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ON MARCINKIEWICZ'S TYPE LAW FOR FUZZY RANDOM SETS

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ABSTRACT. In this paper, we will obtain Marcinkiewicz's type limit laws for fuzzy random sets as follows : Let $\{X_n | n \geq 1\}$ be a sequence of independent identically distributed fuzzy random sets and $E\|X_i\|^r_{\rho_p} < \infty$ with $1 \leq r \leq 2$. Then the following are equivalent: $S_n/n^{\frac{1}{r}} \to \tilde{0}$ a.s. in the metric ρ_p if and only if $S_n/n^{\frac{1}{r}} \to \tilde{0}$ in probability in the metric ρ_p if and only if $S_n/n^{\frac{1}{r}} \to \tilde{0}$ in L_1 if and only if $S_n/n^{\frac{1}{r}} \to \tilde{0}$ in L_r where $S_n = \sum_{i=1}^{n} X_i$.

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1. Introduction

The study of the fuzzy random sets, defined as measurable mappings on a probability space, was initiated by Kwakernaak [12] where useful basic properties were developed. Puri and Ralescu [9] used the concept of fuzzy random variables in generating results for random sets to fuzzy random sets. Kruse [8] proved a strong law of large numbers for independent identically distributed fuzzy random variables. Artstein and Vitale [1] proved a strong law of large numbers(SLLN) for R^p-valued random sets and Cressie [3] proved a SLLN for some paticular class of R^p-valued random sets. Using Rådstrom embedding(e.g. Rådstrom [14]), Puri and Ralescu [12] proved a SLLN for Banach space valued random sets and they also proved SLLN for fuzzy random sets, which generalized all of previous SLLN for random sets. In recent year, Joo, Kim and Kwon [6] proved Chung's type law of large numbers for fuzzy random variables and Kwon and Shim [11] obtained a uniform strong law of large numbers for partial sum processes of fuzzy random sets. In this paper we obtain Marcinkiewicz's type laws for fuzzy random sets in

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the Euclidean space under the assumption that WLLN holds. The proofs of the results are based heavily on isometrical embeddings of the fuzzy sample spaces, endowed with L_p-metrics, into L_p-spaces. Our results give the fuzzy version of Marcinkiewicz's type law of large numbers in general Banach spaces.

2. Preliminaries

Let $\mathcal{K}(R^n)$ ($\mathcal{K}_c(R^n)$) be the collection of nonempty compact (and convex) subsets of Euclidean space R^n. The set can be viewed as a linear structure induced by the scalar multiplication and the Minkowski addition, that is

$$\lambda A = \{\lambda a : a \in A\},\ A + B = \{a + b : a \in A,\ b \in B\}$$

for all A, $B \in \mathcal{K}(R^n)$ and $\lambda \in R$. If d is the Hausdoff metric on $\mathcal{K}(R^n)$ which, for $A,\ B \in \mathcal{K}(R^n)$, is given by

$$d(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} |a - b|, \sup_{b\in B} \inf_{a\in A} |a - b|\}$$

where $|\cdot|$ denotes the Euclidean norm. Then $(\mathcal{K}(R^n), d)$ is a complete separable metric space [4,10].

A fuzzy set of R^n is a mapping $A : R^n \to [0, 1]$. We will denote by A_α the α-level set of A (that is $A_\alpha = \{x \in R^n : A(x) \geq \alpha\}$) for all $\alpha \in (0, 1]$ and by A_0 the closure of the support of A(that is $A_0 = cl\{x \in R^n : A(x) > 0\}$).

Let $\mathcal{F}_c(R^n)(\mathcal{F}_{coc}(R^n))$ be the class of the fuzzy sets A satisfying the following conditions

(1) $A_1 \neq \emptyset$,
(2) A_0 is compact, and
(3) A is upper semi continuous
((4) A_α is convex for all $\alpha \in [0, 1]$).

And $\mathcal{F}^b_c(R^n)(\mathcal{F}^b_{coc}(R^n))$ is the subset of $\mathcal{F}_c(R^n)$($\mathcal{F}_{coc}(R^n)$) with bounded support.

