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ROBUSTLY SHADOWABLE CHAIN COMPONENTS OF C^1 VECTOR FIELDS

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Abstract. Let γ be a hyperbolic closed orbit of a C^1 vector field X on a compact boundaryless Riemannian manifold M, and let $C_X(\gamma)$ be the chain component of X which contains γ. We say that $C_X(\gamma)$ is C^1 robustly shadowable if there is a C^1 neighborhood $\mathcal{U}$ of X such that for any $Y \in \mathcal{U}$, $C_Y(\gamma_Y)$ is shadowable for Y_t, where γ_Y denotes the continuation of γ with respect to Y. In this paper, we prove that any C^1 robustly shadowable chain component $C_X(\gamma)$ does not contain a hyperbolic singularity, and it is hyperbolic if $C_X(\gamma)$ has no non-hyperbolic singularity.

1. Introduction

The main goal of the study of differentiable dynamical systems is to understand the structure of the orbits of diffeomorphisms or vector fields on a compact Riemannian manifold. To describe the dynamics on the underlying manifold, it is usual to use the dynamic properties on the tangent bundle such as hyperbolicity, dominated splitting, partial hyperbolicity, etc.

A fundamental problem in recent years is to study the influence of a robust dynamic property (i.e., a property that holds for a given system and all C^1 nearby systems) on the behavior of the tangent map on the tangent bundle (for more details, see [2, 4, 5, 7, 13, 17, 22]).

Many of the dynamic results for diffeomorphisms can be extended to the case of vector fields, but not always. In particular, the results involving the hyperbolic structure or shadowing property may not be extended to the case of vector fields. For example, it is well known that if a diffeomorphism f has a C^1 neighborhood $\mathcal{U}(f)$ such that every periodic point of $g \in \mathcal{U}(f)$ is hyperbolic, then the nonwandering set $\Omega(f)$ is hyperbolic. However, the result is not true for the case of vector fields (for more details, see [5]).

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Chain components and homoclinic classes are natural candidates to replace the Smale's hyperbolic basic sets in non-hyperbolic theory of dynamical systems. Many recent papers, most of which are for diffeomorphisms only, have explored their hyperbolic-like properties such as dominated splitting, partial hyperbolicity, etc (for more details, see [4, 11, 12, 20, 21, 22]). For instance, Sakai ([20]) proved that if the chain component $C_f(p)$ of a diffeomorphism f containing a hyperbolic periodic point p is C^1 robustly shadowable and the $C_f(p)$-germ of f is expansive, then $C_f(p)$ is hyperbolic. Wen *et al.* [22] showed that the assumption of the $C_f(p)$-germ expansivity of f can be dropped in the above result to show the hyperbolicity of the C^1 robustly shadowable chain component $C_f(p)$. However, it is still an open question whether the above results can be extended to the case of vector fields. In fact, there is no known results for vector fields in this direction.

In this paper, we study the hyperbolic structure on the chain components of C^1 vector fields. More precisely, our main problem can be formally stated as follows.

Problem. *If the chain component of a vector field containing a hyperbolic periodic orbit is C^1 robustly shadowable, then is it hyperbolic?*

There are quite satisfactory answers for the systems (both in diffeomorphisms and in vector fields) given on the whole manifold. Robinson [18] and Sakai [19] showed that the C^1 interior of the set of diffeomorphisms having the shadowing property coincides with the set of Axiom A diffeomorphisms with the strong transversality condition.

Let us recall two recent papers which consider the above results for vector fields instead of diffeomorphisms. The first one is given by Lee and Sakai [13] to prove that a non-singular vector field belongs to the C^1 interior of the set of vector fields with the shadowing property if and only if it satisfies both Axiom A and the strong transversality condition (that is, it is structurally stable). The second one, by Pilyugin and Tikhomirov [17], deals with the *singular* vector fields. In [17], they introduced a special class $\mathcal{B}$ of vector fields that are not structurally stable to describe the C^1 interior, $Int^1(OrientSh)$, of the set of vector fields having the oriented shadowing property. Then they proved that $Int^1(OrientSh \setminus \mathcal{B})$ is characterized by the set of Axiom A vector fields with the strong transversality condition.

In attempting to solve **Problem** mentioned above, we face with several difficulties. For instance, the hyperbolic-like structures near singular points and near regular orbits of a vector field are qualitatively different, the time reparametrization in the shadowing theory of vector fields causes the complexity of the calculations, what kinds of dominated splitting (for flow or linear Poincaré flow) are suitable to get the hyperbolic structure, etc.

In this paper, we give a positive answer of the above **Problem** if the chain component does not contain a non-hyperbolic singularity.

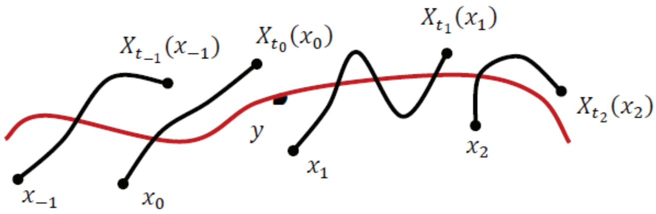

FIGURE 1. The pseudo orbit $\{(x_i, t_i)\}$ is shadowed by the orbit through y.

