i=1}^{n} e^{2\delta_i} \geq \sum_{i=1}^{n}\sum_{k\geq 0}\frac{(2\delta_i)^k}{k!} &= 2n + \sum_{k\geq 2}\frac{\sum_{i=1}^{n}(2\delta_i)^k}{k!} \\ &\geq 2n + 4\sum_{k\geq 2}\frac{\sum_{i=1}^{n}\delta_i^k}{k!} \\ &= 2n + 4(\ell EE(G) - 2n).\end{aligned}$$

In Theorem 3.16, it was shown that $2\sum_{1\leq i<j\leq n} e^{\delta_i}e^{\delta_j} \geq n(n-1)e^2$. Thus,

$$\ell EE(G)^2 \geq 4\ell EE(G) + n(n-1)e^2 - 6n.$$

Note that $e^x \geq (1+x)$, so if $n > 2$ then $n(n-1)e^2 - 6n + 4 \geq 3n(n-1) - 6n + 4 \geq 0$. Therefore,

$$\ell EE(G) \geq 2 + \sqrt{n(n-1)e^2 - 6n + 4}.$$

Since the graph is connected, the equality can not be attained. $\square$

Theorem 3.18. *If G is a connected graph with n vertices, then $\ell EE(G) < n - 1 + e^{\sqrt{\frac{n}{d_{min}}}}$.*

Proof. By definition,

$$\begin{aligned}e^{-1}\ell EE(G) = \sum_{i=1}^{n} e^{\delta_i - 1} &\leq n + \sum_{i=1}^{n}\sum_{k\geq 1}\frac{|\delta_i - 1|^k}{k!} \\ &= n + \sum_{k\geq 1}\frac{1}{k!}\sum_{i=1}^{n}(|\delta_i - 1|^2)^{\frac{k}{2}} \\ &\leq n + \sum_{k\geq 1}\frac{1}{k!}(\sum_{i=1}^{n}|\delta_i - 1|^2)^{\frac{k}{2}} \\ &= n + \sum_{k\geq 1}\frac{1}{k!}(2R_{-1}(G))^{\frac{k}{2}}\end{aligned}$$

$$= n - 1 + \sum_{k\geq 0} \frac{\sqrt{(2R_{-1}(G))}^k}{k!}$$
$$= n - 1 + e^{\sqrt{2R_{-1}(G)}},$$

and by Lemma 1.2, $e^{-1}\ell EE(G) \leq n - 1 + e^{\sqrt{\frac{n}{d_{min}}}}$. Also, the equality occurs if only if $G \cong \overline{K}_n$, which is impossible. □

Corollary 3.19. *If G is an $r-$regular connected graph with n vertices, then*

$$\ell EE(G) < n - 1 + e^{\sqrt{\frac{n}{r}}}.$$

Theorem 3.20. *If G is connected graph with n vertices, then*

$$\ell EE(G) > n - 1 + e^{\sqrt{n+2R_{-1}(G)}} - \sqrt{n + 2R_{-1}(G)}.$$

Proof. Recall that $\sum_{i=1}^{n} \delta_i^2 = n+2R_{-1}(G)$. Using a similar method as [24, Proposition 3.1], for an integer $k \geq 3$, $(\sum_{i=1}^{n} \delta_i^2)^k \geq (\sum_{i=1}^{n} \delta_i^k)^2$, and then $\sum_{i=1}^{n} \delta_i^k \leq (\sum_{i=1}^{n} \delta_i^2)^{\frac{k}{2}} = (n + 2R_{-1}(G))^{\frac{k}{2}}$. It is easily seen that

$$\ell EE(G) = 2n + \sum_{k\geq 2} \frac{1}{k!} \sum_{i=1}^{n} {\delta_i}^k$$
$$\geq 2n + (n - 1) \sum_{k\geq 2} \frac{1}{k!} (\sqrt{n + 2R_{-1}(G)})^k$$
$$= n - 1 - \sqrt{n + 2R_{-1}(G)} + e^{\sqrt{n+2R_{-1}(G)}},$$

with equality if and only if at most one of $\delta_1, \delta_2, \ldots, \delta_n$ is non-zero, or equivalently $G \cong K_2 \cup \overline{K}_{n-2}$ or $G \cong \overline{K}_n$, which is impossible. □

Theorem 3.21. *If G is connected graph with n vertices, then*

$$\ell EE(G) \geq n + 1 + (n - 1)e^{\sqrt{\frac{n+2R_{-1}(G)}{n-1}}} - \sqrt{(n - 1)(n + 2R_{-1}(G))},$$

with equality if and only if $G \cong K_n$.

