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COMPACT INTERTWINING RELATIONS FOR COMPOSITION OPERATORS BETWEEN THE WEIGHTED BERGMAN SPACES AND THE WEIGHTED BLOCH SPACES

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Abstract. We study the compact intertwining relations for composition operators, whose intertwining operators are Volterra type operators from the weighted Bergman spaces to the weighted Bloch spaces in the unit disk. As consequences, we find a new connection between the weighted Bergman spaces and little weighted Bloch spaces through this relations.

1. Introduction

If X and Y are two Banach spaces, the symbol $\mathscr{B}(X,Y)$ denotes the collection of all bounded linear operators from X to Y. Let $\mathcal{K}(X,Y)$ be the collection of all compact elements of $\mathscr{B}(X,Y)$, and $\mathscr{Q}(X,Y)$ be the quotient set $\mathscr{B}(X,Y)/\mathcal{K}(X,Y)$.

For linear operators $A \in \mathscr{B}(X,X)$, $B \in \mathscr{B}(Y,Y)$ and $T \in \mathscr{B}(X,Y)$, the phrase "T intertwines A and B in $\mathscr{Q}(X,Y)$" (or "T intertwines A and B compactly") means that

$$TA = BT \mod \mathcal{K}(X,Y) \quad \text{with} \quad T \neq 0. \tag{1.1}$$

Notation $A \propto_K B$ (T) represents the relation in equation (1.1). In fact, if T is an invertible operator on X, then the relation $\propto_K$ is symmetric.

We denote the class of all holomorphic functions on the complex unit disk $\mathbb{D}$ by $H(\mathbb{D})$, and the collection of all the holomorphic self mappings of $\mathbb{D}$ by $S(\mathbb{D})$. Let $\alpha > -1$, $0 < \beta < \infty$ and $\gamma \geq 0$, and a real number $p > 0$. These settings of α, β, γ and p are valid in the following context unless specifications.

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Recall that the weighted Bergman space on the unit disk $A^p_\alpha = A^p_\alpha(\mathbb{D})$ consists of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z)\right)^{\frac{1}{p}} < \infty,$$

where $\mathrm{d}A$ is the normalized Lebesgue area measure on $\mathbb{D}$. The weighted Bloch space $\mathcal{B}^\beta$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\beta} := \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}^\beta}$ is a complete semi-norm on $\mathcal{B}^\beta$, which is Möbius invariant. The weighted Bloch space is a Banach space under the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\beta}.$$

Let $\mathcal{B}^\beta_0$ denote the subspace of $\mathcal{B}^\beta$ consisting of those $f \in \mathcal{B}^\beta$ for which

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |f'(z)| = 0.$$

The space of weighted bounded analytic functions on $\mathbb{D}$ is

$$H^{\infty,\gamma} = \left\{ f \in H(\mathbb{D}) : \|f\|_\infty = \sup_{z\in\mathbb{D}} (1-|z|^2)^\gamma |f(z)| < \infty \right\}.$$

We write non-weighted bounded analytic functions space $H^{\infty,0}$ as H^∞. And let $H^{\infty,\gamma}_0$ be the subspace of $H^{\infty,\gamma}$ consisting of $f \in H^{\infty,\gamma}$ with

$$\lim_{|z|\to 1} (1-|z|^2)^\gamma |f(z)| = 0.$$

Remark. It is well-known that, for $\gamma > 0$, $H^{\infty,\gamma} = \mathcal{B}^{1+\gamma}$, and $H^{\infty,\gamma}_0 = \mathcal{B}^{1+\gamma}_0$, see, for example, Proposition 7 in [11].

