J. Appl. Math. & Informatics Vol. **32***(2014), No. 1 - 2, pp. 137 - 152*
http://dx.doi.org/10.14317/jami.2014.137

INFINITELY MANY SMALL SOLUTIONS FOR THE $p(x)$-LAPLACIAN OPERATOR WITH CRITICAL GROWTH†

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ABSTRACT. In this paper, we prove, in the spirit of [3, 12, 20, 22, 23], the existence of infinitely many small solutions to the following quasilinear elliptic equation $-\Delta_{p(x)}u+|u|^{p(x)-2}u = |u|^{q(x)-2}u+\lambda f(x,u)$ in a smooth bounded domain Ω of $\mathbb{R}^N$. We also assume that $\{q(x) = p^*(x)\} \neq \emptyset$, where $p^*(x) = Np(x)/(N - p(x))$ is the critical Sobolev exponent for variable exponents. The proof is based on a new version of the symmetric mountain-pass lemma due to Kajikiya [22], and property of these solutions are also obtained.

AMS Mathematics Subject Classification : 35J60, 35B33.
Key words and phrases : $p(x)$-Laplacian, Generalized Lebesgue-Sobolev spaces, Symmetric mountain-pass lemma, Concentration-compactness principle.

1. Introduction

In this paper we deal with quasilinear elliptic problem of the form

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = |u|^{q(x)-2}u + \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary and $p(x), q(x)$ are two continuous functions on $\overline{\Omega}$, $1 < p(x) \ll q(x) < N$, where denote by $p(x) \ll q(x)$ the fact that $\inf_{x\in\Omega}(q(x) - p(x)) > 0$. λ is a positive parameter, $\Delta_{p(x)}u := \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacia operator. On the exponent $q(x)$ we assume that is the critical exponent in the sense that

Received April 3, 2013. Revised June 21, 2013. Accepted June 26, 2013. *Corresponding author. †This work was supported by Research Foundation for Chunmiao project of Department of Education of Jilin Province, China (Ji Jiao Ke He Zi, Grant [2013] No. 252), Foundation for China Postdoctoral Science Foundation (Grant no. 2012M520665), Youth Foundation for Science and Technology Department of Jilin Province (2013), and Natural Science Foundation of Changchun Normal University (2011. No.2).

$\{q(x) = p^*(x)\} \neq \emptyset$, where $p^*(x) = Np(x)/(N - p(x))$ is the critical exponent according to the Sobolev embedding. Our goal will be to obtain infinitely many small weak solutions which tend to zero for (1) in the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$ for the general nonlinearities of the type $f(x, u)$.

The study of differential equations and variational problems involving variable exponent conditions has been a very interesting and important topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, image processing and so on. For example, Chen, Levin and Rao [4] proposed the following model in image processing

$$F(u) = \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)} + f(|u(x) - I(x)|)dx \to \min,$$

where $p(x)$ is a function satisfies $1 \leq p(x) \leq 2$ and f is a convex function. For more information on modelling physical phenomena by equations involving $p(x)$-growth condition we refer to [1, 19, 28, 30]. The appearance of such physical models was facilitated by the development of variable Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. On the variable exponent Sobolev spaces which have been used to study $p(x)$-Laplacian problems, we refer to [5, 21, 29]. On the existence of solutions for elliptic equations with variable exponent, we refer to [2, 6, 7, 8, 9, 10, 11, 16, 17, 31].

In recent years, the existence of infinitely many solutions have been obtained by many papers. When $p(x) \equiv p = 2$ (a constant) with Dirichlet boundary condition, Li and Zou [23] studied a class of elliptic problems with critical exponents, they obtained the existence theorem of infinitely many solutions under suitable hypotheses. He and Zou [20] proved that the existence infinitely many solutions under case the general nonlinearities. When $p(x) \equiv p \neq 2$. Ghoussoub and Yuan [18] obtained the existence of infinitely many nontrivial solutions for Hardy-Sobolev subcritical case and Hardy critical case by establishing Palais-Smale type conditions around appropriate chosen dual sets in bounded domain. Li and Zhang [24] studied the existence of multiple solutions for the nonlinear elliptic problems of $p\&q$-Laplacian type involving the critical Sobolev exponent, they obtained infinitely many weak solutions by using Lusternik-Schnirelman's theory for Z_2-invariant functional.

On the existence of infinitely many solutions for $p(x)$-Laplacian problems have been studied by [2, 7, 9, 31], but they did not give any further information on the sequence of solutions. Moreover, these papers deal with subcritical nonlinearities. Very little is known about critical growth nonlinearities for variable exponent problems [14, 15], since one of the main techniques used in order to deal with such issues is the concentration-compactness principle. This result was recently obtained for the variable exponent case independently in [12, 13]. In both of these papers the proof are similar and both relates to that of the original proof of P.L. Lions [25, 26].

Recently, Kajikiya [22] established a critical point theorem related to the symmetric mountain pass lemma and applied to a sublinear elliptic equation. But there are no such results on $p(x)$-Laplacian problem with critical growth (1).

