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THEORETICAL DEPENDENCE CONCEPTS FOR STOCHASTIC PROCESSES†

SO-YOUN KIM AND JONG-IL BAEK*

Abstract. We in this paper obtained the theoretical results for multivariate stochastic processes which help us to extended negatively orthant dependent(ENOD) structures among hitting times of the processes. In addition, some applications are given to illustrate these concepts.

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1. Introduction

We first present some definitions in this section.

Definition 1.1 (Lehmann(1966)). Random variables X and Y are said to be negatively dependent(ND) if

$$P(X \le x,\ Y \le y) \le P(X \le x)P(Y \le y) \tag{1.1}$$

for all $x,\ y \in R$. A collection of random variables is said to be pairwise negatively dependent(PND) if every pair of random variables in the collection satisfies (1.1). It is important to note that (1.1) implies that

$$P(X > x,\ Y > y) \le P(X > x)P(Y > y) \tag{1.2}$$

for all $x,\ y \in R$. Moreover, it follows that (1.2) implies (1.1), and, hence, (1.1) and (1.2) are equivalent. However, (1.1) and (1.2) are not equivalent for a collection of 3 or more random variables. Consequently, the following definition is needed to define sequences of negatively dependent random variables.

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Definition 1.2 (Joag-Dev and Proschan(1983)). A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be negatively upper orthant dependent(NUOD) if for all real numbers $x_1, \cdots, x_n$,

$$P(X_1 > x_1, \cdots, X_n > x_n) \le \prod_{i=1}^{n} P(X_i > x_i) \tag{1.3}$$

and it is said to be negatively lower orthant dependent(NLOD) if for all real numbers $x_1, \cdots, x_n$,

$$P(X_1 \le x_1, \cdots, X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i) \tag{1.4}$$

A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be negatively orthant dependent(NOD) if it is both (1.3) and (1.4).

Definition 1.3 (Liu(2009)). A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be extended negatively upper orthant dependent(ENUOD) if for all real numbers $x_1, \cdots, x_n$, there exists a constant $M > 0$ such that

$$P(X_1 > x_1, \cdots, X_n > x_n) \le M \prod_{i=1}^{n} P(X_i > x_i) \tag{1.5}$$

and it is said to be extended negatively lower orthant dependent(ENLOD) if for all real numbers $x_1, \cdots, x_n$, there exists a constant $M > 0$ such that

$$P(X_1 \le x_1, \cdots, X_n \le x_n) \le M \prod_{i=1}^{n} P(X_i \le x_i) \tag{1.6}$$

A sequence $\{X_i, 1 \le i \le n\}$ of random variables is said to be extended negatively orthant dependent(ENOD) if it is both (1.5) and (1.6).

Lehmann(1966) introduced various concepts of dependence for two random variables. Esary and Proschan(1972) were later developed the stronger notio ns of bivariate dependence. Ahmed et al.(1978), Ebrahimi and Ghosh(1981), and Joag-Dev and Proschan(1983) obtained multivariate versions of various bivariate positive and negative dependence as described by Lehmann, and Esary and Proschan. In addition, for other related dependence concepts, many authors had been generalized and extended to several directions and their concept has been very useful in reliability theory and applications; (see Baek(1995), Barlow and Proschan(1975), Block et al.(1981, 1983, 1988), Brindly and Thompson(1972), Choi and Baek(2013), Glaz and Johnson(1982)). Recently, Liu(2009) introduced the concepts of extended negatively dependence in the multivariate case. Concepts of dependence have subsequently been extended to stochastic processes in different directions by many authors;(see Berman (1977), (1978) and Cox and Isham(1980), Ebrahimi(1987, 1994), Ebrahimi and Ramallingam(1988, 1989), Friday(1981), Marshall and Shaked(1983)).

