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A NEW VERTEX-COLORING EDGE-WEIGHTING OF COMPLETE GRAPHS

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ABSTRACT. Let $G = (V; E)$ be a simple undirected graph without loops and multiple edges, the vertex and edge sets of it are represented by $V = V(G)$ and $E = E(G)$, respectively. A weighting w of the edges of a graph G induces a coloring of the vertices of G where the color of vertex v, denoted $S_v := \sum_{e \ni v} w(e)$. A k-edge-weighting of a graph G is an assignment of an integer weight, $w(e) \in \{1, 2, \ldots, k\}$ to each edge e, such that two vertex-color S_v, S_u be distinct for every edge uv. In this paper we determine an exact 3-edge-weighting of complete graphs k_{3q+1} $\forall q \in \mathbb{N}$. Several open questions are also included.

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1. Introduction

Let $G = (V; E)$ be a simple undirected graph without loops and multiple edges, the vertex and edge sets of it are represented by $V = V(G)$ and $E = E(G)$, respectively. A weighting w of the edges of a graph G induces a coloring of the vertices of G.

A *k-edge-weighting* of a graph G is an assignment of an integer weight, $w(e) \in \{1, 2, ..., k\}$ to each edge e. The edge-weighting is proper if for every edge $e = uv$ incident a proper vertex-coloring and the colors of two vertices u, v are distinct, where the color of a vertex v is defined as the sum of the weights on the edges incident to that vertex. Clearly a graph cannot have a k-edge-weighting and vertex-coloring if it has a component which is isomorphic to K_2 i.e., an edge component. Throughout this paper, we denoted the color of a vertex v by $S_v := \sum_{u \in V(G)} w(uv)$, such that if vw is not in $V(G)$ $w(vw) = 0$.

In particular a 3-edge-weighting of G called *1-2-3-edge weighting and vertex coloring* of G.

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In 2002 [9], *Karonski, Łuczak* and *Thomason* conjectured that every graph without an edge component permits a 1-2-3-edge weighting and vertex coloring and proved their conjecture for the case of 3-colorable graphs [9]. For $k = 2$ is not sufficient as seen for instance in complete graphs and cycles of length not divisible by 4.

In 2004, they proved a graph without an edge component permits an 213-edge-weighting and vertex coloring. In continue a constant bound of $k = 30$ was proved by *Addario-Berry, et.al* in 2007 [1]. In next year, Addario-Berry's group improved this bound to $k = 16$ [2] and also in 2008, *T. Wang* and *Q. Yu* improved k to 13 [10]. Recently, its new bounds are $k = 5$ and $k = 6$ by *Kalkowski, et.al* [7, 8]. In addition, for further study and more historical details, readers can see the recent papers [3-5].

In this paper, we show that there is a proper 1-2-3-edge weighting and vertex coloring for the complete graphs K_{3q+1} $\forall q \in \mathbb{N}$ and obtain an exact weighting w of all edges $e \in E(K_{3q+1})$ and alternatively a proper color of all vertices $v \in V(K_{3q+1})$. We present these main results in the following theorem:

Theorem 1.1. *For the complete graph K_{3q+1} for every integer number q with the vertex set $V(K_{3q+1}) = \{v_1, v_2, \dots, v_{3q+1}\}$, there are a 3-edge weighting $w : E(K_{3q+1}) \to \{1, 2, 3\}$ and a vertex-coloring $s : V(K_{3q+1}) \to \{9q, 9q-1, \dots, 7q+1, 7q, 7q-2, 7q-5, 7q-9, \dots, 3q+11, 3q+7, 3q+4\}$.*

2. Main Results and Algorithm

For a graph complete graph $K_n = (V(K_n); E(K_n))$, a 3-edge weighting is a function $w : E(K_n) \to \{1, 2, 3\}$ such that $S_v = \sum_{u \in V(K_n)} w(uv) \neq S_u$ is a color for any every vertex $v \in V(K_n)$. The edge weighting w implies that $E(K_n) = E(K_n)_1 \cup E(K_n)_2 \cup E(K_n)_3$. Throughout this paper, we denoted the size of edge sets $E(K_n)_1, E(K_n)_2$ and $E(K_n)_3$ by γ_n, β_n and α_n, respectively. Obviously $\gamma_n + \beta_n + \alpha_n = \frac{n(n-1)}{2}$.