Motivated by reasons above, the aim of this paper is to show that the existence of infinitely many solutions of problem (1), and there exists a sequence of infinitely many arbitrarily small solutions converging to zero by using a new version of the symmetric mountain-pass lemma due to Kajikiya [22]. In order to use the symmetric mountain-pass lemma, there are many difficulties. The main one in solving the problem is a lack of compactness which can be illustrated by the fact that the embedding of $W^{1,p(x)}(\Omega)$ into $L^{p^*(x)}(\Omega)$ is no longer compact. Hence the concentration-compactness principle is used here to overcome the difficulty. The main result of this paper is as follows.

Theorem 1.1. *Suppose that $f(x,u)$ satisfies the following conditions:*

(H_1) $f(x,u) \in C(\Omega \times R, R)$, $f(x,-u) = -f(x,u)$ *for all* $u \in R$;

(H_2) $\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^{q(x)-1}} = 0$ *uniformly for* $x \in \Omega$;

(H_3) $\lim_{|u|\to 0^+} \frac{f(x,u)}{u^{p^- -1}} = \infty$ *uniformly for* $x \in \Omega$.

Then there exists λ^* such that for any $\lambda \in (0,\lambda^*)$, problem (1) *has a sequence of non-trivial solutions $\{u_n\}$ and $u_n \to 0$ as $n \to \infty$.*

Remark 1.1. If without the symmetry condition (i.e. $f(x,-u) = -f(x,u)$) in Theorem 1.1, we get an existence theorem of at least one nontrivial solution to problem (1) by the same method in this paper.

Remark 1.2. In this paper, we use concentration-compactness principle due to [12] which is slightly more general than those in [13], since we do not require $q(x)$ to be critical everywhere.

Remark 1.3. There exist many functions $f(x,t)$ satisfy condition (H_1)-(H_3), for example, $f(x,u) = u^{(p^- -1)/3}$, where $p^- > 1$.

Remark 1.4. Theorem 1.1 is new as far as we know and it generalizes results in [3] for $p(x)$-Laplacian type problem. We mainly follow the way in [3] to prove our main result.

Definition 1.2. We say that $u_0 \in W_0^{1,p(x)}(\Omega)$ is a weak solution of problem (1) if for any $v \in W_0^{1,p(x)}(\Omega)$

$$\int_\Omega \left(|\nabla u_0|^{p(x)-2}\nabla u_0 \cdot \nabla v + |u_0|^{p(x)-2}u_0 v\right) dx - \int_\Omega |u_0|^{q(x)-2}u_0 v dx$$
$$- \lambda \int_\Omega f(x,u_0) v dx = 0.$$

The energy functional corresponding to problem (1) is defined as follows,

$$J(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \int_\Omega \frac{1}{q(x)}|u|^{q(x)} dx - \lambda \int_\Omega F(x,u) dx,$$

then, it is easy to check that as arguments [27] show that $J(u)$ is well defined on $W_0^{1,p(x)}(\Omega)$ and $J \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and the weak solutions for problem

(1) coincides with the critical points of J. We try to use a new version of the symmetric mountain-pass lemma due to Kajikiya [22]. But since the functional $J(u)$ is not bounded from below, we could not use the theory directly. So we follow [3] to consider a truncated functional of $J(u)$. Denote $J' : E \to E^*$ is the derivative operator of J in the weak sense. Then

$$\begin{aligned}\langle J'(u), v\rangle \quad = \quad & \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v + |u|^{p(x)-2}uv\Big)\,dx - \int_\Omega |u|^{q(x)-2}uvdx \\ & - \lambda\int_\Omega f(x,u)vdx, \quad \forall\ u, v\in W_0^{1,p(x)}(\Omega).\end{aligned}$$

Definition 1.3. We say J satisfies Palais-Smale condition ((PS) for short) in $W_0^{1,p(x)}(\Omega)$, if any sequence $\{u_n\}\subset W_0^{1,p(x)}(\Omega)$ which satisfies that $\{J(u_n)\}$ is bounded and $\|J'(u_n)\|_{p(x)}\to 0$ as $n\to\infty$, has a convergent subsequence.

Under assumption (H_2), we have

$$f(x,u)u = o\left(|u|^{q(x)}\right), \qquad F(x,u) = o\left(|u|^{q(x)}\right),$$

which means that, for all $\varepsilon > 0$, there exist $a(\varepsilon),\ b(\varepsilon) > 0$ such that

$$|f(x,u)u| \le a(\varepsilon) + \varepsilon|u|^{q(x)}, \tag{2}$$

$$|F(x,u)| \le b(\varepsilon) + \varepsilon|u|^{q(x)}. \tag{3}$$

Hence, for any constants β we have

$$|F(x,u) - \beta f(x,u)u| \le c(\varepsilon) + \varepsilon|u|^{q(x)}, \tag{4}$$

for some $c(\varepsilon) > 0$.

The remainder of the paper is organized as follows. In Section 2, we shall present some basic properties of the variable exponent Sobolev spaces. In Section 3, we will prove the corresponding energy functional satisfies the (PS) condition. In Section 4, we shall prove our main results.

