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THE IDENTITY-SUMMAND GRAPH OF COMMUTATIVE SEMIRINGS

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Abstract. An element r of a commutative semiring R with identity is said to be identity-summand if there exists $1 \neq a \in R$ such that $r+a=1$. In this paper, we introduce and investigate the identity-summand graph of R, denoted by $\Gamma(R)$. It is the (undirected) graph whose vertices are the non-identity identity-summands of R with two distinct vertices joint by an edge when the sum of the vertices is 1. The basic properties and possible structures of the graph $\Gamma(R)$ are studied.

1. Introduction

One of the associated graphs to a ring R is the zero-divisor graph; it is a simple graph with vertex set $Z(R) \setminus \{0\}$, and two vertices x and y are adjacent if and only if $xy = 0$ which is due to Anderson and Livingston [8]. This graph was first introduced by Beck, in [11], where all the elements of R are considered as the vertices. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions [3, 8-9, 18, 20-21]. Recently, the study of graphs of rings are extended to include semirings as in [16].

As a generalization of rings, structure of semirings have proven to be useful tools in various disciplines. They have important applications in mathematics and theoretical computer science [19, 22]. From now on let R be a commutative semiring with identity. We define another graph on R, $\Gamma(R)$, with vertices as elements of $S^*(R) = S(R) \setminus \{1\}$ (where $S(R) = \{r \in R : r+a = 1$ for some $1 \neq a \in R\}$), where two distinct vertices a and b are adjacent if and only if $a+b=1$. We will make an intensive study on the graph $\Gamma(R)$. We recommend to the reader the references [1-2, 4-7, 10, 13-15] where "+" is used in order to connect edges when R is a commutative ring, however this graph is of a different kind.

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Here is a brief summary of our paper. The main purpose of Section 2 is to investigate some properties of I-semirings and its minimal prime co-ideals which be useful in the later sections. In Section 3, we look at the connectedness, the diameter and the girth of the graph $\Gamma(R)$. We completely characterize the diameter and the girth of this graph. Section 4 is devoted when the graph $\Gamma(R)$ is a complete bipartite graph. Indeed, it is shown that for an I-semiring R, $\Gamma(R)$ is complete bipartite if and only if there exist two distinct prime co-ideals P_1 and P_2 of R such that $P_1 \cap P_2 = \{1\}$. In Section 5, we will investigate chromatic number and clique number of the graph $\Gamma(R)$. For example, it is shown that for an I-semiring R, $\chi(\Gamma(R))$ is finite if and only if the co-ideal $\{1\}$ is a finite intersection of prime co-ideals. Section 6 is devoted to study planar property of $\Gamma(R)$. A number of basic results of planar property of $\Gamma(R)$ are given.

In order to make this paper easier to follow, we recall in this section various notions which will be used in the sequel. For a graph Γ by $E(\Gamma)$ and $V(\Gamma)$ we denote the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $d(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$ and $d(a, a) = 0$). The diameter of graph Γ, denoted by diam(Γ), is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. We denote the complete graph on n vertices by K_n. The girth of a graph Γ, denoted g(Γ), is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise; g(Γ) $= \infty$. An edge for which the two ends are the same is called a loop at the common vertex. For r a nonnegative integer, an r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) with part sizes m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a star graph. We define a coloring of a graph G to be an assignment of colors (elements of some set) to vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as an n-coloring. If there exists an n-coloring of a graph G, then G is called n-colorable. The minimum n for which a graph G is n-colorable is called the chromatic number of G and is denoted by $\chi(G)$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $w(G)$, is called the clique number of G.

A commutative semiring R is defined as an algebraic system $(R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b + c) = ab+ac$ for all $a, b, c \in R$, and there exist $0, 1 \in R$ such that $r+0 = r$, $r0 = 0r = 0$ and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

Definition 1.1. Let R be a semiring.

(1) A non-empty subset I of R is called *co-ideal*, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$. A co-ideal I of R is called strong co-ideal provided that $1 \in I$ (clearly, $0 \in I$ if and only if $I = R$) [17, 19, 22].

(2) A co-ideal I of R is called *subtractive* if $x, xy \in I$, then $y \in I$ for all $x, y \in R$ (so every subtractive co-ideal is a strong co-ideal) [17].

(3) A proper co-ideal P of R is called *prime* if $x + y \in P$, then $x \in P$ or $y \in P$ for all $x, y \in R$ [17].

(4) If D is an arbitrary nonempty subset of R, then the set $F(D)$ consisting of all elements of R of the form $d_1 d_2 \cdots d_n + r$ (with $d_i \in D$ for all $1 \leq i \leq n$ and $r \in R$) is a co-ideal of R containing D [17, 19, 22].

(5) A semiring R is called *co-semidomain*, if $a + b = 1$ ($a, b \in R$), then either $a = 1$ or $b = 1$ [17].

A strong co-ideal I of a semiring R is called a *partitioning strong co-ideal* (= Q-strong co-ideal) if there exists a subset Q of R such that $R = \bigcup\{qI : q \in Q\}$, where $qI = \{qt : t \in I\}$ and if $q_1, q_2 \in Q$, then $(q_1 I) \cap (q_2 I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q-strong co-ideal of a semiring R and let $R/I = \{qI : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 I) \oplus (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 I + q_2 I) \subseteq q_3 I$, and $(q_1 I) \odot (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 q_2) I \subseteq q_3 I$ (note that $q_1 I = q_2 I$ if and only if $q_1 = q_2$). Let q_e be the unique element in Q such that $1 \in q_e I$. Then $q_e I = I$ and $q_e I$ is the identity of R/I [17].

2. Some properties of I-semirings

In this section we collect some properties concerning I-semirings which are useful in the later sections.

Definition 2.1. A semiring R is said to be identity-semiring (= I-semiring) if $1 + x = 1$ for all $x \in R$.