Before proving Theorem 1.1, for a general representation of complete graph K_{3q+1} $\forall q \in \mathbb{N}$, we present a proper 3-edge weighting for all edges incident to a vertex v in $3q + 1$ following steps and obtain all summations S_v.

2.1.Algorithm for 1-2-3-edge weighting and vertex coloring of K_{3q+1} ($q \geq 5$): At first, we denote all vertices of K_{3q+1} by $v_1, v_2, \dots, v_{3q+1}$, respectively. Obviously $E(K_{3q+1}) = \{v_i v_j | i \neq j, i, j = 1, 2, \dots, 3q+1\}$ and this implies that $S_{v_i} = \sum_{v_j \in V(K_{3q+1}), i \neq j, j=1,\dots,3q+1} w(v_i v_j) = (S_i)$. Suppose $\forall i = 1, 2, \dots, 3q+1; w(v_i v_i) = 0$. So, we have

Step(1)- For the vertex v_1 label all its edges with 3 ($\forall v_j \in V(K_{3q+1})$ $w(v_1 v_j) = 3$ and $S_{v_1} = \sum_{u \in V(K_{3q+1})} w(uv) = 3(3q)$).

Step(2)- For v_2 label all v_2's edges with 3, except an edge $v_2 v_{3q+1}$, then $\forall j = 1, 3, 4, \dots, 3q$, $w(v_2 v_j) = 3$ and $w(v_2 v_{3q+1}) = 2$. Thus $S_2 = S_1 - 1 = 9q - 1$.

Step(3)- For v_3 label all edges $v_3 v_j$ ($j = 1, 2, 4, \dots, 3q-1$) with 3 and $v_3 v_{3q}, v_3 v_{3q+1}$ with 2. Thus $S_3 = S_2 - 1 = 9q - 2$.

Step(4)- For v_4 label all edges v_4v_j $(j = 1, 2, 3, 5, \dots, 3q-1)$ with 3, v_4v_{3q} with 2 and v_4v_{3q+1} with 1. Thus $S_4 = S_3 - 1 = 9q - 3$.

Step(s)- $\forall s = 5, 7, \dots, 2q-1$ label all edges v_sv_j $(j = 1, 2, \dots, s-1, s+1, \dots, 3q - [\frac{s}{2}])$ with 3, label $v_sv_{3q-[\frac{s}{2}]+1}, v_sv_{3q-[\frac{s}{2}]+2}$ with 2 and all edge v_sv_j $(j = 3q - [\frac{s}{2}] + 3, \dots, 3q+1)$ with 1. Thus $S_s = 3 \times (3q - [\frac{s}{2}] - 1) + 2 \times 2 + 1 \times ([\frac{s}{2}] - 1) = 9q - 2[\frac{s}{2}] = S_{s-2} - 2$.

Step(r)- $\forall r = 6, 8, \dots, 2q$ label all edges v_rv_j $(j = 1, 2, \dots, r-1, r+1, \dots, 3q+1-[\frac{r}{2}])$ with 3, label $v_rv_{3q-[\frac{r}{2}]+2}$ with 2 and all edge v_rv_j $(j = 3q-[\frac{r}{2}]+3, \dots, 3q+1)$ with 1. Thus $S_r = 3\times(3q+1-[\frac{r}{2}]-1)+2\times1+1\times([\frac{r}{2}]-1) = 9q+1-r = S_{r-2}-2$.

Step(2q+1)- For v_{2q+1}, all edges $v_{2q+1}v_j$ $(j = 1, 2, \dots, 2q)$ were labeled with 3. Thus label all edge $v_{2q+1}v_j$ $(j = 2q+2, \dots, 3q+1)$ with 1. Thus $S_{2q+1} = 3 \times 2q + 2 \times 0 + 1 \times (q) = 7q = S_{2q} - 1$.

