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TRIPLE SOLUTIONS FOR THREE-ORDER PERIODIC BOUNDARY VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITY[†]

HUIXUAN TAN*, HANYING FENG, XINGFANG FENG, YATAO DU

Abstract. In this paper, we consider the periodic boundary value problem with sign changing nonlinearity

$$u''' + \rho^3 u = f(t,u),\ t \in [0, 2\pi],$$

subject to the boundary value conditions:

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2,$$

where $\rho \in (0, \frac{1}{\sqrt{3}})$ is a positive constant and $f(t,u)$ is a continuous function. Using Leggett-Williams fixed point theorem, we provide sufficient conditions for the existence of at least three positive solutions to the above boundary value problem. The interesting point is the nonlinear term f may change sign.

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1. Introduction

In this paper, we are concerned with the multiplicity of positive solutions of the nonlinear three-order periodic boundary value problem

$$\begin{cases} u''' + \rho^3 u = f(t,u), & t \in [0, 2\pi], \\ u^{(i)}(0) = u^{(i)}(2\pi), & i = 0, 1, 2, \end{cases} \tag{1}$$

throughout this paper, we assume that

(H_1) $\rho \in (0, \frac{1}{\sqrt{3}})$, $f : [0, 2\pi] \times [0, \infty] \to R$ is continuous and there exists a constant $L > 0$ such that $F(t,u) := f(t,u) + L \geqslant 0$ for all $(t,u) \in [0, 2\pi] \times [0, \infty]$;

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(H_2)There exist continuous, non-negative and non-decreasing functions $g(x)$, $h(x)$ on $(0,\infty)$ such that $g(x) \le f(t,x) + L \le h(x)$.

Nonlinear periodic boundary value problem have been extensively studied by many authors. The existence of solutions is one of the most important aspects of periodic boundary value problem(see[1]-[7],[9]-[17] and references therein). In recent years, many authors take more interested in the two-order or four-order periodic boundary value problem ([4],[11],[13]-[17]). However, for three-order periodic boundary value problems , a few of authors have studied ([9],[10],[12]). For the periodic boundary value problem (1), different methods and techniques have been employed to discuss the existence of positive solutions. We recall the following three results. In [12], Kong and Wang, by employing Schauder fixed point theorem together with priori estimates and perturbation technique, established the existence of at least one positive solution under suitable conditions of f . In [9], by using Krasnoselskii fixed point theorem together with non-linear alternative of Leray-Schauder, the existence of positive periodic solutions have been discussed. Recently, In [10], Yao obtained existence results for singular and multiple positive periodic solutions by applying Guo-Lakshmikantham fixed point index theory for cones.

Inspired and motivated by the work mentioned above, in this paper, we shall apply Leggett-Williams fixed point theorem to investigate the existence of at least three positive periodic solutions to (1). The interesting point is the nonlinear term f may change sign.

2. Background and definitions

The proof of our main result is based on the Leggett-Williams fixed-point theorem, which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. For the convenience of readers, we present here the necessary definitions from cone theory in Banach spaces. These definitions can be found in the recent literature.

Definition 2.1. Let E be a real Banach space over R. A nonempty closed set $P \subset E$ is said to be a cone provided that
(i) $\alpha u + \beta v \in P$ for all $u, v \in P$ and all $\alpha \ge 0, \beta \ge 0$, and
(ii) $u, -u \in P$ implies $u = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\le$. For $u, v \in P$, we write $u \le v$ if and only if $v - u \in P$.

Definition 2.2. The map ψ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\psi : P \to [0,\infty)$ is continuous and

$$\psi(tx + (1-t)y) \ge t\psi(x) + (1-t)\psi(y)$$

for all $x,\ y \in P$ and $0 \le t \le 1$.

Definition 2.3. Let $0 < a < b$ be given and let ψ be a nonnegative continuous concave functional on the cone P. Define the sets $P_a, \overline{P}_a$ and $P(\psi, a, b)$ by

$$P_a = \{x \in P \mid \|x\| < a\}, \quad \overline{P}_a = \{x \in P \mid \|x\| \leq a\},$$

$$P(\psi, a, b) = \{x \in P \mid a \leq \psi(x), \|x\| \leq b\}.$$

Next we state the Leggett-Williams fixed-point theorem. The proof can be found in Deimling's text [8].