One can easily show that R is an I-semiring if and only if $\{1\}$ is a strong co-ideal of R.

Example 2.2. (1) Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ a semiring with $1_R = X$, where $P(X) =$ the set of all subsets of X. It is easy to see that R is an I-semiring.

(2) Assume that $\mathbb{Z}^+$ is the set of all non-negative integers and let $R = \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $(R, +, \cdot)$ with the usual operations of addition and multiplication is a semiring which is not an I-semiring.

The following proposition gives a source of examples of I-semirings.

Proposition 2.3. *If J is a Q-strong co-ideal of a semiring R, then R/J is an I-semiring.*

Proof. Let q_e be the unique element in Q such that $q_eJ = J$ is the identity in R/J. We need to show that $q_eJ \oplus qJ = q_eJ$ for each $q \in Q$. Since J is a co-ideal of R, $q_eJ + qJ \subseteq q_eJ$. Let $q_eJ \oplus qJ = q'J$, where $q_eJ + qJ \subseteq q'J$ for some $q' \in Q$. Hence $q'J \cap q_eJ \neq \emptyset$. This shows that $q_eJ \oplus qJ = q_eJ$. □

Definition 2.4. An element x of a semiring R is called co-nilpotent if $nx = 1$ for some positive integer n. A semiring R is said to be co-reduced semiring if 1 is the only co-nilpotent element of R.

We will frequently need the following proposition.

Proposition 2.5. *Let R be a commutative I-semiring. Then the following statements hold*:

(1) *R is co-reduced*;

(2) *If J is co-ideal, then J is a strong co-ideal of R. Moreover, if $xy \in J$, then $x, y \in J$ for every $x, y \in R$. In particular, J is subtractive*;

(3) *The set $(1 : x) = \{r \in R : r + x = 1\}$ is a strong co-ideal of R for every $x \in S(R)$.*

Proof. (1) Let $x \in R$. We may assume that $x \neq 1$. By assumption, $1 + x = 1$. If $nx = x + x + \cdots + x = 1$, then $x(1 + 1 + \cdots + 1) = x = 1$ which is a contradiction.

(2) We may assume that $J \neq \{1\}$. So there exists $1 \neq x \in J$. Since J is co-ideal, $1 = 1 + x \in J$. Thus J is a strong co-ideal of R. Now let $xy \in J$ $(x, y \in R)$. Then $x + xy \in J$; so $x = x(1 + y) \in J$. Similarly, $y \in J$. The "in particular" statement is clear.

(3) Clearly, $1 \in (1 : x)$. If $a, b \in (1 : x)$, then $a + x = 1$ and $b + x = 1$, implying $ab + ax + bx + x^2 = 1$. Since $(ab + x)(1 + x)(1 + a)(1 + b) = 1$ and $1 + x = 1 + a = 1 + b = 1$, $ab + x = 1$. Thus $ab \in (1 : x)$. Let $r \in R$ and $a \in (1 : x)$. Since R is an I-semiring and $a + x = 1$, $a + r + x = 1 + r = 1$. Thus $a + r \in (1 : x)$, as required. □

The following example shows that co-reduced semirings are not necessarily I-semiring.

Example 2.6. Let $R = (\mathbb{Z}^+, +, \cdot)$. It is clear that R is a co-reduced semiring which is not an I-semiring.

The remainder of this section is concerned to study the minimal prime co-ideals of I-semirings.

Remark 2.7. Assume that P is a prime co-ideal of a semiring R and let $\sum$ be the set of those prime co-ideals of R which are contained in P. Then $P \in \sum$ and the set $\sum$ of prime co-ideals of R (partially ordered by reverse inclusion and using Zorn's Lemma) has at least one minimal element, and any such minimal element of $\sum$ is a prime co-ideal. Thus P has a minimal prime co-ideal.

Theorem 2.8. *Let R be an I-semiring.*

(1) *If* $\{P_\alpha\}_{\alpha\in\Lambda}$ *is the set of all prime co-ideals of* R, *then* $\cap_{\alpha\in\Lambda}P_\alpha = \{1\}$.

(2) *If* $P_1, \ldots, P_n$ *are the only distinct minimal prime co-ideals of* R, *then* $\cap_{i=1}^{n}P_i = \{1\}$ *and* $\{1\} \neq \cap_{1\leq i\leq n, i\neq j}P_i$ *for each* $1 \leq j \leq n$.

Proof. (1) By Proposition 2.5, it is clear that $\{1\} \subseteq \cap_{\alpha\in\Lambda}P_\alpha$. For the reverse inclusion, let $x \in \cap_{\alpha\in\Lambda}P_\alpha$. Assume that $x \neq 1$. Set $\sum = \{I : x \notin I,\ I \text{ is a co-ideal of } R\}$. Since $\{1\} \in \sum$, $\sum \neq \emptyset$. Of course, the relation of inclusion, $\subseteq$, is a partial order on $\sum$, and then the partially ordered set $(\sum, \subseteq)$ has a maximal element by Zorn's Lemma, say K. Since $x \notin K$, $K \neq R$. We claim that K is prime. Let $a + b \in K$ such that $a \notin K$ and $b \notin K$. Since K is properly contained in $F(K\cup\{a\})$ and $F(K\cup\{b\})$, $x \in F(K\cup\{a\})\cap F(K\cup\{b\})$. Hence $x = r_1 + k_1a^n = r_2 + k_2b^m$ for some $r_1, r_2 \in R$, $k_1, k_2 \in K$ and $n, m \in \mathbb{N}$. Since $k_1(a + b)^n = k_1a^n + bt \in K$ for some $t \in R$, $x + bt = r_1 + k_1a^n + bt \in K$. Thus $(x + b)(1 + t) = x + bt + xt + b \in K$. By Proposition 2.5, we must have $x + b \in K$. Thus $(b + x)^2 \in K$. Hence $b^2 + x = (b^2 + x)(1 + b)(1 + x)(1 + b) = (b+x)^2+c \in K$, where $c \in R$. Inductively $b^i+x \in K$ for each $i \in \mathbb{N}$. In particular $b^m+x \in K$. As $k_2 \in K$, $k_2+x \in K$. Hence $(k_2+x)(b^m+x) \in K$. Therefore $k_2b^m + x = (k_2b^m + x)(k_2 + 1)(x + 1)(b^m + 1) = (k_2 + x)(b^m + x) + d \in K$, where $d \in R$. So $x + r_2 + k_2b^m \in K$; thus $x = x + x \in K$, a contradiction. Therefore K is prime, and so $x \in K$ that is a contradiction. Hence $x = 1$, as needed.