2. Weighted variable exponent Lebesgue and Sobolev spaces

We recall some definitions and properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Set

$$C_+(\overline{\Omega}) = \{h\in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h\in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \quad \text{and} \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

We can introduce the variable exponent Lebesgue space as follows:

$$\begin{aligned}L^{p(\cdot)}(\Omega) = \big\{u : u \text{ is a measurable real-valued function} \\ \text{such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\},\end{aligned}$$

for $p \in C_+(\overline{\Omega})$. Equipping with the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(\cdot)} = \inf\big\{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\,dx \le 1\big\},$$

which is a Banach space, we call it a generalized Lebesgue space.

Proposition 2.1 ([5, 11])**.** (i) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{q(x)}(\Omega)$, where $1/q(x) + 1/p(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$|\int_\Omega uv\,dx| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(\cdot)}|v|_{q(\cdot)}; \tag{5}$$

(ii) *If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous.*

Proposition 2.2 ([5, 11])**.** *The mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by*

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

Then the following relations hold:

$$\begin{gathered}
|u|_{p(\cdot)} < 1\ (= 1;\ > 1) \ \Leftrightarrow\ \rho_{p(\cdot)}(u) < 1\ (= 1;\ > 1),\\
|u|_{p(\cdot)} > 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+},\\
|u|_{p(\cdot)} < 1 \ \Rightarrow\ |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-},\\
|u_n - u|_{p(\cdot)} \to 0 \ \Leftrightarrow\ \rho_{p(\cdot)}(u_n - u) \to 0.
\end{gathered}$$

Next, we define $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall\ u \in W^{1,p(x)}(\Omega).$$

Denote $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_1 = |\nabla u|_{p(x)}.$$

We know that if $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\|u\|$ and $\|u\|_1$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Proposition 2.3 ([5, 11])**.** (i) $W^{1,p(x)}(\Omega)$ *are separable reflexive Banach spaces;* (ii) *If $p \in C_+(\overline{\Omega})$ and $p(x) \le q(x) \le p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous.*

In this paper, we use the following equivalent norm on $W^{1,p(x)}(\Omega)$:

$$\|u\|_{p(\cdot)} = \inf\big\{\mu > 0 : \int_\Omega \Big|\frac{\nabla u}{\mu}\Big|^{p(x)} + \Big|\frac{u}{\mu}\Big|^{p(x)}\,dx \le 1\big\}. \tag{6}$$

Proposition 2.4 ([21, 6]). *Let $I(u) = \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)}dx$. If $u, u_n \in W^{1,p(x)}(\Omega)$, then the following relations hold:*

$$\|u\|_{p(\cdot)} < 1 \ (= 1; > 1) \ \Leftrightarrow \ I(u) < 1 \ (= 1; > 1), \tag{7}$$

$$\|u\|_{p(\cdot)} > 1 \ \Rightarrow \ \|u\|_{p(\cdot)}^{p^-} \le I(u) \le \|u\|_{p(\cdot)}^{p^+}, \tag{8}$$

$$\|u\|_{p(\cdot)} < 1 \ \Rightarrow \ \|u\|_{p(\cdot)}^{p^+} \le I(u) \le \|u\|_{p(\cdot)}^{p^-}, \tag{9}$$

$$\|u_n - u\|_{p(\cdot)} \to 0 \ \Leftrightarrow \ I(u_n - u) \to 0. \tag{10}$$

3. Preliminaries and lemmas

In the following, we always use C and $c_i (i = 1, 2, \cdots)$ to denote positive constants. We give the concentration-compactness principle of the variable exponent due to [12, 15].

Lemma 3.1. *Let $q(x)$ and $p(x)$ be two continuous functions such that*

$$1 < \inf_{x\in\Omega} p(x) \le \sup_{x\in\Omega} p(x) < N \quad and \quad 1 \le q(x) \le p^*(x) \quad in\ \Omega.$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a weakly convergent sequence in $W_0^{1,p(x)}(\Omega)$ with weak limit u, and such that $|\nabla u_j|^{p(x)} \rightharpoonup \mu$ weakly-* in the sense of measures; $|u_j|^{q(x)} \rightharpoonup \nu$ weakly-* in the sense of measures. Assume, moreover that $\Gamma = \{x \in \Omega : q(x) = p^*(x)\} \neq \emptyset$. Then, for some countable index set I we have
(i) $\nu = |u|^{q(x)} + \Sigma_{i\in I}\nu_i\delta_{x_i}, \quad \nu_i > 0$;
(ii) $\mu \ge |\nabla u|^{p(x)} + \Sigma_{i\in I}\mu_i\delta_{x_i}, \quad \mu_i > 0$;
(iii) $S\nu_i^{1/p^*(x_i)} \le \mu_i^{1/p(x_i)}, \quad \forall\, i \in I$;
where $\{x_i\}_{i\in I} \subset \Gamma$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = S_q(\Omega) := \inf_{\phi\in C_0^\infty(\Omega)} \frac{\||\nabla\phi|\|_{L^{p(x)}(\Omega)}}{\|\phi\|_{L^{q(x)}(\Omega)}}.$$

In order to prove the functional J satisfies the local $(PS)_c$ condition, we take continuous function $\eta(x)$ satisfies $p(x) \ll \eta(x) \ll q(x), \forall\, x \in \overline{\Omega}$. Denote

$$d_1 := \inf_{x\in\overline{\Omega}} \left(\frac{1}{p(x)} - \frac{1}{\eta(x)}\right) > 0, \tag{11}$$

$$d_2 := \inf_{x\in\overline{\Omega}} \left(\frac{1}{\eta(x)} - \frac{1}{q(x)}\right) > 0. \tag{12}$$