Theorem 2.4 (Leggett-Williams Fixed-Point Theorem)**.** *Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ is a cone in E. Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on P such that $\psi(u) \leq \|u\|, \forall u \in \overline{P}_c$. Suppose that there exist $0 < r < a < b < c$ such that*
(S_1) $\{u \in P(\psi, a, b) \mid \psi(u) > a\} \neq \emptyset$ and $\psi(Tu) > a$ for $u \in P(\psi, a, b)$,
(S_2) $\|Tu\| < a$ for $u \in \overline{P}_a$,
(S_3) $\psi(Tu) > a$ for $u \in P(\psi, a, c)$ with $\|Tu\| > b$.
Then T has at least three fixed points $u_1,\ u_2,\ u_3 \in \overline{P}_c$, such that $\|u_1\| < r$, $a < \psi(u_2)$, $\|u_3\| > r$, $\psi(u_3) < a$.

3. Some preliminary results

Lemma 3.1 ([12])**.** *If $\rho \in (0, +\infty)$, then the linear problem*

$$\begin{cases} u'' - \rho u' + \rho^2 u = 0, & t \in [0, 2\pi], \\ u(0) - u(2\pi) = 0, & u'(0) - u'(2\pi) = 1, \end{cases} \tag{2}$$

has a unique positive solution

$$w(t) = \frac{2e^{(\rho/2)t}\left[\sin \frac{\sqrt{3}}{2}\rho(2\pi - t) + e^{-\rho\pi} \sin \frac{\sqrt{3}}{2}\rho t\right]}{\sqrt{3}\rho(e^{\rho\pi} + e^{-\rho\pi} - 2\cos\sqrt{3}\rho\pi)}.$$

For every function $u \in C[0, 2\pi]$, we define the operator

$$(Ju)(t) := \int_0^{2\pi} g(t, x)u(x)dx.$$

where

$$g(t, x) := \begin{cases} \dfrac{e^{\rho(2\pi + x - t)}}{e^{2\rho\pi} - 1}, & 0 \leq x \leq t \leq 2\pi, \\ \dfrac{e^{\rho(x - t)}}{e^{2\rho\pi} - 1}, & 0 \leq t \leq x \leq 2\pi. \end{cases}$$

By a direct calculation, we can easily obtain

$$\int_0^{2\pi} g(t, x)dx = \frac{1}{\rho}.$$

Now, we consider the problem

$$\begin{cases} u'' - \rho u' + \rho^2 u = f(t, Ju), \\ u^{(i)}(0) = u^{(i)}(2\pi), \ i = 0, 1. \end{cases} \tag{3}$$

If u is a positive solution of problem (3), i.e. $u(t) > 0$ for $t \in [0, 2\pi]$, it is easy to verify that $y(t) = (Ju)(t)$ is a positive solution of problem (1). Therefore, we will concentrate the problem (3) for which we have the following result.

Lemma 3.2 ([9, 12])**.** *Let $w(t)$ be a unique solution of* (2)*, then problem* (3) *is equivalent to integral equation*

$$u(t) = \int_0^{2\pi} G(t,s) f(s, (Ju)(s)) ds,$$

where

$$\begin{aligned} G(t,s) &= \begin{cases} w(t-s), & 0 \le s \le t \le 2\pi, \\ w(2\pi + t - s), & 0 \le t \le s \le 2\pi, \end{cases} \\ &= \begin{cases} \frac{2e^{(\rho/2)(t-s)}[\sin \frac{\sqrt{3}}{2}\rho(2\pi - t + s) + e^{-\rho\pi} \sin \frac{\sqrt{3}}{2}\rho(t-s)]}{\sqrt{3}\rho(e^{\rho\pi} + e^{-\rho\pi} - 2\cos\sqrt{3}\rho\pi)}, & s \le t, \\ \frac{2e^{(\rho/2)(2\pi + t - s)}[\sin \frac{\sqrt{3}}{2}\rho(s-t) + e^{-\rho\pi} \sin \frac{\sqrt{3}}{2}\rho(2\pi - s + t)]}{\sqrt{3}\rho(e^{\rho\pi} + e^{-\rho\pi} - 2\cos\sqrt{3}\rho\pi)}, & s \ge t. \end{cases} \end{aligned} \tag{4}$$

Lemma 3.3 ([9, 12])**.** *Let $\rho \in (0, \frac{1}{\sqrt{3}})$, then we have the estimates*

$$m = \frac{2\sin\sqrt{3}\rho\pi}{\sqrt{3}\rho(e^{\rho\pi}+1)^2} \le G(t,s) \le \frac{2}{\sqrt{3}\rho\sin\sqrt{3}\rho\pi} = M, \quad t, s \in [0, 2\pi].$$

From [9], we know that $u(t) = y(t) - \rho\omega\big(\omega = \dfrac{L}{\rho^3}\big)$ is a positive solution of (3) when $y(t)$ is the solution of

$$\begin{cases} u'' - \rho u' + \rho^2 u = F(t, (Ju)(t) - \omega), \quad t \in [0, 2\pi], \\ u^{(i)}(0) = u^{(i)}(2\pi), \ i = 0, 1, \end{cases} \tag{5}$$

if the nonlinear term f satisfies the condition (H_2). So, we can transform the problem into discussing the existence of positive solutions for (5).