For $f \in H(\mathbb{D})$, every $\varphi \in S(\mathbb{D})$ induces a composition operator C_φ by

$$C_\varphi f = f \circ \varphi.$$

If $g \in H(\mathbb{D})$, the Volterra operator J_g is defined by

$$J_g f(z) = \int_0^z f(\zeta) g'(\zeta) \mathrm{d}\zeta;$$

and another integral operator I_g is defined by

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) \mathrm{d}\zeta,$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. The operator J_g is actually a generalization of the integral operator (when $g(z) = z$). The operators J_g and I_g are close companions because of their relations to the multiplication operator $M_g f(z) = g(z) f(z)$. To see this, integration by parts gives

$$M_g f = f(0)g(0) + J_g f + I_g f.$$

The theory of composition operators has been developed extensively since the work of E. Nordgren in the mid 1960's. Some properties of composition operators have been studied quite well, just like the boundedness and compactness on several holomorphic functions spaces. For more about these topics, see [4, 7] and the references therein. Discussion of Volterra operators first arose in the connection with semigroups of composition operators, and readers may refer to [8] for the background. Recently, characterizing the boundedness and compactness of J_g and I_g on certain spaces of analytic functions becomes the most active work. The boundedness of J_g on the Hardy spaces, Bergman spaces, BMOA and $\mathcal{Q}_p$ are characterized in [1, 2, 8, 10], respectively.

Based on these results, we consider the compact intertwining relations

$$C_\varphi \propto_K C_\varphi \ (V_g),$$

where V_g represents both J_g and I_g. That is to characterize those $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$ such that

$$V_g(C_\varphi : A^p_\alpha \to A^p_\alpha) = (C_\varphi : \mathcal{B}^\beta \to \mathcal{B}^\beta)V_g \qquad \text{mod } \mathcal{K}(A^p_\alpha, \mathcal{B}^\beta) = \mathcal{K}. \tag{1.2}$$

If C_φ and V_g satisfy (1.2), we say they are *essentially commutative*. An interesting follow-up question is to ask whether there is a non-constant holomorphic function g on $\mathbb{D}$ such that V_g is bounded and essentially commutes with every bounded C_φ. Furthermore, can we characterize the set of all such g? Let $\Omega_{co}(V_g)$ denote the collection consisting of all the $g \in H(\mathbb{D})$ that

$$V_g \in \mathscr{B}(A^p_\alpha, \mathcal{B}^\beta) \text{ and } C_\varphi \propto_K C_\varphi \ (V_g).$$

We call $\Omega_{co}(V_g)$ the *universal set* of V_g. The lower symbol "*co*" stands for "composition operator". In [9], the authors discussed the intertwining relation and compact intertwining relation for Volterra operators on the Bergman spaces, and the concept of universal set has been introduced and characterized in that settings.

The main results of this paper are to characterize $\Omega_{co}(J_g)$ and $\Omega_{co}(I_g)$, and we draw the following conclusion:

Theorem 3.4. $\Omega_{co}(J_g) = \mathcal{B}_0^{\beta - \frac{\alpha+2}{p}}$;

Theorem 4.2. $\Omega_{co}(I_g) = H_0^{\infty, \beta - \frac{\alpha+2}{p} - 1}$.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Preliminaries

Let $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$ and $S_{\varphi,g} = C_\varphi I_g - I_g C_\varphi$, both of which are from A^p_α to $\mathcal{B}^\beta$. The following lemma is the crucial criterion for compactness, whose proof is an easy modification of that of Proposition 3.11 of [4].

Lemma 2.1. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi,g}$ (or $S_{\varphi,g}$) is a compact operator from A^p_α to $\mathcal{B}^\beta$ if and only if for any bounded sequence $\{f_k\}$, $k = 1, 2, \ldots$ in A^p_α which converges to zero uniformly on compact subsets of $\mathbb{D}$, $T_{\varphi,g} f_k$ (or $S_{\varphi,g} f_k$) converges to zero in the $\mathcal{B}^\beta$ norm topology as k tends to infinity.*

Lemma 2.2 (See Lemma 1 and Lemma 2 in [6])**.** *If $f \in A^p_\alpha$, then for every $z \in \mathbb{D}$,*

(1) $|f(z)| \leq C \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}$;

(2) $|f'(z)| \leq C \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}+1}}$.