Lemma 3.2. *Assume condition (H_2) holds. Then for any $\lambda > 0$, there exists positive constant $m^* > 0$ such that the functional J satisfies the local $(PS)_c$ condition in*

$$c \in \left(-\infty, \ \frac{d_2}{4} \cdot S^N - m^*\right)$$

in the following sense: if

$$J(u_n) \to c < \frac{d_2}{4} \cdot S^N - m^*$$

and $J'(u_n) \to 0$ *for some sequence in* $W_0^{1,p(x)}(\Omega)$. *Then* $\{u_n\}$ *contains a subsequence converging strongly in* $W_0^{1,p(x)}(\Omega)$.

Proof. First, we show that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. If $\|u_n\|_{p(x)} \to \infty$ as $n \to \infty$. Thus, we may assume that $\|u_n\|_{p(x)} > 1$ for any integer n.

Then for n sufficiently large, we have

$$\begin{aligned}
&M + o(1)\|u_n\|_{p(x)} \\
&\geq J(u_n) - \langle J'(u_n), \frac{u_n}{\eta}\rangle \\
&= \int_\Omega \Big(\frac{1}{p(x)} - \frac{1}{\eta(x)}\Big) \cdot \Big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big)\, dx + \int_\Omega \Big(\frac{1}{\eta(x)} - \frac{1}{q(x)}\Big) \cdot |u_n|^{q(x)} dx \\
&\quad - \lambda \int_\Omega \Big[F(x,u_n) - \frac{1}{\eta(x)} f(x,u_n)u_n\Big]\, dx + \int_\Omega \frac{|\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\eta}{\eta^2(x)} dx \\
&\geq d_1 \cdot \int_\Omega \Big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big)\, dx + d_2 \cdot \int_\Omega |u_n|^{q(x)} dx \\
&\quad - \lambda \int_\Omega \Big[F(x,u_n) - \frac{1}{\eta(x)} f(x,u_n)u_n\Big]\, dx + \int_\Omega \frac{|\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\eta}{\eta^2(x)} dx. \qquad (13)
\end{aligned}$$

By (4), for any $(x,t) \in \Omega \times \mathbb{R}$, we have

$$\begin{aligned}
&\int_\Omega \Big[F(x,u_n) - \frac{1}{\eta(x)} f(x,u_n)u_n\Big]\, dx \\
&\leq \int_\Omega \Big|F(x,u_n) - \frac{1}{\eta(x)} f(x,u_n)u_n\Big|\, dx \\
&\leq \int_\Omega \max\Big\{\Big|F(x,u_n) - \frac{1}{\eta^+} f(x,u_n)u_n\Big|, \Big|F(x,u_n) - \frac{1}{\eta^-} f(x,u_n)u_n\Big|\Big\}\, dx \\
&\leq c(\varepsilon_1)|\Omega| + \varepsilon_1 \int_\Omega |u_n|^{q(x)} dx. \qquad (14)
\end{aligned}$$

On the other hand, noting that $p(x) \ll q(x)$, by the Young inequality, for any $\varepsilon_2, \varepsilon_3 \in (0,1)$, we get

$$\begin{aligned}
\left|\frac{|\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\eta}{\eta^2(x)}\right| &\leq c_1 |\nabla u_n|^{p(x)-1}|u_n| \\
&\leq c_1 \left(\frac{\varepsilon_2(p(x)-1)}{p(x)}|\nabla u_n|^{p(x)} + \frac{\varepsilon_2^{1-p(x)}}{p(x)}|u_n|^{p(x)}\right) \\
&\leq c_1 \left(\varepsilon_2|\nabla u_n|^{p(x)} + \varepsilon_2^{1-p^+}|u_n|^{p(x)}\right) \qquad (15)
\end{aligned}$$

and

$$
\begin{aligned}
|u_n|^{p(x)} &\le \frac{\varepsilon_3 p(x)}{q(x)}|u_n|^{q(x)} + \frac{q(x)-p(x)}{q(x)}\varepsilon_3^{\frac{p(x)}{p(x)-q(x)}} \\
&\le \varepsilon_3|u_n|^{q(x)} + \varepsilon_3^{-\frac{p^+}{(q-p)^-}}. \qquad (16)
\end{aligned}
$$

Thus, relations (13)-(16) imply that

$$
\begin{aligned}
&M + o(1)\|u_n\|_{p(x)} \\
&\ge (d_1 - c_1\varepsilon_2)\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx + \left(d_2 - \lambda\varepsilon_1 - c_1\varepsilon_2^{1-p^+}\varepsilon_3\right)\int_\Omega |u_n|^{q(x)}dx \\
&\quad - c_1\varepsilon_2^{1-p^+}\varepsilon_3^{-\frac{p^+}{(q-p)^-}} - \lambda c(\varepsilon_1)|\Omega| \\
&= (d_1 - c_1\varepsilon_2)\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx + \left(\frac{d_2}{2} - c_1\varepsilon_2^{1-p^+}\varepsilon_3\right)\int_\Omega |u_n|^{q(x)}dx \\
&\quad - c_1\varepsilon_2^{1-p^+}\varepsilon_3^{-\frac{p^+}{(q-p)^-}} - \lambda c(\frac{d_2}{2\lambda})|\Omega|, \qquad (17)
\end{aligned}
$$