4. Existence of triple positive solutions

Let $E = C[0, 2\pi]$ be endowed with the maximum norm, $\|y\| = \max\limits_{0 \le t \le 2\pi} |y(t)|$. Define the cone $P \subset E$ by

$$P = \left\{ u \in E \mid u \ge 0 \ \ for \ all \ t \in [0, 2\pi] \ and \ \min_{0 \le t \le 2\pi} u(t) \ge \max\{\sigma\|u\|, \rho\omega\} \right\},$$

where $\sigma = m/M$.

Finally, let the nonnegative continuous concave functional $\psi : P \to [0, \infty)$ be defined by $\psi(u) = \min\limits_{0 \le t \le 2\pi} u(t), u \in P$. We notice that, for each $u \in P$, $\psi(u) \le \|u\|$.

Theorem 4.1. *Assume that* (H_1), (H_2) *hold. There exist constants* $0 < r < a < b < c$ $\big(b = \frac{a}{\sigma}$ *and* $L \leq \min\{\sigma a\rho^2, r\rho^2\}\big)$ *such that*

(H_3) $\int_0^{2\pi} f(s,(Ju)(s)-\omega)ds \geq \frac{\rho\omega}{m} - 2\pi L$;

(H_4) $h(\frac{c}{\rho}-\omega) \leq \frac{c}{2\pi M}$;

(H_5) $g(\frac{\sigma a}{\rho}-\omega) > \frac{a}{2\pi m}$;

(H_6) $h(\frac{r}{\rho}-\omega) < \frac{r}{2\pi M}$.

Then the boundary value problem (1) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying* $u_i = J(y_i - \rho\omega)$, $\|y_1\| < r$, $a < \psi(y_2)$ *and* $\|y_3\| > r$ *with* $\psi(y_3) < a$ *where* $y_i(t)$ *is the solution of* (5)(i=1, 2, 3).

Proof. Define the operator $T : P \to P$ by

$$(Tu)(t) = \int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds, \qquad 0 \leq t \leq 2\pi,$$

where $G(t,s)$ is the Green function given by (4). The boundary value problem (5) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies Theorem 2.1.

Firstly, we show that $T : \overline{P}_c \to \overline{P}_c$. In fact, if $u \in P$, then $(Ju)(t) - \omega \geq \rho w/\rho - \omega = 0$. Thus from Lemma 3.3 and (H_1), it follows that $(Tu)(t) \geq 0$, $0 \leq t \leq 2\pi$. Let $u \in P$, then from Lemma 3.3, we have

$$\begin{aligned}\min_{0\leq t\leq 2\pi}(Tu)(t) &= \min_{0\leq t\leq 2\pi}\int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds\\ &\geq m\int_0^{2\pi} F(s,(Ju)(s)-\omega)ds\\ &= \sigma\int_0^{2\pi} MF(s,(Ju)(s)-\omega)ds\\ &\geq \sigma\max_{0\leq t\leq 2\pi}\int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds\\ &= \sigma\|Tu\|.\end{aligned}$$

On the other hand, by Lemma 3.3 and *(H_3)*, for all $u \in P$,

$$\begin{aligned}(Tu)(t) &= \int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds\\ &\geq m\int_0^{2\pi} F(s,(Ju)(s)-\omega)ds\\ &= m\int_0^{2\pi} [f(s,(Ju)(s)-\omega)+L]ds\\ &\geq m(\frac{\rho\omega}{m} - 2\pi L) + m2\pi L = \rho\omega.\end{aligned}$$

So $\min_{0\le t\le 2\pi}(Tu)(t) \ge \max\{\sigma\|u\|, \rho\omega\}$, thus we have $TP \subset P$.