(2) Clearly $\cap_{\alpha\in\Lambda}P_\alpha = \{1\} \subseteq \cap_{i=1}^{n}P_i$. Let $x \in \cap_{i=1}^{n}P_i$. If $x \notin \cap_{\alpha\in\Lambda}P_\alpha$, then there exists $\alpha \in \Lambda$ such that $x \notin P_\alpha$. Obviously, P_α is not minimal (because $x \notin P_\alpha$). By Remark 2.7, P_α contains a minimal prime co-ideal P' of R. Since P' is minimal, $P' = P_i$ for some $1 \leq i \leq n$. Thus $x \in P' \subseteq P_\alpha$ a contradiction. Therefore $x \in \cap_{\alpha\in\Lambda}P_\alpha$. Hence $\cap_{\alpha\in\Lambda}P_\alpha = \{1\} = \cap_{i=1}^{n}P_i$. Now let $\{1\} = \cap_{1\leq i\leq n, i\neq j}P_i$ for some $1 \leq j \leq n$, so $\cap_{1\leq i\leq n, i\neq j}P_i \subseteq P_j$. Since for each $i \neq j$, $P_i \nsubseteq P_j$, there is a $x_i \in P_i$ such that $x_i \notin P_j$. As $\sum_{i\neq j} x_i \in \cap_{1\leq i\leq n, i\neq j}P_i \subseteq P_j$, it is easy to see that $x_i \in P_j$ for some $i \neq j$, which is a contradiction. Therefore $P_i \subseteq P_j$ for some $1 \leq i \leq n$. Thus we have a contradiction to the minimality of P_j, and hence $\{1\} \neq \cap_{1\leq i\leq n, i\neq j}P_i$ for each $1 \leq j \leq n$. □

3. Examples and basic structure of $\Gamma(R)$

We start this section with the following proposition.

Proposition 3.1. *Let* R *be a semiring. Then* $\Gamma(R) = \emptyset$ *if and only if* R *is a co-semidomain.*

Proof. This follows directly from the definitions. □

Here we consider the following question: If R is a semiring, is $\Gamma(R)$ connected? The following is an example of a semiring R where $\Gamma(R)$ is not connected.

Example 3.2. Assume that $\mathbb{Z}^+$ is the set of all nonnegative integers and let $R = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$. Then $(R, +, \cdot)$ with the usual operations is a semiring and $S^*(R) = \{(1,0,0), (0,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,0)\}$. An inspection will show that $\Gamma(R)$ is not connected.

It is important for us to be able to recognize when the identity-summand graph of semirings are connected. The concept of "R is an I-semiring" helps us to do this. The next theorem gives a more explicit description of the diameter of this graph.

Theorem 3.3. *Let R be an I-semiring. Then the following statements hold*:
(1) $\Gamma(R)$ *is a connected graph with* $\mathrm{diam}(\Gamma(R)) \leq 3$;
(2) $\Gamma(R)$ *contains no loop*;
(3) *If* $|S^*(R)| \geq 3$, *then* $\Gamma(R)$ *is not a complete graph*;
(4) *If* $|S^*(R)| \geq 3$, *then* $\mathrm{diam}(\Gamma(R)) = 2$ *or* 3.

Proof. (1) Let $x, y \in S^*(R)$ be distinct. If $x + y = 1$, then $d(x, y) = 1$. So suppose that $x + y \neq 1$. By Proposition 2.5, $x + x \neq 1$ and $y + y \neq 1$. Since $x, y \in S^*(R)$, $x + a = 1$ and $y + b = 1$ for some $a, b \in S^*(R) \setminus \{x, y\}$. If $a = b$, then $x - a - y$ is a path of length 2 in $\Gamma(R)$. Thus we may assume that $a \neq b$. If $a + b = 1$, then $x - a - b - y$ is a path of length 3, and hence $d(x, y) \leq 3$. If $a + b \neq 1$, then $a + b \neq x$ (otherwise, $x + x = x + a + b = 1$, a contradiction). Similarly, $a + b \neq y$. Therefore, $x - a + b - y$ is a path of length 2. Hence $\mathrm{diam}(\Gamma(R)) \leq 3$.

(2) Follows from Proposition 2.5.

(3) Let $\Gamma(R)$ be complete, and let a, b and c be distinct elements of $S^*(R)$. Since $\Gamma(R)$ is complete, $a + b = 1$ and $a + c = 1$. Thus $bc \in (1 : a)$, since $(1 : a)$ is a strong co-ideal of R by Proposition 2.5. If $bc = 1$, then $c = c(1 + b) = c + bc = 1$, a contradiction. So assume $bc \neq 1$. Hence $bc \in S^*(R)$. If $bc = c$, then $1 = b + c = bc + b = b(c + 1) = b$ that is a contradiction. So $bc \neq c$. Since $\Gamma(R)$ is complete, $bc + c = 1$; hence $c = 1$ which is a contradiction. Therefore $\Gamma(R)$ is not complete.

(4) Follows from (1), (2) and (3). □

The following is an example of a semiring R where $\Gamma(R)$ is connected but R is not an I-semiring.