Step(2q+2)- For v_{2q+2}, all edges $v_{2q+2}v_j$ $(j = 1, 2, \dots, 2q-2)$ were labeled with 3, the edge $v_{2q+2}v_{2q-1}$, $v_{2q+2}v_{2q}$ were labeled with 2 and $v_{2q+2}v_{2q+1}$ were labeled with 1. Thus label all edges $v_{2q+2}v_j$ $(j = 2q+3, \dots, 3q+1)$ with 1 and $S_{2q+2} = 3 \times (2q-2) + 2 \times 2 + 1 \times (q) = 7q - 2 = S_{2q+1} - 3$.

Step(t)- $\forall t = 2q+3, \dots, 3q-2$ all edges v_tv_j $(j = 1, \dots, 6q+2-2t)$ were labeled with 3, three edges $v_tv_{6q+2-2t+i}$ for $i = 1, 2, 3$ were labeled with 2 and all edges v_tv_j $(j = 6q-2t+6, \dots, t-1)$ were labeled with 1. Thus label all edges v_tv_j $(j = t+1, \dots, 3q+1)$ with 1 and $S_t = 3 \times (6q+2-2t) + 2 \times 3 + 1 \times (2t-3q-5) = 15q + 7 - 4t = S_{t-1} - 4$.

Step(3q-1)- v_{3q-1}, $v_{3q-1}v_j$ $(j = 1, 2, 3, 4)$ were labeled with 3 and $v_{3q-1}v_5$, $v_{3q-1}v_6, v_{3q-1}v_7$ were labeled with 2 and all edge $v_{3q-1}v_j$ $(j = 8, \dots, 3q-2)$ were labeled with 1. Thus label $v_{3q-1}v_{3q}, v_{3q-1}v_{3q+1}$ with 1 and $S_{3q-1} = 3 \times 4 + 2 \times 3 + 1 \times (3q-7) = 3q + 11 = S_{3q-2} - 4$.

Step(3q)- For v_{3q}, $v_{3q}v_1, v_{3q}v_2$ were labeled with 3 and $v_{3q}v_3, v_{3q}v_4, v_{3q}v_5$ were labeled with 2 and all edge $v_{3q}v_j$ $(j = 6, \dots, 3q-1)$ were labeled with 1. Thus label the edge $v_{3q+1}v_{3q}$ with 1 and $S_{3q} = 3 \times 2 + 2 \times 3 + 1 \times (3q-5) = 3q + 7 = S_{3q-1} - 4$.

Step(3q+1)- Obviously, for the vertex v_{3q+1}, the edge $v_{3q+1}v_1$ were labeled with 3 and $v_{3q+1}v_2, v_{3q+1}v_3$ were labeled with 2 and all edge $v_{3q+1}v_j$ $(j = 4, \dots, 3q)$ were labeled with 1. Thus $S_{3q+1} = 3 + 2 \times 2 + 1 \times (3q-3) = 3q + 4 = S_{3q} - 3$.

Now, we start the proof of main theorem as follow.

Proof. Let K_{3q+1} be a complete graph as order $3q+1$ for every integer number q, with the vertex set $V(K_{3q+1}) = \{v_1, v_2, \dots, v_{3q+1}\}$ and the edge set $E(K_{3q+1}) = \{e_{ij} = v_iv_j | v_i, v_j \in V(K_{3q+1})\}$ $(|V(K_{3q+1})| = 3q+1$ and $|E(K_{3q+1})| = \frac{3q(3q+1)}{2})$. It is easy to see that the above edge-weighting is a nice and proper 3-edge weighting w (or 1-2-3-edge weighting and vertex coloring) of K_{3q+1} $(q \geq 1)$. Because $\forall v_i \in V(K_{3q+1})$ and for every edge $e_{ij} = v_iv_j$ incident to v_i, we have an integer weight $w(v_iv_j) \in \{1, 2, 3\}$ such that this weighting naturally induces two distinct vertex coloring S_{v_i} and S_{v_j} to vertices v_i, v_j.

An every edge $e_{ij} = v_iv_j$ $(i, j = 1, 2, \dots, 3q + 1, i \geq j)$ weighted in $Step(i)$ of above 3-edge weighting w. Also, from above 3-edge weighting w, one can see that all vertex color belong to the color set $\{9q, 9q-1, \dots, 7q+1, 7q, 7q-2, 7q-5, 7q-9, \dots, 3q+11, 3q+7, 3q+4\}$ (For example, see Figure 1, 2 and 3). In Figure 1, 2 and 3, a proper 1-2-3-edge weighting and vertex coloring of complete graphs K_4, K_7, K_{10}, K_{13} and K_{16} are shown.