Multivariate stochastic processes arise when instead of observing a single process we observe several processes, say $X_1(t), \cdots, X_n(t)$, simultaneously. For example, we may want to study the simultaneous variation of current and voltage, or temperature, pressure and volume over time and we also may be interested in studying inflation rates and money supply, unemployment and interest rates. We could, of course, study each quantity on its own and treat each as a separate univariate process. Although this would give us some information about each quantity it could never give information about the interrelationship between various quantities. This leads us to introduce some concepts of negative dependence for multivariate stochastic processes. To introduce the new ideas involved in the study of multivariate processes we consider the multivariate stochastic processes.

The main purpose of this paper introduce various concepts of extended negative dependence for multivariate stochastic processes, namely extended negatively orthant dependent(ENOD), negatively associated(NA), right corner set decreasing(RCSD), and right tail decreasing in sequence(RTDS). The theoretical results of these concepts are studied in Section 3. Some applicat ions of these concepts are developed in Section 4.

2. Preliminaries

In this section we present definitions and notations which will be used through out this paper. Suppose that $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are stochastic processes. The state space of $X_i(t)$ will be taken to be a subset E_i of real line $R_+ = [0, \infty]$, $i = 1, 2, \cdots, n$. For any state $a_i \in E_i, i = 1, 2, \cdots, n$, we now define the random times as follows,

$$S_i(a_i) = inf\{t|X_i(t) \geq a_i,\ 0 \leq t \leq \infty\},$$

that is, $S_i(a_i)$ is the first hitting times that the process $X_i(t)$ reaches a_i.

Definition 2.1. Stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are said to be extended negatively upper orthant dependent(ENUOD) if for all t_i and $a_i,\ i = 1, 2, \cdots, n$, there exists a constant $M > 0$ such that

$$P(S_1(a_1) > t_1, \cdots, S_n(a_n) > t_n) \leq MP(S_1(a_1) > t_1) \cdots P(S_n(a_n) > t_n) \quad (2.1)$$

and they are said to be extended negatively lower orthant dependent (ENLOD) if for all t_i and $a_i,\ i = 1, 2, \cdots, n$, there exists a constant $M > 0$ such that

$$P(S_1(a_1) \leq t_1, \cdots, S_n(a_n) \leq t_n) \leq MP(S_1(a_1) \leq t_1) \cdots P(S_n(a_n) \leq t_n) \quad (2.2)$$

Stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are said to be extended negatively orthant dependent(ENOD) if it is both ENUOD and ENLOD.
It is clear that stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are called NOD if (2.1) and (2.2) hold when $M = 1$. Obviously, an NOD stochastic processes must be an ENOD stochastic processes. Therefore, the ENOD structure is substantially more comprehensive than the NOD structure and ENOD structure can reflect not only a negative dependence structure but also positive one, to some extend.

Definition 2.2. Stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are said to be negatively associated if for all $a_i,\ i = 1, 2, \cdots, n$, and increasing functions f and g for which the covariance exists,

$$Cov(f(S_1(a_1), \cdots, S_n(a_n)),\ g(S_1(a_1), \cdots, S_n(a_n))) \leq 0.$$

Definition 2.3. Stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are said to be right corner set decreasing(RCSD) if for all a_i and $t_i,\ i = 1, \cdots, n$,

$$P(\bigcap_{i=1}^{n} S_i(a_i) > t_i | \bigcap_{i=1}^{n} S_i(a_i) > t'_i)$$

decreasing in $t'_1, \cdots, t'_n$ for every choice of $t_1, \cdots, t_n$.

Definition 2.4. Stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are said to be right tail decreasing in sequence(RTDS) if for all a_i and $t_i,\ i = 1, 2, \cdots, n$,

$$P(S_i(a_i) > t_i | S_1(a_1) > t_1, \cdots, S_{i-1}(a_{i-1}) > t_{i-1})$$

decreasing in $t_1, \cdots, t_{i-1}$.