Proof. Using a similar method as given in [19, Proposition 3.4] and by an inequality from [16, p. 26], where $a_1, a_2, \ldots, a_p$ are non-negative numbers and $m \leq k$ with $m, k \neq 0$, we have:

$$\left(\frac{1}{p}\sum_{i=1}^{p} a_i^m\right)^{\frac{1}{m}} \leq \left(\frac{1}{p}\sum_{i=1}^{p} a_i^k\right)^{\frac{1}{k}}.$$

Equality is attained if and only if $a_1 = a_2 = \cdots = a_p$. In above inequality, we substitute $m = 2$, $p = n-1$, $a_i = \delta_i$, $2 \le i \le n$ and $k \ge 2$. Then we have:

$$\sum_{i=2}^{n} \delta_i^k \ge (n-1)\left(\frac{1}{n-1}\sum_{i=2}^{n}\delta_i^2\right)^{\frac{k}{2}} = (n-1)\left(\sqrt{\frac{n+2R_{-1}(G)}{n-1}}\right)^k,$$

which is an equality for $k = 2$ whereas equality holds for $k \ge 3$ if and only if $\delta_2 = \cdots = \delta_n$. By Lemma 1.3, this is equivalent to $G \cong \overline{K}_n$ or $G \cong K_n$. Since G is a connected graph, $G \cong K_n$. Clearly,

$$\begin{aligned}
\ell EE(G) &= 2n + \sum_{k\ge 2}\frac{1}{k!}\sum_{i=1}^{n}{\delta_i}^k \\
&\ge 2n + (n-1)\sum_{k\ge 2}\frac{1}{k!}(\sqrt{\frac{n+2R_{-1}(G)}{n-1}})^k \\
&= 2n + (n-1)(e^{\sqrt{\frac{n+2R_{-1}(G)}{n-1}}} - \sqrt{\frac{n+2R_{-1}(G)}{n-1}} - 1) \\
&= n + 1 + (n-1)e^{\sqrt{\frac{n+2R_{-1}(G)}{n-1}}} - \sqrt{(n-1)(n+2R_{-1}(G))},
\end{aligned}$$

with equality if and only if the lower bound for $\sum_{i=2}^{n}\delta_i^k$ above is attained for $k \ge 3$, if and only if $G \cong K_n$. □

4. Bounds for the $\ell-$Estrada Index

We recall that the normalized Laplacian energy of the graph G is defined as $E_\ell(G) = \sum_{i=1}^{n}|\delta_i - 1|$ [8]. In this section, the relationship between the $\ell-$Estrada index and the normalized Laplacian energy of graphs are investigated.

Theorem 4.1. *If G is connected, then $\ell EE(G) < e(n - 1 + e^{E_\ell(G)})$.*

Proof. By definition, we have

$$\begin{aligned}
e^{-1}\ell EE(G) &= \sum_{i=1}^{n} e^{\delta_i - 1} \\
&= \sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{1}{k!}(\delta_i - 1)^k \\
&= n + \sum_{i=1}^{n}\sum_{k\ge 2}\frac{1}{k!}(\delta_i - 1)^k \\
&\le n + \sum_{k\ge 2}\frac{1}{k!}(\sum_{i=1}^{n}|\delta_i - 1|)^k
\end{aligned}$$

$$= n - 1 + e^{E_\ell(G)},$$

with equality if and only if $\sum_{i=1}^{n}(\delta_i - 1)^k = (\sum_{i=1}^{n}|\delta_i - 1|)^k$ if and only if $\delta_i = 0$, $1 \leq i \leq n$, if and only if G is an empty graph with n vertices, which is impossible. □

In [5], the authors introduced the notion of the Randić matrix of a graph G as $R(G) = [R_{i,j}]_{n\times n}$, where

$$R_{i,j} = \begin{cases} \frac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } v_i \text{ is and adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}.$$

The Randić energy of G is defined by $E_R(G) = \sum_{i=1}^{n}|\tau_i|$, where τ_i's are the eigenvalues of Randić matrix $R(G)$.