Example 3.4. Consider the semiring $R = (\mathbb{Z}^+ \times \mathbb{Z}^+, +, \cdot)$. Then $S^*(R) = \{(1, 0), (0, 1)\}$ gives $\Gamma(R)$ is connected but R is not an I-semiring.

A cycle of a graph is a path such that the start and end vertices are the same. For a graph G, it is well-known that if G contains a cycle, then $\mathrm{g}(G) \leq 2\ \mathrm{diam}(G) + 1$.

Theorem 3.5. *Let R be an I-semiring. If $\Gamma(R)$ contains a cycle, then*

$$\mathrm{g}(\Gamma(R)) \leq 4.$$

Proof. Suppose that $\Gamma(R)$ contains a cycle. We may assume that $\mathrm{g}(\Gamma(R)) \leq 7$. Suppose that $\mathrm{g}(\Gamma(R)) = n$, where $n \in \{5, 6, 7\}$ and let $x_1 - x_2 - \cdots - x_n - x_1$ be a cycle of minimum length. Since x_1 is not adjacent to x_3, $x_1 + x_3 \neq 1$. Let $x_1 + x_3 \neq x_i$ for $1 \leq i \leq n$. Then $x_2 - x_3 - x_4 - x_1 + x_3 - x_2$ is a 4-cycle, a contradiction. Therefore $x_1 + x_3 = x_i$ for some $1 \leq i \leq n$. We divide the proof into five cases.

Case 1: If $x_1 + x_3 = x_1$, then $x_1 - x_2 - x_3 - x_4 - x_1$ is a 4-cycle, a contradiction.

Case 2: If $x_1 + x_3 = x_2$, then $x_2 - x_3 - x_4 - x_2$ is a 3-cycle, a contradiction.

Case 3: If $x_1 + x_3 = x_3$, then $x_1 - x_2 - x_3 - x_n - x_1$ is a 4-cycle, a contradiction.

Case 4: If $x_1 + x_3 = x_4$, then $x_2 - x_3 - x_4 - x_2$ is a 3-cycle, a contradiction.

Case 5: If $x_1 + x_3 = x_n$, then $x_2 - x_3 - x_4 - x_n - x_2$ is a 4-cycle, a contradiction.

Therefore, there must be a shorter cycle in $\Gamma(R)$, and $g(\Gamma(R)) \leq 4$. □

A cycle graph is a graph that consists of a single cycle. Our next result characterizes the identity-summand graphs that are a cycle graph.

Theorem 3.6. *The only cycle identity-summand graph of a commutative I-semiring is $K_{2,2}$.*

Proof. By Theorem 3.3, there is no 3-cycle graph and there are no cycle graphs with five or more vertices by Theorem 3.5. Thus the only cycle graph is $K_{2,2}$. □

A vertex x of a connected graph G is a cut-point of G if there are vertices a and b of G such that x is in every path from a to b (and $x \neq a, x \neq b$). Equivalently, for a connected graph G, x is a cut-point of G if $G - \{x\}$ is not connected. The following example shows that there exist I-semirings R_1 and R_2 such that $\Gamma(R_1)$ has cut-points and $\Gamma(R_2)$ is a star graph.

Example 3.7. (1) Let $R_1 = \{0, 1, 2, 3, 5, 6, 10, 15, 30\}$. Then $(R_1, \mathrm{gcd}, \mathrm{lcm})$ (take $\mathrm{gcd}(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$) is an I-semiring and $S^*(R_1) = \{2, 3, 5, 6, 10, 15\}$. In $\Gamma(R_1)$, an inspection will show that $2, 3$ and 5 are cut-points.

(2) Let $R_2 = \{0, 1, 2, 5, 10, 25, 50\}$. Then $(R_2, \mathrm{gcd}, \mathrm{lcm})$ (take $\mathrm{gcd}(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$) is an I-semiring. It can be seen than $\Gamma(R_2)$ is a star graph.

4. Bipartite graphs

In this section, we want to determine when the identity-summand graph $\Gamma(R)$ is a complete bipartite graph. Now we start with the following definition.

Definition 4.1. Let P be a prime co-ideal of an I-semiring R. We say that P is an associated prime co-ideal of R precisely when there exists $x \in R$ with $(1 : x) = P$.

The set of associated prime co-ideals of R is denoted by co-Ass(R).

Example 4.2. Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ a semiring with $1_R = X$, where $P(X) =$ the set of all subsets of X. Then $S^*(R) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and an inspection will show that

$$P_1 = (1 : \{a, b\}) = \{\{c\}, \{b, c\}, \{a, c\}, X\},$$

$$P_2 = (1 : \{a, c\}) = \{\{b\}, \{a, b\}, \{c, b\}, X\},$$

$$P_3 = (1 : \{b, c\}) = \{\{a\}, \{a, c\}, \{a, b\}, X\}$$

are prime strong co-ideals of R and co-Ass$(R) = \{P_1, P_2, P_3\}$.

We are now in a position to show a finer relationship between vertices which are adjacent to all the other vertex and co-Ass of a semiring.

Theorem 4.3. *Let R be an I-semiring, and let $a \in S^*(R)$ be a vertex of $\Gamma(R)$ which is adjacent to every other vertex in $\Gamma(R)$. Then $(1 : a)$ is a maximal element in $\Delta = \{(1 : x) : x \in S^*(R)\}$ and $(1 : a) \in$ co-Ass(R).*

Proof. Suppose $(1 : a)$ is not maximal in Δ. So there is a vertex x in $\Gamma(R)$ such that $(1 : a) \subset (1 : x)$. Since a is adjacent to every other vertex in $\Gamma(R)$, $x \in (1 : a)$. Since $(1 : a) \subset (1 : x)$, $x = x + x = 1$ which is a contradiction. It remains to show that $(1 : a)$ is a prime strong co-ideal of R. Let $x + y \in (1 : a)$ such that $x, y \notin (1 : a)$. Thus $x + y + a = 1$ and $y + a \neq 1$. So $x, y + a \in S^*(R)$. Since a is adjacent to every other vertex in $\Gamma(R)$, we have $a + x = 1$ which is a contradiction. □

Theorem 4.4. *Let R be an I-semiring.*

(1) If $\Gamma(R)$ is a complete r-partite graph with $r \geq 3$, then at most one part has more than one vertex.