3. Theoretical Results for Multivariate Stochastic Processes

Theorem 3.1. *Suppose that stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are RCSD. Then $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are RTDS.*

Proof. Let stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are RCSD. Then $P(\bigcap_{i=1}^n S_i(a_i) > t_i | \bigcap_{i=1}^n S_i(a_i) > t'_i)$ is decreasing in $t'_1, \cdots, t'_n$ for all choices of $t_1, \cdots, t_n$. Therefore, for fixed j,

$$\frac{P(S_j(a_j) > t_j, \bigcap_{i=1}^n S_i(a_i) > t'_i)}{P(\bigcap_{i=1}^n S_i(a_i) > t'_i)}$$

is decreasing in $t'_1, \cdots, t'_n$ for all choices of t_j. Now putting $t'_i \to 0$ for all $i = 1, \cdots, j-1$, we can obtain that

$$P(S_j(a_j) > t_j | S_{j-1}(a_{j-1}) > t'_{j-1}, \cdots, S_1(a_1) > t'_1)$$

is decreasing in $t'_1, \cdots, t'_{j-1}$ for all choices of t_j. Since j is arbitrary, $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are RTDS. $\square$

Next, we now show that RTDS implies ENOD.

Theorem 3.2. *Suppose that the stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)| t \geq 0\}$ are RTDS, then they are ENOD.*

Proof. We prove only this result for RTDS implies ENUOD.

$$\begin{aligned}
&P(S_1(a_1) > t_1, \cdots, S_n(a_n) > t_n) \\
&= P(S_1(a_1) > t_1 | S_2(a_2) > t_2, \cdots, S_n(a_n) > t_n) P(S_2(a_2) > t_2, \cdots, S_n(a_n) > t_n) \\
&\leq MP(S_1(a_1) > t_1) \prod_{i=2}^{n} P(S_i(a_i) > t_i,\ \bigcap_{j=1}^{n-1} S_j(a_j) > t_j) \text{ by RTDS}
\end{aligned}$$

$$= M \prod_{i=1}^{n} P(S_i(a_i) > t_i), \text{ by taking } t_j \to 0 (j = 1, \cdots, i-1).$$

The proof of the ENLOD is similar to that of ENUOD. □

Theorem 3.3. *Suppose that stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are ENOD and let $f_1, \cdots, f_n$ are nonnegative increasing functions. Let $Y_1,\ Y_2, \cdots, Y_n$ be independent of $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ and $P(S_1(a_1) > t_1), \cdots, P(S_n(a_n) > t_n)$ be increasing or decreasing in both $a_1, a_2, \cdots, a_n$ and $t_1, t_2, \cdots, t_n$, then $\{f_1(X_1(t)) + Y_1|t \geq 0\}, \cdots \{f_n(X_n(t)) + Y_n|t \geq 0\}$ are ENOD.*

Proof. We prove only this result for ENUOD. Let $S_i(a_i) = inf\{s|f_i(X_i(s)) + Y_i \geq a_i\}$ and $T_i(b_i){=}inf\{t|X_i(t) \geq b_i\},\ i = 1, \cdots, n$. Then
$P(S_1(a_1) > t_1, \cdots, S_n(a_n) > t_n)$
$= P((inf\{s|f_1(X_1(s)) + Y_1 \geq a_1\}) > t_1, \cdots, (inf\{s|f_n(X_n(s)) + Y_n \geq a_n\}) > t_n)$
$= P((inf\{s|X_1(s) \geq f_1^{-1}(a_1 - y_1)\}) > t_1, \cdots, (inf\{s|X_n(s) \geq f_n^{-1}(a_n - y_n)\}) > t_n)$
$= P(T_1(f_1^{-1}(a_1 - y_1)) > t_1, \cdots, T_n(f_n^{-1}(a_n - y_n)) > t_n)$
$\leq M \prod_{i=1}^{n} P(T_i(f_i^{-1}(a_i - y_i)) > t_i) = M \prod_{i=1}^{n} (S_i(a_i) > t)$ for every t_i's and a_i's.
The proof of ENLOD is similar to that proof of ENUOD. □

Remark 3.1. Theorem 3.3 can be proved for NA(RCSD(RTDS)) concepts of dependence.