where $\varepsilon_1 = \frac{d_2}{2\lambda}$. Thus, we choose $\varepsilon_2, \varepsilon_3$ be so small that $d_1 - c_1\varepsilon_2 > 0$ and $\frac{d_2}{2} - c_1\varepsilon_2^{1-p^+}\varepsilon_3 > 0$. It follows from (8) and (17) that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Therefore we can assume that

$$
|u_n|^{q(x)} \rightharpoonup \nu = |u|^{q(x)} + \sum_{i\in I}\nu_i\delta_{x_i}, \quad \nu_i > 0, \qquad (18)
$$

$$
|\nabla u_n|^{p(x)} \rightharpoonup \mu \ge |\nabla u|^{p(x)} + \sum_{i\in I}\mu_i\delta_{x_i}, \quad \mu_i > 0, \qquad (19)
$$

$$
S\nu_i^{1/p^*(x_i)} \le \mu_i^{1/p(x_i)}. \qquad (20)
$$

Note that if $I = \emptyset$ then $u_n \to u$ strongly in $L^{q(x)}(\Omega)$. If not, let x_i be a singular point of the measures μ and ν, define a function $\phi(x) \in C_0^\infty(\Omega)$ such that $\phi(x) = 1$ in $B(x_i, \varepsilon)$, $\phi(x) = 0$ in $\Omega \setminus (x_i, 2\varepsilon)$ and $|\nabla\phi| \le 2/\varepsilon$ in Ω. As $J'(u_n) \to 0$ in $(W_0^{1,p(x)}(\Omega))'$, we obtain that

$$
\lim_{n\to\infty}\langle J'(u_n), \phi u_n\rangle \to 0,
$$

i.e.

$$
\begin{aligned}
\lim_{n\to\infty}\Bigg\{&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(\phi u_n)dx + \int_\Omega |u_n|^{p(x)-2}u_n\phi u_n dx \\
&- \int_\Omega |u_n|^{q(x)-2}u_n\phi u_n dx - \lambda\int_\Omega f(x,u_n)\phi u_n dx\Bigg\} = 0.
\end{aligned}
$$

On the other hand, by Hölder inequality and boundedness of $\{u_n\}$, we have that

$$
0 \quad \le \quad \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi dx\right|
$$

$$\begin{aligned}
&\leq C\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega |u_n|^{p(x)}|\nabla\phi|^{p(x)}dx\right)^{\frac{1}{p(x)}}\left(\int_\Omega |\nabla u_n|^{p(x)}dx\right)^{\frac{p(x)-1}{p(x)}}\\
&\leq C\lim_{\varepsilon\to 0}\left(\int_{B(x_i,\varepsilon)} |u|^{p(x)}|\nabla\phi|^{p(x)}dx\right)^{\frac{1}{p(x)}}\\
&\leq C\lim_{\varepsilon\to 0}\left(\int_{B(x_i,\varepsilon)} |\nabla\phi|^{N}dx\right)^{\frac{1}{N}}\left(\int_{B(x_i,\varepsilon)} |u|^{p^*(x)}dx\right)^{\frac{1}{p^*(x)}}\\
&= 0. \qquad (21)
\end{aligned}$$

From $(18),(19)$ and (21), we get that

$$0 = \lim_{\varepsilon\to 0}\left[\int_\Omega \phi d\mu - \int_\Omega \phi d\nu - \lambda\int_\Omega f(x,u_n)u_n\phi dx\right] = \mu_i - \nu_i. \qquad (22)$$

Combing this with Lemma 2.1 (iii), we obtain $\nu_i^{\frac{1}{p(x_i)}} \geq S\nu_i^{\frac{1}{p^*(x_i)}}$. This result implies that

$$\nu_i = 0 \quad \text{or} \quad \nu_i \geq S^N.$$

If the second case $\nu_i \geq S^N$ holds, for some $i \in I$, then by using Lemma 2.1 and selecting $\varepsilon_2, \varepsilon_3$ in (17) such that $d_1 - c_1\varepsilon_2 > \frac{d_1}{2}$ and $\frac{d_2}{2} - c_1\varepsilon_2^{1-p^+}\varepsilon_3 > \frac{d_2}{4}$, we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J(u_n) - \langle J'(u_n), \frac{u_n}{\eta}\rangle\right)\\
&\geq \frac{d_2}{4}\cdot\int_\Omega |u_n|^{q(x)}dx - c_1\varepsilon_2^{1-p^+}\varepsilon_3^{-\frac{p^+}{(q-p)^-}} - c(\frac{d_2}{2\lambda})|\Omega|\\
&= \frac{d_2}{4}\cdot\int_\Omega d\nu - c_1\varepsilon_2^{1-p^+}\varepsilon_3^{-\frac{p^+}{(q-p)^-}} - c(\frac{d_2}{2\lambda})|\Omega|\\
&\geq \frac{d_2}{4}\cdot\int_\Omega |u|^{q(x)}dx + \frac{d_2}{4}\cdot S^N - c_1\varepsilon_2^{1-p^+}\varepsilon_3^{-\frac{p^+}{(q-p)^-}} - c(\frac{d_2}{2\lambda})|\Omega|\\
&\geq \frac{d_2}{4}\cdot S^N - c_1\varepsilon_2^{1-p^+}\varepsilon_3^{-\frac{p^+}{(q-p)^-}} - c(\frac{d_2}{2\lambda})|\Omega|\\
&= \frac{d_2}{4}\cdot S^N - m^*,
\end{aligned}$$