(2) If $\Gamma(R)$ is a complete r-partite graph, then $\Gamma(R)$ is a complete bipartite graph.

Proof. (1) Let $V_1, V_2, \ldots, V_r$ be parts of $\Gamma(R)$. Without loss of generality, let V_1, V_2 have more than one vertex. Let $a \in V_1$ and $b \in V_2$. Since $r \geq 3$, there exists a part, say V_3, such that $V_3 \neq V_1$ and $V_3 \neq V_2$. Let $z \in V_3$. Since $\Gamma(R)$ is complete r-partite, $(1 : z) \subseteq (1 : a) \cup (1 : b)$. Now we show that $(1 : z) \subseteq (1 : a)$ or $(1 : z) \subseteq (1 : b)$. Let $x \in (1 : z) \setminus (1 : a)$ and $y \in (1 : z) \setminus (1 : b)$. Since $(1 : z)$ is a strong co-ideal of R, $xy \in (1 : z) \subseteq (1 : a) \cup (1 : b)$. If $xy \in (1 : a)$, then $x, y \in (1 : a)$, by Proposition 2.5 which is contradiction. If $xy \in (1 : b)$, then $x, y \in (1 : b)$, by Proposition 2.5, another contradiction. Thus we may assume that $(1 : z) \subseteq (1 : a)$. Since $|V_1| \geq 2$, there exists $c \in V_1$ with $a \neq c$. Thus $c \in (1 : z) \setminus (1 : a)$ that is a contradiction.

(2) Let $V_1, V_2, \ldots, V_r$ be parts of $\Gamma(R)$ and $r \geq 3$. By (1), at most one parts of $\Gamma(R)$ has more than one vertex. Let $|V_1| > 1$, and $|V_i| = 1$ for $2 \leq i \leq r$. Let $a, b \in V_1$ and $c_i \in V_i$. Since $\Gamma(R)$ is a complete r-partite graph, $a + c_3 = 1$ and $c_3 + c_2 = 1$. Since $a, c_2 \in (1 : c_3)$ and $(1 : c_3)$ is a strong co-ideal of R, $ac_2 \in (1 : c_3)$; so $ac_2 + c_3 = 1$. If $ac_2 = 1$ then $a = ac_2 + a = 1 + a = 1$, a contradiction. Thus $ac_2 \in S^*(R)$. Since $ac_2 + a = a \neq 1$, ac_2, a are not

adjacent. This implies $ac_2 \in V_1$. Since $\Gamma(R)$ is a complete r-partite graph, $ac_2 \in V_1$ and $c_2 \in V_2$, $ac_2 + c_2 = 1$, where $ac_2 + c_2 = c_2(a+1) = c_2$, which is a contradiction. □

Theorem 4.5. *Let R be an I-semiring. Then $\Gamma(R)$ is complete bipartite if and only if there exist two distinct prime co-ideals P_1 and P_2 of R such that $P_1 \cap P_2 = \{1\}$.*

Proof. Let P_1, P_2 be two prime co-ideals of R such that $P_1 \cap P_2 = \{1\}$. Since $P_1 + P_2 \subseteq P_1 \cap P_2, P_1 + P_2 = \{1\}$. It follows that $P_1 = (1 : a)$ and $P_2 = (1 : b)$ for each $1 \neq a \in P_2, 1 \neq b \in P_1$. Thus $P_1, P_2 \in$ co-Ass(R); hence $P_1 \cup P_2 \subseteq S(R)$. Now we show that $S(R) \subseteq P_1 \cup P_2$. If $1 \neq a \in S(R) \setminus P_1 \cup P_2$, then there exists $1 \neq b \in R$ such that $a + b = 1 \in P_1 \cap P_2$. Since P_1, P_2 are prime co-ideals of R and $a \notin P_1 \cup P_2$, $b \in P_1 \cap P_2 = \{1\}$, a contradiction. Therefore $S(R) = P_1 \cup P_2$. Set $V_1 = P_1 \setminus \{1\}$ and $V_2 = P_2 \setminus \{1\}$. Let $a, b \in V_1$. If $a + b = 1 \in P_2$, then $a \in P_2$ or $b \in P_2$, which is a contradiction. Thus non of elements of V_1 are adjacent together. Similarly, non of elements of V_2 are adjacent. This means that $\Gamma(R)$ is a bipartite graph with parts V_1 and V_2.

Conversely, let $\Gamma(R)$ be a partite graph with two parts V_1 and V_2. First of all, we show that $V_1 \cup \{1\}$ is a co-ideal of R. Let $a, b \in V_1 \cup \{1\}$. Then $a + c = 1$ and $b + c = 1$ for each $c \in V_2 \cup \{1\}$. Since $(1 : c)$ is a co-ideal of R by Proposition 2.5 and $a, b \in (1 : c)$, $ab \in (1 : c)$. So $ab + c = 1$. Now let $r \in R$ and $a \in V_1 \cup \{1\}$. Let $a + r \neq 1$. Since $\Gamma(R)$ is a complete bipartite graph, $a + c = 1$ for each $c \in V_2 \cup \{1\}$. Thus $a + r + c = 1$, this means $a + r \in V_1 \cup \{1\}$. Now we show that $P_1 = V_1 \cup \{1\}$ is a prime co-ideal of R. Let $a + b \in P_1$ such that $a \notin P_1$ and $b \notin P_1$. Hence $a + (b + c) = 1$, $a + c \neq 1$ and $b + c \neq 1$ for each $c \in V_2$. Since $a \notin P_1$, so $a \neq 1$ and $a \in V_2$. Thus $b + c \in V_1$. Since $\Gamma(R)$ is a complete bipartite graph, $b + c = b + c + c = 1$ (because R is an I-semiring and $c + c = c$). Since $c \in V_2$, $b \in V_1$, a contradiction. Thus P_1 is a prime strong co-ideal of R. Similarly, P_2 is a prime strong co-ideal of R. □