Theorem 3.4. *Suppose that stochastic processes (a) $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are ENOD, (b) $\{X_1(t)|t \geq 0\}, \cdots, \{X_n(t)|t \geq 0\}$ are multivariate stochastic processes with state spaces $\{a_1, b_1\}, \{a_2, b_2\}, \cdots, \{a_n, b_n\}$, respectively, and (c) $max(a_1, b_1), max(a_2, b_2), \cdots, max(a_n, b_n)$ are absorbing states. Then for all $a_1, a_2, \cdots, a_n$ and $t_1, t_2, \cdots, t_n$,*

$$P(X_1(t_1) > a_1, X_2(t_2) > a_2, \cdots, X_n(t_n) > a_n)$$
$$\leq MP(X_1(t_1) > a_1)P(X_2(t_2) > a_2) \cdots P(X_n(t_n) > a_n).$$

Proof. We prove only this result for ENUOD. Without loss of generality assume that $a_1 < b_1, a_2 < b_2, \cdots, a_n < b_n$. Now,

$$\begin{aligned}
&P(X_1(t_1) > a_1, X_2(t_2) > a_2, \cdots, X_n(t_n) > a_n)\\
&= P(X_1(t_1) = b_1,\ X_2(t_2) = b_2,\ \cdots, X_n(t_n) = b_n)\\
&\leq MP(S_1(b_1) \leq t_1,\ S_2(b_2) \leq t_2, \cdots S_n(b_n) \leq t_n)\\
&\leq MP(S_1(b_1) \leq t_1)P(S_2(b_2) \leq t_2) \cdots P(S_n(b_n) \leq t_n)\\
&= MP(X_1(t_1) = b_1)P(X_2(t_2) = b_2) \cdots P(X_n(t_n) = b_n)\\
&= MP(X_1(t_1) > a_1)P(X_2(t_2) > a_2) \cdots P(X_n(t_n) > a_n).
\end{aligned}$$

□

Next, we prove that the next limit theorem demonstrates preservation of the ENOD among the hitting times.

Theorem 3.5. *Let $\{X_{1n}(t)|t \geq 0\}, \cdots, \{X_{pn}(t)|t \geq 0\}$ be RTDS stochastic processes with distribution functions H_n such that H_n weakly converges to H as $n \to \infty$ where H is the distribution functions of stochastic processes $\{X_1(t)|t \geq 0\}, \cdots, \{X_p(t)|t \geq 0\}$. Then $\{X_1(t)|t \geq 0\}, \cdots, \{X_p(t)|t \geq 0\}$ are ENOD.*

Proof. We prove only this result for ENUOD.
$P(S_1(a_1) > t_1, \cdots, S_p(a_p) > t_p)$
$= lim_{n\to\infty}(P(S_{1n}(a_{1n}) > t_{1n}|S_{2n}(a_{2n}) > t_{2n}, \cdots, S_{pn}(a_{pn}) > t_{pn}))$
$\cdot (P(S_{2n}(a_{2n}) > t_{2n}, \cdots, S_{pn}(a_{pn}) > t_{pn}))$
$\leq MP(S_1(a_1) > t_1)\ \lim_{n\to\infty} \prod_{i=2}^{p} P(S_{in}(a_{in}) > t_{in}, \bigcap_{j=1}^{p-1} S_{jn}(a_{jn}) > t_{jn})$ by RTDS
$= MP(S_1(a_1) > t_1) \prod_{i=2}^{p} \lim_{n\to\infty} P(S_{in}(a_{in}) > t_{in})$ by taking $t_j \to\ (0 \leq j \leq i-1)$
$= M \prod_{i=1}^{p} P(S_i(a_i) > t_i)$.
The proof of the ENLOD is similar to that of ENUOD. □

Remark 3.2. If we change RTDS to ENOD, we can get results of ENOD.