Given a measurable space $(\Omega, \mathcal{A})$ and the metric space $(\mathcal{K}(R^n), d)$, a random set (or as a random compact set) is associated with a Borel measurable mapping $X : \Omega \to \mathcal{K}(R^n)$. If $X : \Omega \to \mathcal{K}(R^n)$ is a set-valued mapping, then X is a random set if and only if $X^{-1}(C) = \{\omega \in \Omega : X(\omega) \cap C \neq \emptyset\} \in \mathcal{A}$ for all $C \in \mathcal{K}(R^n)$.

If X is a random set, the mapping denoted by $\|X\|_d$ and defined by

$$\|X(\omega)\|_d = d(X, \tilde{0})(\omega) = \sup_{x\in X(\omega)} |x|$$

for all $\omega \in \Omega$, is a random variable, where $\tilde{0}$ is the fuzzy set where $\tilde{0}(0) = 1$ and $\tilde{0}(x) = 0$ otherwise.

A support function of a non-void bounded subset K of R^n is defined by

$$s_K : R^n \to R\ :\ x \mapsto \sup_{y\in K} < x, y >$$

where $< x, y >$ denotes the standard scalar product of the vectors x and y. Support functions s_K are uniquely associated with the subsets $K \in \mathcal{K}_c(R^n)$

and preserve addition and nonnegative scalar multiplication when we restricted ourselves to $\mathcal{K}(R^n)$, i.e.

$$s_{K+L} = s_K + s_L,\ s_{\lambda K} = \lambda s_K$$

Now we endow $\mathcal{F}_c(R^n)$ with the initial topology generated by the mappings

$$\pi_\alpha : \mathcal{F}_c(R^n) \to \mathcal{K}(R^n), A \mapsto A_\alpha, \alpha \in (0,1] \cap Q$$

then the topology mentioned above enables us to introduce a measurability concept for defining fuzzy random variable. We call a mapping $X : \Omega \to \mathcal{F}_c(R^n)$ fuzzy random variable over $(\Omega, \mathcal{A}, \mu)$ if it is $\mathcal{A}$-measurable over the initial topology.

For a real number $p \geq 1$ and $A,\ B \in \mathcal{F}_c(R^n)$, define

$$d_p(A,B) = (\int_0^1 d(A_\alpha, B_\alpha)^p d\alpha)^{1/p}$$

and

$$\rho_p(A,B) = (\int_0^1 \int_{S^{n-1}} |s_{A_\alpha} - s_{B_\alpha}|^p \lambda^{S^{n-1}} d\alpha)^{1/p}$$

where $\lambda^{S^{n-1}}$ denotes the unit Lebesgue measure on the unit sphere in R^n. Then $d_p(\rho_p)$ becomes a separable metric on $\mathcal{F}_c^b(R^n)$ $(\mathcal{F}_{coc}^b(R^n))$ with the relation $\rho_p \leq d_p$ which induce the same topology

Now consider $L_p([0,1] \times S^{n-1})$, the L_p-space with respect to $[0,1] \times S^{n-1}$, the obvious product σ-algebra and the product measure $\lambda \otimes \lambda^{S^{n-1}}$. Then under the L_p-norm $\| \|_p$ we obtain $L_p([0,1] \times S^{n-1})$ as a separable Banach space. Next we can embed $\mathcal{F}_{coc}^b(R^n)$ isometrically isomorphic into $L_p([0,1] \times S^{n-1})$ as a positive cone (for details see [5,7]). Embedding $\mathcal{F}_{coc}^b(R^n)$ into $L_p([0,1] \times S^{n-1})$, we draw a convergence theorem in Banach space. For $1 \leq p < \infty$, $L_p([0,1] \times S^{n-1})$ is so called separable Banach space of type $\min(p,2)$. It is known that separable Banach spaces of type 2 is exactly those separable Banach space where the classical strong law of large numbers for independent non-identically distributed random variables holds.

3. Main Results

To prove the main theorem we will need the following lemmas. Lemma 1 connects two metric spaces $\mathcal{F}_{coc}^b(R^n)$ and $L_p([0,1] \times S^{n-1})$ isometrically.