Theorem 4.6. *Let R be an I-semiring.*

(1) *If $P_1 = (1 : x_1)$ and $P_2 = (1 : x_2)$ are two distinct elements of* co-Ass(R), *then we have $x_1 + x_2 = 1$.*

(2) *If* co-Ass$(R) = \{P_1, P_2\}$, $|P_i| \geq 3$ $(i = 1, 2)$ *and* $P_1 \cap P_2 = \{1\}$, *then* $\mathrm{g}(\Gamma(R)) = 4$.

(3) *If* $|$co-Ass$(R)| \geq 3$, *then* $\mathrm{g}(\Gamma(R)) = 3$.

Proof. (1) By assumption, we may assume that $P_2 \nsubseteq P_1$. Then there exists $1 \neq a \in P_2 \setminus P_1$ such that $a + x_2 = 1$; so $x_2 \in P_1$ since P_2 is a prime co-ideal of R. Thus $x_1 + x_2 = 1$.

(2) By Theorem 4.5, $\Gamma(R)$ is a complete bipartite graph with parts $P_1 \setminus \{1\}$ and $P_2 \setminus \{1\}$. Since for each i, $|P_i| \geq 3$, we must have $\mathrm{g}(\Gamma(R)) = 4$.

(3) If $P_1 = (1 : x_1)$, $P_2 = (1 : x_2)$ and $P_3 = (1 : x_3)$ are three distinct elements of co-Ass(R), then $x_1 - x_2 - x_3 - x_1$ is a cycle of length 3 in $\Gamma(R)$ by (1), as required. □

For a graph G and vertex $x \in V(G)$, the degree of x, denoted $\deg(x)$, is the number of edges of G incident with x. For every nonnegative integer r, the graph G is called r-regular if the degree of each vertex of G is equal to r. Also for a given vertex $x \in V(G)$, the neighborhood set of x is the set $N(x) = \{a \in V(G) : a \text{ is adjacent to } x\}$. We need the following lemma proved in [21, Lemma 1.9].

Lemma 4.7. *Let G be a finite, simple graph with the property that two distinct vertices v and w of G are non-adjacent if and only if $N(v) = N(w)$. Then G is a complete r-partite graph for some positive integer r.*

Theorem 4.8. *Let R be an I-semiring, and let $\Gamma(R)$ be a finite regular graph. Then $\Gamma(R)$ is $K_{n,n}$ for some $n \in \mathbb{N}$.*

Proof. Let $\Gamma(R)$ be a regular graph of degree n. If $\Gamma(R)$ contains only two vertices, then $\Gamma(R) = K_{1,1}$. So let $\Gamma(R)$ contains at least 3 vertices. Let $a, b \in S^*(R)$ be non-adjacent in $\Gamma(R)$. Since $N(b) \subseteq N(a+b)$ and $\deg(b) = \deg(a+b)$, we have $N(b) = N(a+b)$. By similar argument as before, $N(a) = N(a+b)$; hence $N(a) = N(b)$. Therefore, any two non adjacent vertices on the graph have the same neighborhood, and clearly, the converse is true. So by Lemma 4.7, $\Gamma(R)$ is complete r-partite and hence $\Gamma(R)$ is bipartite by Theorem 4.4(2). Since $\Gamma(R)$ is regular, $\Gamma(R) = K_{n,n}$. □

Anderson and Mulay [9, Theorem 2.8] proved that for direct product of integral domains and their subrings, the diameter is at most 2. We generalize this result to identity-summand graph of semirings.

Theorem 4.9. *Assume that R_1 and R_2 are I-co-semidomains and let $R \subseteq R_1 \times R_2$ be an I-semiring with $|S^*(R)| \geq 3$. Then $\Gamma(R)$ is a complete bipartite graph with $\operatorname{diam}(\Gamma(R)) = 2$.*

Proof. Let $(r_1, r_2) \in S^*(R)$. Then $(r_1, r_2) + (s_1, s_2) = (1, 1)$ for some non-identity element (s_1, s_2) of R. By assumption, either $r_1 = 1$ or $s_1 = 1$ and either $r_2 = 1$ or $s_2 = 1$; hence either $r_1 = 1$ or $r_2 = 1$. Set $V_1 = \{(1, a) : 1 \neq a \in R_2\}$ and $V_2 = \{(b, 1) : 1 \neq b \in R_1\}$. We show that any two elements of V_1 are non-adjacent. Suppose, on the contrary, that $x = (1, a_1), y = (1, a_2) \in V_1$ are adjacent in $\Gamma(R)$. So $x + y = (1, a_1) + (1, a_2) = (1, 1)$; hence $a_1 + a_2 = 1$ which implies $a_1 = 1$ or $a_2 = 1$ (because R_2 is I-co-semidomain), a contradiction. By the similar way, non of elements of V_2 are adjacent in $\Gamma(R)$. Also for every $x = (1, a) \in V_1$ and $y = (b, 1) \in V_2$, $x + y = (1, a) + (b, 1) = (1, 1)$ which means every element in V_1 is adjacent to every element in V_2. Hence $\Gamma(R)$ is a complete bipartite graph and $\operatorname{diam}(\Gamma(R)) = 2$. □

5. Chromatic number and clique number of $\Gamma(R)$

In this section, we will investigate chromatic number and clique number of the graph $\Gamma(R)$.