If $u \in \overline{P}_c$, then $(Ju)(t) - \omega \le c/\rho - \omega$. Therefore, by ($H_2$), *($H_4$)* and Lemma 3.3, for $0 \le t \le 2\pi$,

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 2\pi}\int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds\\ &\le M\int_0^{2\pi} F(s,(Ju)(s)-\omega)ds\\ &\le M\int_0^{2\pi} h((Ju)(s)-\omega)ds\\ &\le M\int_0^{2\pi} h(\frac{c}{\rho}-\omega)ds = c.\end{aligned}$$

Then $T : \overline{P}_c \to \overline{P}_c$ is well defined. It is easy to see that T is continuous and completely continuous since $f : [0, 2\pi] \times [0, \infty] \to R$ is a continuous function.

In the same way, we can obtain that $T : \overline{P}_r \to P_r$ by *(H_6)*. So, condition *(S_2)* of Theorem 2.1 is satisfied.

Next we prove *(S_1)* of Theorem 2.1 holds. Choose $u_0(t) = \dfrac{a+b}{2}$, $0 \le t \le 2\pi$. It is easy to see that $u_0(t) \in P(\psi, a, b)$ and $\psi(u_0) = \min_{0\le t\le 2\pi} u_0(t) = \dfrac{a+b}{2}$, so $\{u \in P(\psi, a, b) \mid \psi(u) > a\} \ne \emptyset$.

In fact, if $u \in P(\psi, a, b)$, then $\min_{0\le t\le 2\pi} u(t) \ge \sigma\|u\| \ge \sigma\psi(u) \ge \sigma a$.

Thus

$$(Ju)(t) - \omega = \int_0^{2\pi} g(t,x)u(x)dx - \omega \ge \frac{\sigma a}{\rho} - \omega.$$

As a result, it follows from (H_2), *(H_5)* and Lemma 3.3 that, for $0 \le t \le 2\pi$,

$$\begin{aligned}\psi(Tu) &= \min_{0\le t\le 2\pi}\int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds\\ &\ge m\int_0^{2\pi} F(s,(Ju)(s)-\omega)ds\\ &\ge m\int_0^{2\pi} g((Ju)(s)-\omega)ds\\ &\ge m\int_0^{2\pi} g(\frac{\sigma a}{\rho}-\omega)ds\\ &= 2\pi m g(\frac{\sigma a}{\rho}-\omega) > a.\end{aligned}$$

Consequently condition *(S_1)* of Theorem 2.1 is satisfied.
We finally show that *(S_3)* of Theorem 2.1 also holds.

Suppose that $u \in P(\psi, a, c)$ with $\|Tu\| > b$. Then by Lemma 3.3, we have

$$\begin{aligned}
\psi(Tu) &= \min_{0\le t\le 2\pi} \int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds \\
&\ge m \int_0^{2\pi} F(s,(Ju)(s)-\omega)ds \\
&\ge \sigma \max_{0\le t\le 2\pi} \int_0^{2\pi} G(t,s)F(s,(Ju)(s)-\omega)ds \\
&= \sigma\|Tu\| > \sigma b = a.
\end{aligned}$$

Thus condition *(S_3)* of Theorem 2.1 is also satisfied. Therefore an application of Theorem 2.1 leads to the conclusion that the boundary value problem (5) has at least three positive solutions y_1, y_2 *and* y_3 satisfying $\|y_1\| < r$, $a < \psi(y_2)$ and $\|y_3\| > r$ with $\psi(y_3) < a$. Thus the boundary value problem (1) has at least three positive solutions u_1, u_2 *and* u_3 satisfying $u_i = J(y_i - \rho\omega)$. □

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Huixuan Tan received M.SC. from Dalian University of Thechnology. She is currently a instructor in the Department of Mathematics at Shijiazhuang Mechanical Engineering College. Her research interests focus on the theory and applications of differential equations.

Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, China.
e-mail: tanhuixuan_2004@163.com

Hanying Feng received M.SC. from Anhui University and Ph.D. from Beijing Institute of Thechnology.He is currently a professor in the Department of Mathematics at Shijiazhuang Mechanical Engineering College. His research interests focus on the theory and applications of differential equations.

Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, China.
e-mail: fhanying@yahoo.com.cn

Xingfang Feng received M.SC. from Hebei Normal University. She is currently a instructor in Department of Mathematics at Shijiazhuang Mechanical Engineering College. Her research interests focus on the theory and applications of differential equations.

Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, China.
e-mail: fxfg651@163.com

Yatao Du received M.SC. and Ph.D. from Hebei Normal University. She is currently a instructor in Department of Mathematics at Shijiazhuang Mechanical Engineering College. Her research interests focus on the theory and applications of differential equations.

Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, China.
e-mail: dyt77@sina.com