where $m^* = c_1\varepsilon_2^{1-p^+}\varepsilon_3^{-\frac{p^+}{(q-p)^-}} + c(\frac{d_2}{2\lambda})|\Omega|$. This is impossible. Consequently, $\nu_i = 0$ for all $i \in I$ and hence

$$\int_\Omega |u_n|^{q(x)}dx \to \int_\Omega |u|^{q(x)}dx.$$

Since $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$ we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $W_0^{1,p(x)}(\Omega)$ such that $\{u_n\}$ converges weakly to $W_0^{1,p(x)}(\Omega)$. Note that

$$\langle J'(u_n)-J'(u_0),u_n-u_0\rangle\to 0\quad \text{as } n\to\infty.$$

On the other hand, we have

$$\begin{aligned}&\int_\Omega(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u_0|^{p(x)-2}\nabla u_0)\cdot(\nabla u_n-\nabla u_0)dx\\&+\int_\Omega(|u_n|^{p(x)-2}u_n-|u_0|^{p(x)-2}u_0)(u_n-u_0)dx\\&=\langle J'(u_n)-J'(u_0),u_n-u_0\rangle+\int_\Omega(|u_n|^{q(x)-2}u_n-|u_0|^{q(x)-2}u_0)(u_n-u_0)dx\\&\quad+\lambda\int_\Omega(f(x,u_n)-f(x,u_0))(u_n-u_0)dx.\end{aligned}$$

Using the fact that $\{u_n\}$ converges strongly to u_0 in $L^{q(x)}(\Omega)$ and inequality (5), we have

$$\begin{aligned}&\Big|\int_\Omega(f(x,u_n)-f(x,u_0))(u_n-u_0)dx\Big|\\&\le\Big|\int_\Omega|f(x,u_n)|(u_n-u_0)dx\Big|+\Big|\int_\Omega|f(x,u_0)|(u_n-u_0)dx\Big|\\&\le\Big|\int_\Omega a(\varepsilon)(u_n-u_0)dx\Big|+\varepsilon\Big|\int_\Omega|u_n|^{q(x)-1}(u_n-u_0)dx\Big|\\&\quad+\Big|\int_\Omega a(\varepsilon)(u_n-u_0)dx\Big|+\varepsilon\Big|\int_\Omega|u_0|^{q(x)-1}(u_n-u_0)dx\Big|\\&\le c_1\cdot|u_n-u_0|_{q(x)}+c_2\cdot\||u_n|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot|u_n-u_0|_{q(x)}\\&\quad+c_3\cdot\||u_0|^{q(x)-1}|_{\frac{q(x)}{q(x)-1}}\cdot|u_n-u_0|_{q(x)},\end{aligned}$$

where c_1 c_2 and c_3 are positive constants. Using $|u_n-u_0|_{q(x)}\to 0$ as $n\to\infty$, we deduce that

$$\lim_{n\to\infty}\int_\Omega(f(x,u_n)-f(x,u_0))(u_n-u_0)dx=0,\tag{23}$$

$$\lim_{n\to\infty}\int_\Omega(|u_n|^{q(x)-2}u_n-|u_0|^{q(x)-2}u_0)(u_n-u_0)dx=0.\tag{24}$$

By (23) and (24), we obtain

$$\begin{aligned}&\int_\Omega(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u_0|^{p(x)-2}\nabla u_0)\cdot(\nabla u_n-\nabla u_0)dx\\&+\int_\Omega(|u_n|^{p(x)-2}u_n-|u_0|^{p(x)-2}u_0)(u_n-u_0)dx=0.\end{aligned}\tag{25}$$

It is known that

$$\big(|s|^{p-2}s-|t|^{p-2}t,s-t\big)\ge\begin{cases}C_p|s-t|^p, & \forall\, p\ge 2,\\ C_p\frac{|s-t|^2}{(|s|+|t|)^{2-p}}, & \forall\, p\le 2,\end{cases}\quad s,t\in\mathbb{R}^N,\tag{26}$$

where $(\cdot\,,\,\cdot)$ is the standard scalar product in $\mathbb{R}^N$. Relations (25) and (26) yield

$$\lim_{n\to\infty}\int_\Omega \Big(|\nabla u_n - \nabla u_0|^{p(x)} + |u_n - u_0|^{p(x)}\Big)\,dx = 0.$$

This fact and relation (10) imply $\|u_n - u_0\|_{p(x)} \to 0$ as $n \to \infty$. The proof is complete. □

4. Existence of a sequence of arbitrarily small solutions

In this section, we prove the existence of infinitely many solutions of (1) which tend to zero. Let X be a Banach space and denote

$$\Sigma := \{A \subset X \setminus \{0\} :\ A \text{ is closed in } X \text{ and symmetric with respect to the orgin}\}.$$

For $A \in \Sigma$, we define genus $\gamma(A)$ as

$$\gamma(A) := \inf\{m \in N :\ \exists\, \varphi \in C(A, R^m \setminus \{0\}), -\varphi(x) = \varphi(-x)\}.$$

If there is no mapping φ as above for any $m \in N$, then $\gamma(A) = +\infty$. Let Σ_k denote the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$. We list some properties of the genus (see [22]).