Lemma 5.1. *Let R be an I-semiring with $w(\Gamma(R))$ finite. Then R has a.c.c on co-ideals of the form $(1 : x)$, where $x \in R$.*

Proof. Let $(1 : a_1) \subseteq (1 : a_2) \subseteq \cdots$ be an ascending chain of co-ideals of R. Let $x_i \in (1 : a_i) \setminus (1 : a_{i-1})$, where $i \geq 2$; so $x_i + a_{i-1} \neq 1$. Since $x_i + a_{i-1} + x_j + a_{j-1} = 1$ for each $i \neq j$, the set $\{x_i + a_{i-1}\}_{i \geq 2}$ is a clique. Also, for each $i \neq j$, $x_i + a_{i-1} \neq x_j + a_{j-1}$ (if not, by Proposition 2.5, $x_i + a_{i-1} = x_i + a_{i-1} + x_i + a_{i-1} = x_i + a_{i-1} + x_j + a_{j-1} = 1$, a contradiction); thus $w(\Gamma(R))$ is not finite, which is a contradiction. □

Theorem 5.2. *Let R be an I-semiring. Then the following are equivalent.*
(1) *$\chi(\Gamma(R))$ is finite*;
(2) *$w(\Gamma(R))$ is finite*;
(3) *The co-ideal $\{1\}$ is a finite intersection of prime co-ideals.*

Proof. (1) ⇒ (2) Clearly $w(G) \leq \chi(G)$ for any graph G, so the result is trivial.

(2) ⇒ (3) Let $w(\Gamma(R)) = n$. By Lemma 5.1, the partially ordered set $(\Delta, \subseteq)$ has a maximal element, where $\Delta = \{(1 : a) : a \neq 1,\ a \in R\}$. Let $(1 : x_i)(i \in J)$ be the different maximal members of the set Δ. By a usual argument, for each $i \in J$, $(1 : x_i)$ is prime. Therefore $\{x_i\}_{i \in J}$ is a clique in $\Gamma(R)$ by Theorem 4.6(1). Since $w(\Gamma(R))$ is finite, J is a finite set; we show that $\{1\} = \cap_{i \in J}(1 : x_i)$. Let $1 \neq x \in R$. Then $(1 : x) \subseteq (1 : x_i)$ for some $i \in I$. We claim that $x \notin (1 : x_i)$. Otherwise, $x + x_i = 1$, so $x_i \in (1 : x) \subseteq (1 : x_i)$; hence $1 = x_i + x_i = x_i$ which is a contradiction. Therefore $x \notin (1 : x_i)$, and so $x \notin \cap_{i \in I}(1 : x_i)$. Hence $\cap_{i \in I}(1 : x_i) \subseteq \{1\}$, and so we have equality.

(3) ⇒ (1) Let $\cap_{i=1}^{n} P_i = \{1\}$, where for each $1 \leq i \leq n$, P_i is a prime co-ideal of R. If x is adjacent to y in $\Gamma(R)$, then there is no $1 \leq i \leq n$ such that $x \notin P_i$ and $y \notin P_i$. Therefore we can label any vertex x of $\Gamma(R)$ by $\min\{i :\ x \notin P_i\}$. This implies that $\chi(\Gamma(R)) \leq n$. □

Theorem 5.3. *Let R be an I-semiring with $w(\Gamma(R))$ finite. Then* $\min(R) \subseteq$ co-Ass(R), *where* $\min(R)$ *is the set of minimal prime co-ideals of R.*

Proof. Let $P \in \min(R)$. By an argument like that in Theorem 5.2, $\{1\} = \cap_{i=1}^{n} Q_i$ where $Q_i \in$ co-Ass(R). Hence $\cap_{i=1}^{n} Q_i \subseteq P$. By an argument similar to the proof in Theorem 2.8(2), $Q_j \subseteq P$ for some $1 \leq j \leq n$. As P is minimal, $Q_j = P$. Thus $P \in$ co-Ass(R) and so $\min(R) \subseteq$ co-Ass(R). □

Theorem 5.4. *Let R be an I-semiring. Then $w(\Gamma(R)) = |\min(R)|$.*

Proof. First we show that $|\min(R)|$ is finite if and only if $w(\Gamma(R))$ is finite. Let $|\min(R)|$ be finite. Then $\{1\}$ is a finite intersection of prime co-ideals by Theorem 2.8; so by Theorem 5.2, $w(\Gamma(R))$ is finite. Now suppose that $w(\Gamma(R))$ is finite. Hence $\{1\} = \cap_{i=1}^{n} P_i$ for some prime co-ideals P_i of R by Theorem 5.2. Let $\{Q_\alpha\}_{\alpha \in \Lambda}$ be the set of all minimal prime co-ideals of R. For each $\alpha \in \Lambda$, $1 \in Q_\alpha$, so $\cap_{i=1}^{n} P_i \subseteq Q_\alpha$ for each $\alpha \in \Lambda$. This implies that $P_i \subseteq Q_\alpha$ for some $1 \leq i \leq n$. Since Q_α is minimal, $P_i = Q_\alpha$. This gives Λ is finite, and so $|\min(R)|$ is finite.