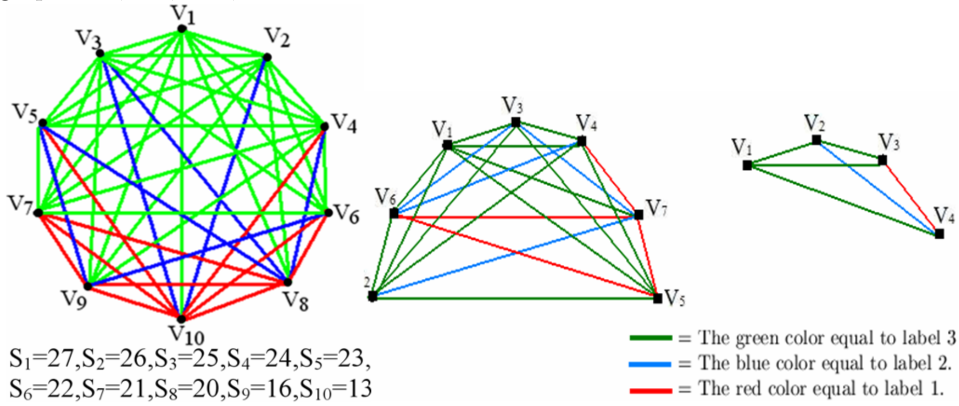

FIGURE 1. The 1-2-3-edge weighting and vertex coloring of complete graphs K_4, K_7 and K_{10}.

Thus, the 3-edge weighting $w : E(K_{3q+1}) \to \{1, 2, 3\}$ and the vertex coloring $s : V(K_{3q+1}) \to \{9q, 9q-1, \dots, 7q+1, 7q, 7q-2, 7q-5, 7q-9, \dots, 3q+11, 3q+7, 3q+4\}$ is a proper 1-2-3-edge weighting and vertex coloring of K_{3q+1} $\forall q \in \mathbb{N}$ and this complete the proof of theorem. $\square$

By using the proof of Theorem 1.1 (the 3-edge weighting w and the vertex coloring s), one can see that the number of all edge weigh 3, 2 and 1 are equal to $\alpha_{3q+1} = 3q^2 + 1 = (|E(K_{3q+1})_3|)$, $\beta_{3q+1} = 3q - 2 = (|E(K_{3q+1})_2|)$ and $\gamma_{3q+1} = \frac{3q^2-3q+2}{2} = (|E(K_{3q+1})_1|)$. For example, $E(K_{3q+1})_2 = \{v_2v_{3q+1}, v_4v_{3q}, v_5v_{3q-1}, v_5v_{3q}, v_6v_{3q-1}, v_7v_{3q-2}, v_sv_{3q-1}, v_8v_{3q-2}, \dots, v_{2q-1}v_{2q+2}, v_{2q-1}v_{2q+3}, v_{2q}v_{2q+2}, v_{2q+2}v_{2q-3}, v_{2q+2}v_{2q-2}, v_{2q+2}v_{2q-1}, v_{2q+3}v_{2q-3}, v_{2q+3}v_{2q-2}, v_{2q+3}v_{2q-1}, v_{2q+4}v_{2q-5}, v_{2q+4}v_{2q-4}, v_{2q+4}v_{2q-3}, \dots, v_{3q-2}v_6, v_{3q-2}v_7, v_{3q-2}v_8, v_{3q-1}v_5, v_{3q-1}v_6, v_{3q-1}v_7, v_{3q}v_3v_{3q}v_4, v_{3q}v_5, v_{3q+1}v_2, v_{3q+1}v_3\}$.