The following asymptotic estimates for certain important integrals support the construction of our test functions. Here we list the general version on the unit ball B_N and the unit sphere S_N of N-dimensional complex Euclidean space $\mathbb{C}^N$, which appears in [12], Theorem 1.12.

Lemma 2.3. *Suppose c is real and $t > -1$. Let $d\sigma$ and dv denote the normalized surface measure on S_N and normalized volume measure on B_N. Then the integrals*

$$E_c(z) = \int_{S_N} \frac{d\sigma(\zeta)}{|1 - \langle z, \zeta \rangle|^{n+c}}, \quad z \in B_N,$$

and

$$F_{c,t}(z) = \int_{B_N} \frac{(1 - |w|^2)^t \, dv(w)}{|1 - \langle z, w \rangle|^{n+1+t+c}}, \quad z \in B_N,$$

have the following asymptotic properties.

(i) *If $c < 0$, then E_c and $F_{c,t}$ are both bounded in B_N.*

(ii) *If $c = 0$, then*

$$E_c(z) \sim F_{c,t}(z) \sim \log \frac{1}{1 - |z|^2}$$

as $|z| \to 1^-$.

(iii) *If $c > 0$, then*

$$E_c(z) \sim F_{c,t}(z) \sim (1 - |z|^2)^{-c}$$

as $|z| \to 1^-$.

The author of [6] defined two operators $J_{g,\varphi}$ and $I_{g,\varphi}$ as

$$(J_{g,\varphi} f)(z) = \int_0^z (f \circ \varphi)(\xi)(g \circ \varphi)'(\xi) \mathrm{d}\xi$$

and

$$(I_{g,\varphi} f)(z) = \int_0^z (f \circ \varphi)'(\xi)(g \circ \varphi)(\xi) \mathrm{d}\xi.$$

When $\varphi = \mathrm{id}$, then $J_{g,\varphi} = J_g$, $I_{g,\varphi} = I_g$. When $g \equiv 1$, then $I_{g,\varphi} = C_\varphi$. Next lemma is the main result of [6].

Lemma 2.4. *Let $\alpha > -1$ and $0 < \beta < \infty$, and assume that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then,*

(1) *$J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*
$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |g'(\varphi(z))| < \infty;$$

(2) *$I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*
$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}} |g(\varphi(z))| < \infty.$$

Thus $J_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if
$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta - \frac{\alpha+2}{p}} |g'(z)| < \infty, \tag{2.1}$$
and I_g is bounded if and only if
$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\beta - \frac{\alpha+2}{p} - 1} |g(z)| < \infty. \tag{2.2}$$

Lemma 2.5 (Corollary 2.40 in [4])**.** *If $\varphi \in S(\mathbb{D})$, then $|\varphi(z)| \le \frac{|z|+|\varphi(0)|}{1+|z||\varphi(z)|}$.*

3. J_g as intertwining operator

To characterize the set $\Omega_{co}(J_g)$, the boundedness and compactness of $T_{\varphi,g}$ should be discussed first.

Proposition 3.1. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi,g}$ is a bounded operator from A^p_α to $\mathcal{B}^\beta$ if and only if*
$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |(g\circ\varphi)'(z) - g'(z)| < \infty. \tag{3.1}$$

Proof.
$$\begin{aligned}(C_\varphi J_g - J_g C_\varphi) f(z) &= C_\varphi \left(\int_0^z f(w) g'(w) \mathrm{d}w \right) - J_g (f\circ\varphi)(z) \\ &= \int_0^{\varphi(z)} f(w) g'(w) \mathrm{d}w - \int_0^z f(\varphi(w)) g'(w) \mathrm{d}w.\end{aligned}$$
Thus
$$\begin{aligned}\|T_{\varphi,g}(f)\|_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \left|(g\circ\varphi)'(z) - g'(z)\right| |f(\varphi(z))| \\ &\le C\|f\|_{A^p_\alpha} \cdot \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \left|(g\circ\varphi)'(z) - g'(z)\right|.\end{aligned}$$
Sufficiency is followed as an obvious consequence.