Proposition 4.1. *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold.*

(1) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$;*
(2) *If there exists an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$;*
(3) *If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;*
(4) *Then n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam Theorem;*
(5) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $U_\delta(A) \in \Sigma$ and $\gamma(U_\delta(A)) = \gamma(A)$, where $U_\delta(A) = \{x \in X :\ \|x - A\| \leq \delta\}$.*

The following version of the symmetric mountain-pass lemma is due to Kajikiya [22].

Lemma 4.2. *Let E be an infinite-dimensional space and $J \in C^1(E, R)$ and suppose the following conditions hold.*

(C_1) *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the Palais-Smale condition;*
(C_2) *For each $k \in N$, there exists an $A_k \in \Sigma_k$ such that $\sup_{u\in A_k} J(u) < 0$.*

Then either (R_1) or (R_2) below holds.

(R_1) *There exists a sequence $\{u_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$ and $\{u_k\}$ converges to zero.*
(R_2) *There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $J'(u_k) = 0$, $J(u_k) < 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $J'(v_k) = 0$, $J(v_k) < 0$, $\lim_{k\to\infty} v_k = 0$, and $\{v_k\}$ converges to a non-zero limit.*

Remark 4.1. From Lemma 4.2 we have a sequence $\{u_k\}$ of critical points such that $J(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to get infinitely many solutions we need some lemmas. We first point out that we have

$$\|u(x)\|_{p(x)}^{q^-} + \|u(x)\|_{p(x)}^{q^+} \geq \|u(x)\|_{p(x)}^{q(x)}, \quad \forall\, x \in \overline{\Omega}. \tag{27}$$

Proposition 2.3 (ii) imply that

$$|u(x)|_{q(x)} \leq c_4 \|u(x)\|_{p(x)}, \quad \forall\, x \in \overline{\Omega}, \tag{28}$$

where $c_4 > 0$.
Next, we focus our attention on the case when $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\|_{p(x)} < 1$. For such a u by relation (9) we obtain

$$\int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} dx \geq \|u\|_{p(x)}^{p^+}. \tag{29}$$

Using (3) and (27)-(29), we deduce that

$$\begin{aligned}
J(u) &= \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx - \lambda \int_\Omega F(x,u) dx \\
&\geq \frac{1}{p^+} \cdot \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} dx - \frac{1}{q^-} \cdot \int_\Omega |u|^{q(x)} dx - \lambda b(\varepsilon)|\Omega| - \lambda\varepsilon \int_\Omega |u|^{q(x)} dx \\
&= \frac{1}{p^+} \cdot \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} dx - \frac{2}{q^-} \cdot \int_\Omega |u|^{q(x)} dx - \lambda b(\frac{1}{q^-\lambda})|\Omega| \\
&\geq \frac{1}{2p^+} \cdot \|u\|_{p(x)}^{p^+} - \frac{2c_4}{q^-} \cdot \|u\|_{p(x)}^{q^+} - \frac{2c_4}{q^-} \cdot \|u\|_{p(x)}^{q^-} - \lambda b(\frac{1}{q^-\lambda})|\Omega| \\
&\geq A\|u\|_{p(x)}^{p^+} - B\|u\|_{p(x)}^{q^+} - \lambda C,
\end{aligned} \tag{30}$$

where $\varepsilon = \frac{1}{q^-\lambda}$, $A = \frac{1}{2p^+}$, $B = \frac{2c_4}{q^-}$, $C = \frac{2c_4}{q^-\lambda} + b(\frac{1}{q^-\lambda})|\Omega|$, for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\|_{p(x)} < 1$. If we define

$$Q(s) = As^{p^+} - Bs^{q^+} - \lambda C.$$

Then

$$J(u) \geq Q(\|u\|_{p(x)}).$$

From the definition of $Q(s)$ and the fact that $p^+ < q^+$, we konw that there exists λ^* such that for $\lambda \in (0, \lambda^*)$, $Q(t)$ attains its positive maximum, that is, there exists

$$R_1 = \left(\frac{p^+A}{q^+B}\right)^{1/(q^+-p^+)}$$

such that

$$e_1 = Q(R_1) = \max_{t\geq 0} Q(t) > 0.$$

Therefore, for $e_0 \in (0, e_1)$, we may find $R_0 < R_1$ such that $Q(R_0) = e_0$. Now we define

$$\chi(t) = \begin{cases} 1, & 0 \le t \le R_0, \\ \frac{At^{p^+} - \lambda C - e_1}{Bt^{q^+}}, & t \ge R_1, \\ C^\infty, \quad \chi(t) \in [0,1], & R_0 \le t \le R_1. \end{cases}$$