Theorem 3.6. *Let $Y_1(t) = \sum_{j=1}^{N(t)} X_{1j}, \cdots, Y_k(t) = \sum_{j=1}^{N(t)} X_{kj}$ be processes and let $\{(X_{1n}, \cdots, X_{kn}|n \geq 1\}$ be a k-variate stochastic processes such that (a) $(X_{11}, \cdots, X_{k1}, (X_{12}, \cdots, X_{k2}), \cdots$ are independent and (b) $X_{1i}, \cdots, X_{ki}$ are ENOD, $i = 1, 2, \cdots,$ (c) $N(t)$ is a Poisson process which is independent of $X_{1i}'s, X_{2i}'s, \cdots, X_{ki}'s,\ i = 1, 2, \cdots$. Then $\{Y_1(t)|t \geq 0\}, \cdots \{Y_k(t)|t \geq 0\}$ are ENOD.*

Proof. We prove only this result for ENLOD.
$P(S_1(a_1) \leq t_1, \cdots, S_k(a_k) \leq t_k)$
$= P(\{\sum_{j=1}^{N(s)} X_{1j} \geq a_1, t_1 \leq s < \infty\}, \cdots \{\sum_{j=1}^{N(s)} X_{kj} \geq a_k, t \leq s < \infty\})$
$= P((\sum_{j=1}^{N(t_1)} X_{1j} \geq a_1), \cdots, (\sum_{j=1}^{N(t_k)} X_{kj} \geq a_k)$
$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (P(\sum_{j=1}^{k_1} X_{1j} \geq a_1, \cdots, \sum_{j=1}^{k_n} X_{kj} \geq a_k|N(t_1) = k_1, \cdots,$
$N(t_k) = k_n)) \cdot (P(N(t_1) = k_1, \cdots, N(t_k) = k_n))$
$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (P(\sum_{j=1}^{k_1} X_{1j} \geq a_1, \cdots, \sum_{j=1}^{k_n} X_{kj} \geq a_n))\cdot$
$(P(N(t_1) = k_1, \cdots, N(t_k) = k_n))$ by (c)
$\leq M \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (P(\sum_{j=1}^{k_1} X_{1j} \geq a_1) P(\sum_{j=1}^{k_n} X_{kj} \geq a_n) P(N(t_1) = k_1, \cdots,$
$N(t_k) = k_n))$ by (a) and (b)
$= M(\sum_{k_1=1}^{\infty} P(\sum_{j=1}^{k_1} X_{1j} \geq a_1|(N(t_1) = k_1)) P(N(t_1 = k_1))) \cdots$
$(\sum_{k_n=0}^{\infty} P(\sum_{j=1}^{k_n} X_{kj} \geq a_n|(N(t_k) = k_n) P(N(t_k) = k_n)))$
$= MP(\{\sum_{j=1}^{N(s)} X_{1j} \geq a_1,\ t_1 \leq s < \infty\}) \cdots P(\{\sum_{j=1}^{N(s)} X_{kj} \geq a_k,\ t_k \leq s < \infty\})$
$= MP(S_1(a_1) \leq t_1) \cdots P(S_k(a_k) \leq t_k)$.
The proof of the ENUOD is similar to that of ENLOD. □

Remark 3.3. Let (a) and (c) hold, and if $X_{1i}, \cdots, X_{ki}$ are NA(RCSD(RTDS)), $i = 1, 2, \cdots$, then we can get that $Y_1(t) = \sum_{j=1}^{N(t)} X_{1j}, \cdots, Y_k(t) = \sum_{j=1}^{N(t)} X_{kj}$ are NA(RCSD(RTDS)).