Let $|\min(R)| = n$. By Theorem 2.8, for each $1 \leq j \leq n$ there exists $x_j \in \cap_{1\leq i\leq n, i\neq j} P_i$ with $x_j \notin P_j$. Since each P_i is a co-ideal of R, $x_i + x_j \in P_s$ for each $1 \leq s \leq n$, and so $x_i + x_j = 1$ (because $\{1\} = \cap_{i=1}^{n} P_i$) for each $1 \leq i \neq j \leq n$; thus $X = \{x_1, x_2, \ldots, x_n\}$ is a clique in $\Gamma(R)$. Hence $w(\Gamma(R) \geq n$. Now we show that $w(\Gamma(R)) \leq n$. Let $w(\Gamma(R)) = m$. Then there is $\{y_1, \ldots, y_m\}$ such that it is a clique in $\Gamma(R)$. Since $y_i \neq 1$ and $\{1\} = \cap_{1\leq i\leq n} P_i$, there is $1 \leq s \leq n$ such that $y_i \notin P_s$. If $m > n$, then by Pigeon hole principle, there is at least one P_s $(1 \leq s \leq n)$ such that $y_i \notin P_s$ and $y_j \notin P_s$ for some $(1 \leq i \neq j \leq m)$. Since P_s is prime, $y_i + y_j \notin P_s$ which is a contradiction (because $y_i + y_j = 1 \in P_s$). Hence $m \leq n$. Thus $w(\Gamma(R)) = n$. □

Example 5.5. Let R, P_1, P_2 and P_3 be as in Example 4.2. It is easy to see that $P_1 \cap P_2 \cap P_3 = X = \{1\}$, $X \neq \cap_{1\leq i\leq 3, i\neq j} P_i$ and $w(\Gamma(R)) = 3$.

Theorem 5.6. *If R is an I-semiring, then $\chi(\Gamma(R)) = w(\Gamma(R))$.*

Proof. It is clear that $w(\Gamma(R)) \leq \chi(\Gamma(R))$. Let $w(\Gamma(R)) = n$. Then $\{1\} = P_1 \cap \cdots \cap P_n$, where for each i, P_i is a minimal prime co-ideal. By an argument like that in the proof of Theorem 5.2 ((3) $\Rightarrow$ (1)), $\chi(\Gamma(R)) \leq n$. Therefore $\chi(\Gamma(R)) = w(\Gamma(R))$. □

6. Planar property of $\Gamma(R)$

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths. In this section, the planar property of graph $\Gamma(R)$ is investigated.

Theorem 6.1. *Let R be an I-semiring.*
(1) *If $|\min(R)| \geq 5$, then $\Gamma(R)$ is not planar.*
(2) *If $|\min(R)| = 4$, then $\Gamma(R)$ is not planar.*

Proof. (1) This follows from Theorem 5.4.

(2) By Theorem 5.4, $w(\Gamma(R)) = 4$. Thus there exist $x_1, x_2, x_3, x_4 \in S(R)^*$ such that $\{x_1, x_2, x_3, x_4\}$ forms a clique in $\Gamma(R)$. Let $x_{ij} = x_i x_j$, where $1 \leq i \neq j \leq 4$. Assume that $1 \leq k \neq i, j \leq 4$. Since $x_i, x_j \in (1 : x_k)$, $x_{ij} \in (1 : x_k)$. If $x_{ij} = 1$, then $x_i = x_{ij} + x_i = 1$ which is a contradiction. This implies that $x_{ij} \in S^*(R)$. If $x_{ij} = x_s$ for some $1 \leq s \leq 4$, then we split the proof into two cases:

Case 1. If $s = i$, then $x_{ij} + x_j = 1$. This implies that $x_j = 1$ which is a contradiction. Similarly, for $s = j$.

Case 2. If $s \neq j$ and $s \neq i$, then $x_{ij} + x_s = 1$; hence $x_s + x_s = 1$. It follows that $x_s = 1$ by Proposition 2.5, a contradiction.

Therefore $x_{ij} \notin \{x_1, x_2, x_3, x_4\}$. Let $s \neq k$ and $s, k \in \{1, 2, 3, 4\} - \{i, j\}$. Since $x_{ij} + x_s = 1$ and $x_{ij} + x_k = 1$, we have $x_s, x_k \in (1 : x_{ij})$; thus $x_{sk} \in (1 : x_{ij})$. Set $V_1 = \{x_1, x_2, x_{12}\}$ and $V_2 = \{x_{34}, x_3, x_4\}$. Then V_1 and V_2 are two parts of a complete bipartite subgraph of $\Gamma(R)$. Therefore $K_{3,3}$ is a subgraph of $\Gamma(R)$, and so $\Gamma(R)$ is not planar. □

Example 6.2. Let $R = \{2^i 3^j 5^k : i \in \{0,1,2,3\},\ j \in \{0,1,2,3\},\ k \in \{0,1\}\} \cup \{0\}$. Then $(R, \gcd, \text{lcm})$ is an I-semiring. An inspection shows that $\{2,3,5\}$ is a clique in $\Gamma(R)$ and $w(\Gamma(R)) = 3$. Hence $|\min(R)| = 3$ by Theorem 5.4. Set $V_1 = \{2, 2^2, 2^3\}$ and $V_2 = \{3, 3^2, 3^3\}$. Then $K_{3,3}$ is a subgraph of $\Gamma(R)$ with two parts V_1 and V_2. Hence $\Gamma(R)$ is not planar.

Remark 6.3. Let R be an I-semiring. Then:

(1) If $|\min(R)| = 1$, then by Theorem 2.8(2), $\{1\}$ is the only minimal prime co-ideal of R. Hence R is co-semidomain and so $\Gamma(R) = \emptyset$ by Proposition 3.1.

(2) If $|\min(R)| = 2$, then by Theorem 4.5, $\Gamma(R)$ is $K_{n,m}$ for some integer n,m, where $|P_1| - 1 = n$ and $|P_2| - 1 = m$. If $n, m \geq 3$, then $\Gamma(R)$ is not planar.

(3) If $|\min(R)| \geq 4$, then by Theorem 6.1, $\Gamma(R)$ is not planar

(4) If R is the I-semiring as in Example 4.1, then $|\min(R)| = 3$ and R is planar. However there exist I-semirings that have only three minimal prime co-ideals and their identity-summand graphs are not planar as Example 4.2 shows. It is not entirely clear for us which semirings with $|\min(R)| = 3$, the $\Gamma(R)$ is planar.

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