Lemma 3.1. *Let $1 \leq p < \infty$ be fixed. Then $j : \mathcal{F}_{coc}^b(R^n) \to L_p([0,1] \times S^{n-1})$ by $A \mapsto < \tilde{s}_A >$ defines an injection mapping satisfying*

(1) $\|j(A) - j(B)\|_p = \rho_p(A,B)$

(2) $j(A+B) = j(A) + j(B)$

(3) $j(\lambda A) = \lambda j(A)$

for any $A, B \in \mathcal{F}_{coc}^b(R^n)$ and $\lambda \geq 0$(where $\tilde{s}_A : [0,1] \times R^n \to R$ by $(\alpha, x) \mapsto s_{A_\alpha}(x)$).

The following is a generalization of a classical result [15, p. 127-128].

Lemma 3.2. *Let $\{X_n | n \geq 1\}$ be a sequence of fuzzy random sets stochastically dominated by X with $E\|X\|_{\rho_p}^r < \infty$ for $0 < r < \infty$, that is, for any $t > 0$, $P(||X_n||_{\rho_p} \geq t) \leq P(\|X\|_{\rho_p} \geq t)$. Then*

(i)$\sum_{n=1}^{\infty} \frac{1}{n^{\beta/r}} E\|X_n\|_{\rho_p}^{\beta} I(\|X_n\|_{\rho_p} \leq n^{\frac{1}{n}}) < \infty$ for $0 < r < \beta$

(ii)$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha/r}} E\|X_n\|_{\rho_p}^{\alpha} I(\|X_n\|_{\rho_p} \geq n^{\frac{1}{n}}) < \infty$ for $0 < \alpha < r$

Proof. . Notice that $\{\|X_n\|_{\rho_p} | n \geq 1\}$ is a sequence of random variables stochastically dominated by $\|X\|_{\rho_p}$. Now apply Stout's result. □

Lemma 3.3 ([5]). *Let $\{X_k | 1 \leq k \leq n\}$ be $\mathcal{F}_{coc}^b(R^n)$-valued independent random variables. Let $S_i = \sum_{k=1}^{i} X_k$ for $i = 1, 2, 3, \cdots, n$ and $t > 0$. Then*

$$P(\max_{1 \leq i \leq n} \|S_i\|_{\rho_p}) \leq 4 \max_{1 \leq i \leq n} P(\|S_i\|_{\rho_p} > t/4).$$

Lemma 3.4 ([5]). *Let $\{X_k | 1 \leq k \leq n\}$ be independent $\mathcal{F}_{coc}^b(R^n)$-valued random variables with $E\|X_k\|_{\rho_p}^r < \infty$ for $k = 1, 2, \cdots n$ and $1 \leq r \leq 2$. Then we have*

$$E|\|S_n\|_{\rho_p} - E\|S_n\|_{\rho_p}|^r \leq C_r \sum_{k=1}^{n} E\|X_k\|_{\rho_p}^r.$$

where C_r is a positive constant depending only on r ; if $r = 2$ then it is possible to take $C_2 = 4$.

Theorem 3.5. *Let $\{X_n | n \geq 1\}$ be a sequence of independent identically distributed $\mathcal{F}_{coc}^b(R^n)$-valued fuzzy random variables with $E\|X_i\|\rho_p^r < \infty$ for $1 \leq r \leq 2$ and let $S_n = \sum_{i=1}^{n} X_i$. Then the following are equivalent:*

(i) $S_n/n^{\frac{1}{r}} \to \tilde{0}$ a.s. in the metric ρ_p;

(ii) $S_n/n^{\frac{1}{r}} \to \tilde{0}$ in probability in the metric ρ_p;

(iii) $S_n/n^{\frac{1}{r}} \to \tilde{0}$ in L_1

(iv) $S_n/n^{\frac{1}{r}} \to \tilde{0}$ in L_r

Proof. Let $j : \mathcal{F}_{coc}^b(R^n) \to L_p([0,1] \times S^{n-1})$ be an isometry. Then $\{j \circ X_n | n \geq 1\}$ be a sequence of independent identically distributed random element in a Banach space $L_p([0,1] \times S^{n-1})$. Since $\|X\|_{\rho_p} = \|j \circ X\|_p$, in what follow we use X and $j \circ X$ interchangeably.

Let $X_n' = X_n 1(\|X_n\|_{\rho_p} \leq n^{\frac{1}{n}})$, $X_n'' = X_n 1(\|X_n\|_{\rho_p} > n^{\frac{1}{n}})$, $S_n' = \sum_{i=1}^{n} X_i'$ and $S_n'' = \sum_{i=1}^{n} X_i''$.