Theorem 3.7. *Let $\{(X_{1n}, X_{2n}, X_{3n}, \cdots, X_{kn})|n \geq 1\}$ be stochastic processes such that $(X_{11}, X_{21}, X_{31}, \cdots, X_{k1})$, $(X_{12}, X_{22}, X_{32}, \cdots, X_{k2}), \cdots$ are independent and $X_{1i}, X_{2i}, X_{3i}, \cdots, X_{ki}, i = 1, 2, \cdots$ are ENOD. Then $\{(X_{1n}, X_{2n}, X_{3n}, \cdots, X_{kn})|n \geq 1\}$ is ENOD.*

Proof. Let $P(T_1(x_1) > n_1, T_2(x_2) > n_2, \cdots, T_n(x_n) > n_k)$

$$= P(X_{11} > x_1, X_{12} > x_1, \cdots, X_{1n} > x_1, X_{21} > x_2, X_{22} > x_2, \cdots, X_{2n} > x_2, \\ \cdots, X_{kn_1} > x_1, X_{kn_2} > x_2, \cdots, X_{kn_k} > x_n).$$

□

Now consider the following cases.

Case 1. When $n_1 = n_2 = n_3 = \cdots = n_k$,

$$P(T_1(x_1) > n_1, T_2(x_2) > n_2, \cdots, T_n(x_n) > n_k)$$
$$= P(X_{11} > x_1, X_{12} > x_1, \cdots, X_{1n_1} > x_1, X_{21} > x_2, X_{22} > x_2, \cdots, \\ X_{2n_2} > x_2, \cdots, X_{kn_1} > x_n, X_{kn_2} > x_n, \cdots, X_{kn_k} > x_n)$$
$$= P(X_{11} > x_1, X_{12} > x_1, X_{13} > x_1, \cdots, X_{kn_1} > x_n)P(X_12 > x_1, \\ X_{22} > x_2, \cdots, X_{kn_2} > x_n) \cdots P(X_{1n_1} > x_1, X_{2n_2} > x_2, \cdots, X_{kn_k} > x_n)$$
$$\leq MP(X_{11} > x_1, X_{12} > x_2, X_{13} > x_1, \cdots, X_{1n_1} > x_1) \cdot P(X_{21} > x_2, \\ X_{22} > x_2, \cdots, X_{2n_2} > x_2) \cdots P(X_{1n_1} > x_1, X_{2n_2} > x_2, \cdots, X_{kn_k} > x_n)$$
$$= MP(T_1(x_1) > n_1)P(T_2(x_2) > n_2) \cdots P(T_n(x_n) > n_k).$$

Case 2. When $n_1 < n_2 < n_3 < \cdots < n_k$,

$$P(T_1(x_1) > n_1, T_2(x_2) > n_2, \cdots, T_n(x_n) > n_k)$$
$$P(X_{11} > x_1, X_{12} > x_2, \cdots, X_{1n_1} > x_1, X_{21} > x_2, X_{22} > x_2, \cdots, X_{2n_1} > x_2, \\ X_{2n_1+1} > x_2, \cdots, X_{2n_2} > x_2, \cdots, X_{kn_1} > x_n, X_{kn_2} > x_n, \cdots, X_{kn+1+1} > x_n, \\ \cdots, X_{kn_k} > x_n)$$
$$\leq MP(X_{11} > x_1, X_{21} > x_2, \cdots, X_{kn_1} > x_n)P(X_{2n_1+1} > x_2, X_{2n_1+2} > x_2, \cdots, \\ X_{kn_1+1} > x_n, \cdots, X_{kn_k} > x_n)$$
$$= MP(T_1(x_1) > n_1)P(T_2(x_2) > n_2) \cdots P(T_n(x_n) > n_k).$$

Case 3. The proof of $n_1 > n_2 > n_3 > \cdots > n_k$ is similar to Case 2.