3. Conclusions and Conjectures

We conclude our paper with the following open questions and conjectures:

Corollary 3.1 (The 1-2-3-conjecture [6, 9])**.** *Every connected graph $G = (V, E)$ non-isomorph to K_2 (with at least two edges) has an edge labeling $f : E \longrightarrow \{1, 2, 3\}$ and vertex coloring $S : V \longrightarrow \{n-1, ..., 3n-3\}$.*

Corollary 3.2 (n vertex coloring)**.** *There are distinct numbers of S_v's, $v \in V(G)$, of a graph G of order n, for a 1-2-3-edge labeling and vertex coloring.*

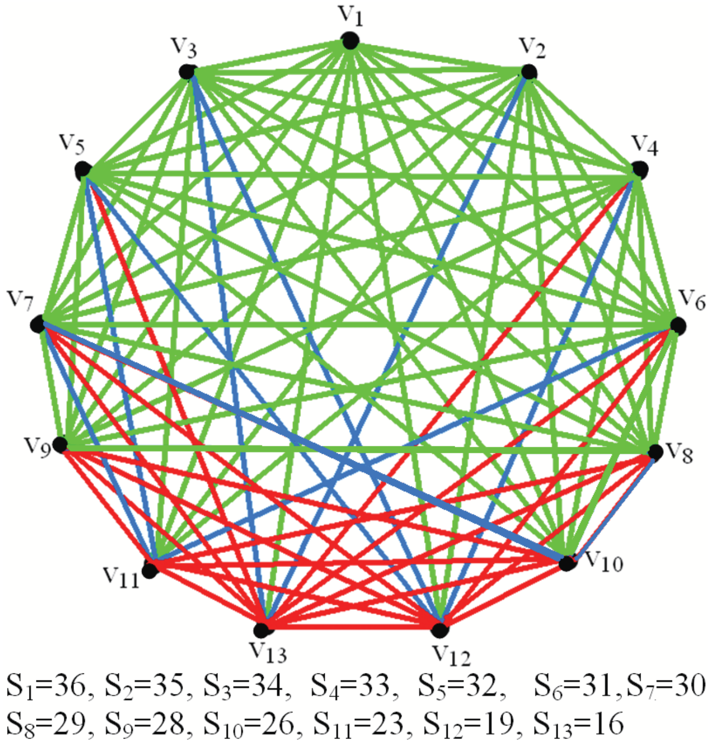

FIGURE 2. The 1-2-3-edge weighting and vertex coloring of K_{13}.

Corollary 3.3 (Proper vertex coloring)**.** *For all graph G of order n, there are the $\chi(G)$ numbers of S_v's, $v \in V(G)$, with this 1-2-3-edge labeling and vertex Coloring. Where $\chi(G)$ is the number colors of the vertices on the graph G.*

In this parer, we show that the complete graph K_{3q+1} $(q \geq 1)$, recognize in three Conjecture 3.1, 3.2 and 3.3.

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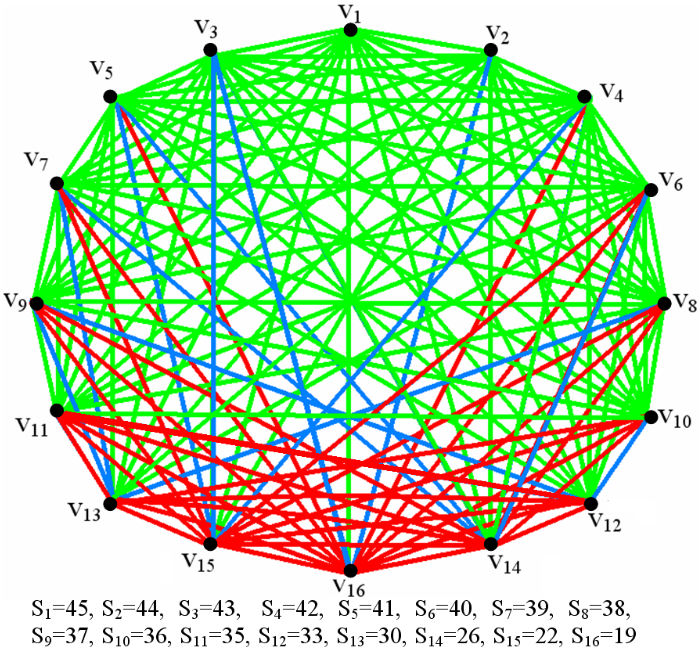

FIGURE 3. The 1-2-3-edge weighting and vertex coloring of complete graph K_{16}.

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