Now we assume $T_{\varphi,g}$ is bounded. And suppose that (3.1) dose not hold, then there is a sequence $\{z_n\} \subset \mathbb{D}$ satisfying

$$\lim_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}} |(g\circ\varphi)'(z_n) - g'(z_n)| = \infty.$$

To find a contradiction, we choose a sequence of test functions $\{f_n\}$ as follow:

$$f_n(z) = \left(\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^2}\right)^{\frac{\alpha+2}{p}}.$$

By (iii) in Lemma 2.3, it is easy to see that $f_n \in A^p_\alpha$, and moreover $\sup_n \|f_n\|_{A^p_\alpha} \le C$. Hence we have

$$\begin{aligned}
\|T_{\varphi,g}f_n\|_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |(g\circ\varphi)'(z) - g'(z)||f_n(\varphi(z))| \\
&\ge (1-|z_n|^2)^\beta |(g\circ\varphi)'(z_n) - g'(z_n)| \cdot \left(\frac{1-|\varphi(z_n)|^2}{|1-\overline{\varphi(z_n)}\varphi(z_n)|^2}\right)^{\frac{\alpha+2}{p}} \\
&\ge \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\frac{\alpha+2}{p}}} |(g\circ\varphi)'(z_n) - g'(z_n)| \to \infty.
\end{aligned}$$

That is impossible since $T_{\varphi,g}$ is bounded. The proof of this proposition is complete. □

Theorem 3.2. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $T_{\varphi,g}$ is a compact operator from A^p_α to $\mathcal{B}^\beta$ if and only if $T_{\varphi,g}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |(g\circ\varphi)'(z) - g'(z)| = 0. \tag{3.2}$$

Proof. Suppose (3.2) holds. To prove $T_{\varphi,g}$ is a compact operator from A^p_α to $\mathcal{B}^\beta$. For any bounded sequence $\{f_k\}$ in A^p_α converging to zero uniformly on compact subsets of $\mathbb{D}$ with $\|f_k\|_{A^p_\alpha} \le M$. From (3.2), there is a $\delta > 0$ for any small $\varepsilon > 0$ such that

$$\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |(g\circ\varphi)'(z) - g'(z)| < \frac{\varepsilon}{M}$$

when $\varphi(z) \in \mathbb{D} \setminus (1-\delta)\mathbb{D}$.

$$\begin{aligned}
&\|T_{\varphi,g}f_k\|_{\mathcal{B}^\beta} \\
= &\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} |(g\circ\varphi)'(z) - g'(z)| \cdot (1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}} |f_k(\varphi(z))| \\
= &\max\{\sup_{\varphi(z)\in(1-\delta)\mathbb{D}} Q, \sup_{\varphi(z)\in\mathbb{D}\setminus(1-\delta)\mathbb{D}} Q\},
\end{aligned}$$

where

$$Q = \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|(g\circ\varphi)'(z) - g'(z)| \cdot (1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}|f_k(\varphi(z))|.$$

The first supremum of Q can be smaller than ε for sufficient large k since equation (3.1) holds and f_k converges to zero on compact subsets of $\mathbb{D}$. Note that

$$\sup_{\varphi(z)\in\mathbb{D}\setminus(1-\delta)\mathbb{D}} Q < \frac{\varepsilon}{M}\cdot\sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}|f_k'(\varphi(z))| = \frac{\varepsilon}{M}C\|f_k\|_{A^p_\alpha} < \varepsilon C.$$

Thus we have clarify the sufficiency of the theorem.