First we show that $(i) \iff (ii) \iff (iii)$ in L^1. Since $E\|X_i\|\rho_p^r < \infty$, we have $S_n''/n^{\frac{1}{r}} \to \tilde{0}$ a.s. and $S_n''/n^{\frac{1}{r}} \to \tilde{0}$ in L_1

Hence it is enough to show that

$$S_n'/n^{\frac{1}{r}} \to \tilde{0} \ a.s. \iff S_n'/n^{\frac{1}{r}} \to \tilde{0} \quad \text{in probability}$$
$$\iff S_n'/n^{\frac{1}{r}} \to \tilde{0} \quad in \quad L_1$$

.

Since $\sum_{n=1}^{\infty} E\|X_n'\|_{\rho_p}/n^{\frac{2}{n}} < \infty$ by lemma 1, these equivalence hold by applying Theorem 5 in [5] to $\{X_n'\}$ with $\phi(x) = x^2$.

Now it remains to show that (iii) $\Longrightarrow$ (iv). Assume that $S_n/n^{\frac{1}{r}} \to \tilde{0}$ in L_1. Then now

$$E\|S_n\|_{\rho_p} < 2^{r-1}E|\|S_n\|_{\rho_p} - E\|S_n\|_{\rho_p}|^r + 2^{r-1}(E\|S_n\|_{\rho_p})^r.$$

Thus it is enough to show that

$$\frac{1}{n}E|\|S_n\|_{\rho_p} - E\|S_n\|_{\rho_p}|^r \longrightarrow 0$$

From Lemma 4

$$E|\|S_n\|_{\rho_p} - E\|S_n\|_{\rho_p}|^r = E|\|S_n' + S_n''\|_{\rho_p} - E\|S_n" + S_n''\|_{\rho_p}|^r$$

$$\leq E(|\|S_n'\|_{\rho_p} - E\|S_n'\|_{\rho_p}| + |\|S_n''\|_{\rho_p} - E\|S_n''\|_{\rho_p}| + 2E\|S_n''\|_{\rho_p})^r$$

$$\leq 2^{2r-2}E|\|S_n'\|_{\rho_p} - E\|S_n'\|_{\rho_p}|^r + 2^{2r-2}E|\|S_n''\|_{\rho_p} - E\|S_n''\|_{\rho_p}|^r + 2^{2r-1}(E\|S_n''\|_{\rho_p})^r$$

$$\leq 2^{2r-2}(\sum_{i=1}^{n} E\|X_i\|_{\rho_p}^2)^{\frac{r}{2}} + 2^{2r-2}C_r(\sum_{i=1}^{n} E\|X_i\|_{\rho_p}^r + 2^{2r-1}(\sum_{i=1}^{n} E\|X_i''\|_{\rho_p})^r$$

By a standard calculation, we have $\sum_{i=1}^{n} E\|X_i'\|_{\rho_p}^2/n^{2/r} \longrightarrow 0$, $\sum_{i=1}^{n} E\|X_i''\|_{\rho_p}^2/n \longrightarrow 0$ and $\sum_{i=1}^{n} E\|X_i''\|_{\rho_p}^1/n^{1/r} \longrightarrow 0$. Thus the proof is completed. □

Remark 3.1. (1) For i.i.d real valued random variables, Pyke and Root [13] showed that

$$E|X_1|^r < \infty \Longleftrightarrow S_n/n^{\frac{1}{r}} \to 0 \quad \text{a.s} \Longleftrightarrow S_n/n^{\frac{1}{r}} \to 0 \quad \text{in} \quad L_r.$$

(2) For i.i.d. B-valued random variables with $E\|X_1\| < \infty$ for $1 \leq r < 2$, Choi and Sung [2] showed that

$$S_n/n^{\frac{1}{r}} \to 0 \quad \text{a.s.} \Longleftrightarrow S_n/n^{\frac{1}{r}} \to 0 \quad \text{in probability}$$

$$\Longleftrightarrow S_n/n^{\frac{1}{r}} \to 0 \quad \text{in} \quad L_r \Longleftrightarrow S_n/n^{\frac{1}{r}} \to 0 \quad \text{in} \quad L_r.$$

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