Then it is easy to see $\chi(t) \in [0,1]$ and $\chi(t)$ is C^∞. Let $\varphi(u) = \chi(\|u\|_{p(x)})$ and consider the perturbation of $J(u)$:

$$\begin{aligned} G(u) &= \int_\Omega \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \varphi(u) \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx \\ &\quad - \lambda \varphi(u) \int_\Omega F(x,u) dx. \end{aligned} \tag{31}$$

Then

$$\begin{aligned} G(u) &\ge A\|u\|_{p(x)}^{p^+} - B\varphi(u)\|u\|_{p(x)}^{q^+} - \lambda C \\ &= \overline{Q}(\|u\|_{p(x)}), \end{aligned}$$

where $\overline{Q}(t) = At^{p^+} - B\chi(t)t^{q^+} - \lambda C$ and

$$\overline{Q}(t) = \begin{cases} Q(t), & t \le R_0, \\ e_1, & t \ge R_1. \end{cases}$$

From the above arguments, we have the following:

Lemma 4.3. *Let $G(u)$ be defined as in* (31). *Then*

(i) *$G \in C^1(E, R)$ and G is even and bounded from below;*
(ii) *If $G(u) < e_0$, then $\overline{Q}(\|u\|_{p(x)}) < e_0$, consequently, $\|u\|_{p(x)} < R_0$ and $J(u) = G(u)$;*
(iii) *There exist $m^* > 0$ such that $S^N - m^* > 0$, and λ^* such that for $\lambda \in (0, \lambda^*)$, G satisfies a local $(PS)_c$ condition for $c < e_0 \in \left(0, \min\left\{e_1, \frac{d_2}{4} \cdot S^N - m^*\right\}\right)$.*

Proof. It is easy to see (i) and (ii). (iii) are consequences of (ii) and Lemma 3.2. □

Lemma 4.4. *Assume that (H_3) of Theorem 1.1 holds. Then for any $k \in N$, there exists $\delta = \delta(k) > 0$ such that $\gamma(\{u \in W_0^{1,p(x)}(\Omega) : G(u) \le -\delta(k)\} \setminus \{0\}) \ge k$.*

Proof. First, by (H_3) of Theorem 1.1, for any fixed $u \in W_0^{1,p(x)}(\Omega), u \neq 0$, we have

$$F(x, \rho u) \ge M(\rho)(\rho u)^{p^-} \quad \text{with} \quad M(\rho) \to \infty \quad \text{as} \quad \rho \to 0. \tag{32}$$

Next, given any $k \in N$, let E_k be a k-dimensional subspace of $W_0^{1,p(x)}(\Omega)$. We take $u \in E_k$ with norm $\|u\|_{p(x)} = 1$, for $0 < \rho < \min\{R_0, 1\}$, we get

$$G(\rho u) = J(\rho u)$$

$$\begin{aligned} &= \int_\Omega \rho^{p(x)} \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx - \int_\Omega \rho^{q(x)} \frac{1}{q(x)} |u|^{q(x)} dx \\ &\qquad -\lambda \int_\Omega F(x, \rho u) dx \\ &\le \frac{1}{p^-} \rho^{p^-} \int_\Omega |\nabla u|^{p(x)} + |u|^{p(x)} dx \\ &\qquad - \frac{1}{q^+} \rho^{q^+} \int_\Omega |u|^{q(x)} dx - \lambda M(\rho) \rho^{p^-} \int_\Omega |u|^{p^-} dx. \end{aligned}$$

Since E_k is a space of finite dimension, all the norms in E_k are equivalent. If we define

$$A_k = \inf \left\{ \int_\Omega |u|^{q(x)} dx : \ u \in E_k, \ \|u\|_{p(x)} = 1 \right\} > 0,$$

$$B_k = \inf \left\{ \int_\Omega |u|^{p^-} dx : \ u \in E_k, \ \|u\|_{p(x)} = 1 \right\} > 0.$$

It follows from (32)that

$$\begin{aligned} G(\rho u) &\le \frac{1}{p^-} \rho^{p^-} - \frac{1}{q^+} \rho^{q^+} A_k - \lambda M(\rho) \rho^{p^-} B_k \\ &\le \rho^{p^-} \left(\frac{1}{p^-} - \lambda M(\rho) B_k \right) - \frac{1}{q^+} \rho^{q^+} A_k \\ &= -\delta(k) < 0, \text{ as } \ \rho \to 0, \end{aligned}$$

since $\lim_{|\rho| \to 0} M(\rho) = +\infty$. That is,

$$\{u \in E_k : \ \|u\|_{p(x)} = \rho\} \subset \{u \in W_0^{1,p(x)}(\Omega) : \ G(u) \le -\delta(k)\} \setminus \{0\}.$$

This completes the proof. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Recall that

$$\Sigma_k = \{A \in E \setminus \{0\} : A \text{ is closed and } \ A = -A, \gamma(A) \ge k\}$$

and define

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} G(u).$$

By Lemmas 4.3 (i) and 4.4, we know that $-\infty < c_k < 0$. Therefore, assumptions (C_1) and (C_2) of Lemma 4.2 are satisfied. This means that G has a sequence of solutions $\{u_n\}$ converging to zero. Hence, Theorem 1.1 follows by Lemma 4.3 (ii). □

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