Suppose $T_{\varphi,g}$ is compact on $\mathcal{B}^\beta$, certainly $T_{\varphi,g}$ is a bounded operator from A^p_α to $\mathcal{B}^\beta$, then equation (3.1) holds by Proposition 3.1. Suppose that (3.2) does not hold, there exists a sequence $\{w_n\}$ in $\mathbb{D}$ with $|\varphi(w_n)| \to 1$ and

$$\frac{(1-|w_n|^2)^\beta}{(1-|\varphi(w_n)|^2)^{\frac{\alpha+2}{p}}}|(g\circ\varphi)'(w_n) - g'(w_n)| > \varepsilon > 0$$

with sufficient large n for any given $\varepsilon > 0$. Let

$$f_n(z) = \left(\frac{1-|\varphi(w_n)|^2}{(1-\overline{\varphi(w_n)}z)^2}\right)^{\frac{\alpha+2}{p}}. \tag{3.3}$$

It is easy to show that $f_n \in A^p_\alpha$, moreover $\sup_n \|f_n\|_{A^p_\alpha} \leq C$ and f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Since $T_{\varphi,g}$ is compact, we have $\|T_{\varphi,g}f_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$, by Lemma 2.1. But according to our assumption of $\{w_n\}$, we have

$$\begin{aligned}
\|T_{\varphi,g}f_n\|_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(g\circ\varphi)'(z) - g'(z)||f_n(\varphi(z))| \\
&\geq (1-|w_n|^2)^\beta|(g\circ\varphi)'(w_n) - g'(w_n)| \cdot \left(\frac{1-|\varphi(w_n)|^2}{|1-\overline{\varphi(w_n)}\varphi(w_n)|^2}\right)^{\frac{\alpha+2}{p}} \\
&\geq \frac{(1-|w_n|^2)^\beta}{(1-|\varphi(w_n)|^2)^{\frac{\alpha+2}{p}}}|(g\circ\varphi)'(w_n) - g'(w_n)| > \varepsilon.
\end{aligned}$$

That is impossible. Thus we proved the necessity. □

Let $J_g \in \mathscr{B}(A^p_\alpha, \mathcal{B}^\beta)$, and C_φ be in both $\mathscr{B}(A^p_\alpha, A^p_\alpha)$ and $\mathscr{B}(\mathcal{B}^\beta, \mathcal{B}^\beta)$. We look into the properties of symbols $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$ when $C_\varphi \propto_K C_\varphi$ (J_g).

It is well known that C_φ is bounded on $\mathcal{B}^\beta$ $(\beta > 0)$ if and only if

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}|\varphi'(z)| < \infty. \tag{3.4}$$

On $\mathcal{B}^\beta$ when $\beta \geq 1$, all C_φ's are bounded since the former condition holds naturally by Schwarz inequality and Lemma 2.5.

Corollary 3.3. *Let φ be an analytic self map of $\mathbb{D}$ and $g \in H(\mathbb{D})$. Then C_φ and J_g are essentially commutative if and only if* (2.1), (3.2) *and* (3.4) *holds.*

Proof. Combine Lemma 2.4 with Theorem 3.2. □

According to the compact criterion of $T_{\varphi,g} : A^p_\alpha \to \mathcal{B}^\beta$, which is stated in Theorem 3.2, we can draw the following conclusion:

Theorem 3.4. *Let $\Omega_{co}(J_g)$ be the universal set of J_g. Then*

$$\Omega_{co}(J_g) = \mathcal{B}_0^{\beta-\frac{\alpha+2}{p}}.$$

Proof. Firstly we prove $\mathcal{B}_0^{\beta-\frac{\alpha+2}{p}} \subset \Omega_{co}(J_g)$. For every $g \in \mathcal{B}_0^{\beta-\frac{\alpha+2}{p}}$, we have

$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-\frac{\alpha+2}{p}}|g'(z)| = 0.$$