4. Applications

Application 4.1. We consider the uniformly modulated model(see Priestly (1988)) such that non-stationary processes $X(t) = \{X_1(t), X_2(t), \cdots, X_n(t)|t \geq 0\}$ and $Y(t) = \{Y_1(t), Y_2(t), \cdots, Y_n(t)|t \geq 0\}$ given by

$$X(t) = a(t)Y(t), \ t \geq 0,$$

where $a(t)$ is a deterministic continuous function such that $a(t) \geq 0$ and $Y(t)$ is non-negative stationary process. If $Y(t)$ is ENOD, we can show that $X(t)$ is ENOD. Using the Theorem 3.3, if $f_i,\ i = 1, \cdots, n$ are increasing functions, we can obtain that $f_1(S_1(a_1)), \cdots, f_n(S_n(a_n))$ are ENOD, where $S_i(a_i) = inf\{n|X_i(n) \geq a_i\},\ i = 1, \cdots, n.$

Application 4.2. Suppose that we are given a system with n components which is subjected to shocks and assume that $N(t)$ be the number of shocks received by time t and let $Y_1(t) = \sum_{i=1}^{N(t)} X_{1i} + Z_{1i},\ Y_2(t) = \sum_{i=1}^{N(t)} X_{2i} + Z_{2i}, \cdots, Y_n(t) = \sum_{i=1}^{N(t)} X_{ni} + Z_{ni}$ be total damages to components $1, 2, \cdots, n$ by time t, respectively, where $X_{1i}, X_{2i}, \cdots, X_{ni}$ and $Z_{1i}, Z_{2i}, \cdots, Z_{ni}$ are damages to components $1, 2, \cdots$, and n by shocks, respectively. Let $(X_{1i}, \cdots, X_{ni})$ and $(Z_{1i}, \cdots, Z_{ni})$ be ENOD respectively, $i = 1, 2, \cdots$, and let $(X_11, \cdots, X_{n1}),\ (X_{12},$ $\cdots, X_{n2}) \cdots$ and $(Z_{11}, \cdots, Z_{n1}),\ (Z_{12}, \cdots, Z_{n2}) \cdots$ are independent and have increasing paths. Then by Theorem 3.6, we can get that $\{Y_1(t)|t \geq 0\}, \cdots \{Y_n(t)|t$ $\geq 0\}$ are ENOD.

Application 4.3. Let $Z_i(t),\ i = 1, 2, \cdots, n$ be the strength of system i at time t and let $D_i, i = 1, 2, \cdots, n$ be i.i.d. positive random variables denoting the damage to either system due to the ith shock and $N(t)$ be the number of shocks occurring by time t. Then the stress experienced by either system at time t is given by the process $X(t) = \sum_{i=1}^{N(t)} D_i$. Suppose that $Z_1(t), \cdots, Z_n(t)$ are independent processes with decreasing sample paths and that the processes $X(t), Z_1(t), \cdots, Z_n(t)$ are independent, then we can obtain using Theorem 3.6 and Application 4.2 that $X(t)-Z_1(t),\ X(t)-Z_2(t), \cdots, X(t)-Z_n(t)$ are ENOD processes. Thus the lifetimes of the multivariate systems, namely, $S_i(a_i) = inf\{t|X(t)-Z_i(t) \geq a_i\},\ i = 1, 2, \cdots, n$ are ENOD random variables. Therefore, we can get that

$$P(S_1(a_i) > t_1, \cdots, S_n(a_i) > t_n) \leq M \prod_{i=1}^{n} P(S_i(a_i) > t_i)$$

and

$$P(S_1(a_i) \leq t_1, \cdots, S_n(a_i) \leq t_n) \leq M \prod_{i=1}^{n} P(S_i(a_i) \leq t_i)$$

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Kim So Youn is working at School of Mathematrics and Informational statistics. Their research interests focus on the probability theory, reliability theory, and time series.

School of mathematics and Informational statistics and Institute of Basic Natural Science, Wonkwang University, Iksan 570-749, Korea.

Baek Jong Il is working at School of Mathematrics and Informational statistics. Their research interests focus on the probability theory, reliability theory, and time series.

School of mathematics and Informational statistics and Institute of Basic Natural Science, Wonkwang University, Iksan 570-749, Korea.
e-mail: jibaek@wku.ac.kr