Corollary 4.2. *If G is connected then $\ell EE(G) < e(n - 1 + e^{E_R(G)})$.*

Proof. The proof is follows from [5, Theorem 2] and Theorem 4.1. □

Theorem 4.3. *If G is a connected graph with n vertices, then*

$$e^{-1}\ell EE(G) - E_\ell(G) < n - 1 - \sqrt{\frac{n}{d_{min}}} + e^{\sqrt{\frac{n}{d_{min}}}}.$$

Proof. In the proof of Theorem 3.18, the following inequality is proved:

$$e^{-1}\ell EE(G) \leq n + \sum_{i=1}^{n}\sum_{k\geq 1}\frac{|\delta_i - 1|^k}{k!}.$$

On the other hand, by definition of the normalized Laplacian energy,

$$e^{-1}\ell EE(G) \leq n + E_\ell(G) + \sum_{i=1}^{n}\sum_{k\geq 2}\frac{|\delta_i - 1|^k}{k!}.$$

Thus,

$$\begin{aligned} e^{-1}\ell EE(G) - E_\ell(G) &\leq n + \sum_{i=1}^{n}\sum_{k\geq 2}\frac{|\delta_i - 1|^k}{k!} \\ &\leq n - 1 - \sqrt{2R_{-1}(G)} + e^{\sqrt{2R_{-1}(G)}}. \end{aligned}$$

We now apply Lemma 1.2, to get $e^{-1}\ell EE(G) - E_\ell(G) \leq n - 1 - \sqrt{\frac{n}{d_{min}}} + e^{\sqrt{\frac{n}{d_{min}}}}$. The equality is attained if and only if $G \cong \overline{K}_n$, which is impossible. □

Corollary 4.4. *If G is an $r-$regular $n-$vertex graph, then*

$$e^{-1}\ell EE(G) - E_\ell(G) < n - 1 - \sqrt{\frac{n}{r}} + e^{\sqrt{\frac{n}{r}}},$$

$$e^{-1}\ell EE(G) - E_R(G) < n - 1 - \sqrt{\frac{n}{r}} + e^{\sqrt{\frac{n}{r}}}.$$

Theorem 4.5. *Let p, q and s be, respectively, the numbers of normalized Laplacian eigenvalues which are greater than, equal to, and less than* 1*. Then*

$$\ell EE(G) \geq e(q + pe^{\frac{E_\ell(G)}{2p}} + se^{-\frac{E_\ell(G)}{2s}}).$$

Proof. Let $\delta_1, \ldots, \delta_p$ be the normalized Laplacian eigenvalues of G greater than 1, and $\delta_{n-s+1}, \ldots, \delta_n$ be the normalized Laplacian eigenvalues less than 1. Since the sum of normalized Laplacian eigenvalues of a connected graph G is n and

$$E_\ell(G) = 2\sum_{i=1}^{p}(\delta_i - 1) = -2\sum_{i=n-s+1}^{n}(\delta_i - 1),$$

by the arithmetic-geometric mean inequality, we have:

$$\sum_{i=1}^{p} e^{\delta_i} \geq pe^{\frac{\delta_1+\cdots+\delta_p}{p}} = pe^{\frac{E_\ell(G)}{2p}+1} \; ; \quad \sum_{i=n-s+1}^{n} e^{\delta_i} \geq pe^{\frac{\delta_{n-s+1}+\cdots+\delta_n}{s}} = se^{-\frac{E_\ell(G)}{2s}+1}$$

and for eigenvalues equal to 1, $\sum_{i=p+1}^{n-s} e^{\delta_i} = qe$. Now, the result is obtained by combining these inequalities. □

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