Then equation (2.1) is obvious, and we compute that

$$\begin{aligned}
&\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|(g\circ\varphi)'(z) - g'(z)| \\
\le\ & \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}(|g'(\varphi(z))||\varphi'(z)| + |g'(z)|) \\
\le\ & \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}|\varphi'(z)|(1-|\varphi(z)|^2)^{\beta-\frac{\alpha+2}{p}}|g'(\varphi(z))| \\
&+ \left(\frac{2(1-|z|)}{1-|\varphi(z)|}\right)^{\frac{\alpha+2}{p}}(1-|z|^2)^{\beta-\frac{\alpha+2}{p}}|g'(z)| \\
\le\ & \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\beta}|\varphi'(z)|(1-|\varphi(z)|^2)^{\beta-\frac{\alpha+2}{p}}|g'(\varphi(z))| \\
&+ 2^{\frac{\alpha+2}{p}}\left(\frac{1+|z||\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{\alpha+2}{p}}(1-|z|^2)^{\beta-\frac{\alpha+2}{p}}|g'(z)|.
\end{aligned}$$

Both two items are converging to 0 since our condition and boundedness of C_φ on $\mathcal{B}^\beta$ and $|\varphi(0)| \ne 1$. And equation (3.2) is obtained. Hence $g \in \Omega_{co}(J_g)$.

To prove $\Omega_{co}(J_g) \subset \mathcal{B}_0^{\beta-\frac{\alpha+2}{p}}$, suppose $g \in \Omega_{co}(J_g)$, then equation (2.1) and (3.2) hold for every $\varphi \in S(\mathbb{D})$ with (3.4). Putting $\varphi(z) = e^{\mathbf{i}\theta}z$ in (3.2) where $\forall\theta \in [0, 2\pi]$, we have

$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-\frac{\alpha+2}{p}}|e^{\mathbf{i}\theta}g'(e^{\mathbf{i}\theta}z) - g'(z)| = 0. \tag{3.5}$$

It is necessary to estimate the upper bound of left formula in (3.5),

$$\begin{aligned}
&(1-|z|^2)^{\beta-\frac{\alpha+2}{p}}\left|e^{\mathbf{i}\theta}g'(e^{\mathbf{i}\theta}z) - g'(z)\right| \\
\le\ & (1-|z|^2)^{\beta-\frac{\alpha+2}{p}}\left|e^{\mathbf{i}\theta}g'(e^{\mathbf{i}\theta}z)\right| + (1-|z|^2)^{\beta-\frac{\alpha+2}{p}}|g'(z)| \\
=\ & (1-|e^{\mathbf{i}\theta}z|^2)^{\beta-\frac{\alpha+2}{p}}\left|g'(e^{\mathbf{i}\theta}z)\right| + (1-|z|^2)^{\beta-\frac{\alpha+2}{p}}|g'(z)|
\end{aligned}$$

$$\leq 2\|g\|_{\mathcal{B}^{\beta-\frac{\alpha+2}{p}}}.$$

Equation (2.1) implies $\|g\|_{\mathcal{B}^{\beta-\frac{\alpha+2}{p}}}$ is finite, thus the left formula in (3.5) is bounded independent of θ.

We write $g(z) = \sum_{n=0}^{\infty} a_n z^n$, and integrate the left side of (3.5) with respect to θ from 0 to 2π

$$\begin{aligned}
0 &= \int_0^{2\pi} \lim_{|z|\to 1} (1-|z|^2)^{\beta-\frac{\alpha+2}{p}} \left|e^{\mathbf{i}\theta} g'(e^{\mathbf{i}\theta} z) - g'(z)\right| \mathrm{d}\theta \\
&= \lim_{|z|\to 1} \int_0^{2\pi} (1-|z|^2)^{\beta-\frac{\alpha+2}{p}} \left|e^{\mathbf{i}\theta} g'(e^{\mathbf{i}\theta} z) - g'(z)\right| \mathrm{d}\theta \\
&= \lim_{|z|\to 1} \int_0^{2\pi} (1-|z|^2)^{\beta-\frac{\alpha+2}{p}} \left|\sum_{n=1}^{\infty} n a_n z^{n-1} \left(e^{\mathbf{i}n\theta} - 1\right)\right| \mathrm{d}\theta \\
&\geq \lim_{|z|\to 1} (1-|z|^2)^{\beta-\frac{\alpha+2}{p}} \left|\sum_{n=1}^{\infty} n a_n z^{n-1} \int_0^{2\pi} \left(e^{\mathbf{i}n\theta} - 1\right) \mathrm{d}\theta\right| \\
&= 2\pi \lim_{|z|\to 1} (1-|z|^2)^{\beta-\frac{\alpha+2}{p}} |g'(z)|,
\end{aligned}$$

where Dominant Convergent Theorem is applied in second line. Thus $g \in \mathcal{B}_0^{\beta-\frac{\alpha+2}{p}}$. □

Corollary 3.5. *If $\alpha > -1$, $0 < \beta < \infty$ and $p > 0$ satisfy $\beta - \frac{\alpha+2}{p} \leq 0$, then*

$$\Omega_{co}(J_g) = \mathbb{C}.$$

Proof. We have $g'(z) \to 0$ at the boundary of $\mathbb{D}$ from Theorem 3.4, and that implies $g' \equiv 0$ on $\mathbb{D}$ by Maximum Modular Theorem. □

4. I_g as intertwining operator

When I_g serves as the intertwining operator of composition operators, the discussions are similar to those in the former section.

Proposition 4.1. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. And $S_{\varphi,g} = C_\varphi I_g - I_g C_\varphi$ acts from A^p_α to $\mathcal{B}^\beta$, then*

(i) *$S_{\varphi,g}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}} |\varphi'(z)||g(\varphi(z)) - g(z)| < \infty;$$

(ii) *$S_{\varphi,g}$ is compact if and only if $S_{\varphi,g}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}} |\varphi'(z)||g(\varphi(z)) - g(z)| = 0.$$

Proof. Note that

$$S_{\varphi,g}f(z) = \int_0^{\varphi(z)} f'(\zeta)g(\zeta)\mathrm{d}\zeta - \int_0^{z} (f\circ\varphi)'(\zeta)g(\zeta)\mathrm{d}\zeta,$$

and

$$\begin{aligned}\|S_{\varphi,g}f\|_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \left|f'(\varphi(z))g(\varphi(z))\varphi'(z) - f'(\varphi(z))\varphi'(z)g(z)\right| \\ &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)||f'(\varphi(z))||g(\varphi(z)) - g(z)|.\end{aligned}$$

By Lemma 2.2, we can get necessity. And by choosing test function similarly as (3.3), sufficiency will be clarified. □

Now we concern the compact intertwining relation for C_φ from A^p_α to $\mathcal{B}^\beta$ by intertwining I_g. The symbols φ and g should make $I_g \in \mathscr{B}(A^p_\alpha, \mathcal{B}^\beta)$ and $C_\varphi \propto_K C_\varphi\ (I_g)$. Those two items imply that $g \in H^{\infty,\beta-\frac{\alpha+2}{p}-1}$ by item (2) in Lemma 2.4 and conditions (ii) in Proposition 4.1.

Theorem 4.2. *Let $\Omega_{co}(I_g)$ be the universal set of I_g. Then*

$$\Omega_{co}(I_g) = H_0^{\infty,\beta-\frac{\alpha+2}{p}-1}.$$

Proof. The proof is similar as Theorem 3.4. □

Similarly, we have next corollary.

Corollary 4.3. *If $\alpha > -1$, $0 < \beta < \infty$ and $p > 0$ satisfy $\beta - \frac{\alpha+2}{p} - 1 \le 0$, then*

$$\Omega_{co}(I_g) = \{0\}.$$

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