We shall describe the main definitions and results now. Let M be a compact boundaryless Riemannian manifold. Denote by $\mathcal{X}^1(M)$ the set of all C^1 vector fields of M endowed with the C^1 topology. Then every $X \in \mathcal{X}^1(M)$ generates a C^1 flow $X_t : M \times \mathbb{R} \to M$, that is, a family of diffeomorphisms on M such that $X_s \circ X_t = X_{t+s}$ for all $t, s \in \mathbb{R}$, $X_0 = Id$ and $dX_t(p)/dt|_{t=0} = X(p)$ for any $p \in M$. In this paper, for $X, Y, \ldots \in \mathcal{X}^1(M)$, we always denote the generated flows by $X_t, Y_t, \ldots$, respectively. For $x \in M$, let us denote the orbit $\{X_t(x), t \in \mathbb{R}\}$ of the flow X_t (or X) through x by $orb(x, X_t)$, or $orb(x)$ if no confusion is likely. We say that a point $x \in M$ is a *singularity* of X if $X(x) = 0$; and an orbit $orb(x)$ is *closed* (or *periodic*) if it is diffeomorphic to a circle S^1.

Let d be the distance induced from the Riemannian structure on M. A sequence $\{(x_i, t_i) : x_i \in M; t_i \geq 1; a < i < b\}$ $(-\infty \leq a < b \leq \infty)$ is called a *δ-pseudo orbit* or a *δ-chain* of X_t if for any $a < i < b-1$, $d(X_{t_i}(x_i), x_{i+1}) < \delta$. We say that a compact invariant set Λ of X_t is *shadowable for* X_t if for any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following property: given any δ-pseudo orbit $\{(x_i, t_i) : -\infty \leq i \leq \infty\}$ in Λ, there exist a point $y \in M$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that

$$d(X_{h(t)(y)}, X_{t-T_i}(x_i)) < \varepsilon, \quad T_i \leq t < T_{i+1},$$

where

$$T_i = \begin{cases} t_0 + t_1 + \cdots + t_{i-1} & \text{for} \quad i > 0, \\ 0 & \text{for} \quad i = 0, \\ -(t_{-1} + t_{-2} + \cdots + t_i) & \text{for} \quad i < 0. \end{cases}$$

Note that the above concept of pseudo orbit is slightly different from that of pseudo orbit in [13, 17]. However we point out here that a compact invariant set Λ is shadowable for X_t under the above definition if and only if it is shadowable for X_t under the definition in [13, 17].

A point $x \in M$ is called *chain recurrent* if for any $\delta > 0$, there exists a δ-pseudo orbit $\{(x_i, t_i) : 0 \leq i < n\}$ with $n > 1$ such that $x_0 = x$ and $d(X_{t_{n-1}}(x_{n-1}), x) < \delta$. The set of all chain recurrent points of X_t is called the *chain recurrent set* of X_t, denoted by $CR(X_t)$. It is easy to see that this set is closed and X_t-invariant. For any $x, y \in M$, we say that $x \sim y$, if for any $\delta > 0$, there are a δ-pseudo orbit $\{(x_i, t_i) : 0 \le i < n\}$ with $n > 1$ such that $x_0 = x$ and $d(X_{t_{n-1}}(x_{n-1}), y) < \delta$ and a δ-pseudo orbit $\{(x'_i, t'_i) : 0 \le i < m\}$ with $m > 1$ such that $x'_0 = y$ and $d(X_{t'_{n-1}}(x'_{n-1}), x) < \delta$. It is easy to see that $\sim$ gives an equivalence relation on the set $CR(X_t)$. An equivalence class of $\sim$ is called a *chain component* of X_t (or X).

A compact invariant set Λ of X_t is called *hyperbolic* if there are constants $C > 0$ and $\lambda > 0$ such that the tangent flow $DX_t : T_\Lambda M \to T_\Lambda M$ leaves a continuous invariant splitting $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ satisfying

$$\left\| DX_t|_{E^s(x)} \right\| \le Ce^{-\lambda t} \quad \text{and} \quad \left\| DX_{-t}|_{E^u(x)} \right\| \le Ce^{-\lambda t}$$

for any $x \in \Lambda$ and $t > 0$, where $\langle X \rangle$ denotes the subspace generated by the vector field X (for more details, see [7, 15]).

Let γ be a hyperbolic closed orbit of X_t. Then we know that there are a C^1 neighborhood $\mathcal{U}$ of X and a neighborhood U of γ such that for any $Y \in \mathcal{U}$, there is a unique hyperbolic closed orbit γ_Y which equals to $\bigcap_{t\in\mathbb{R}} Y_t(U)$. The hyperbolic closed orbit γ_Y is called the *continuation* of γ with respect to Y_t. Contrary to the diffeomorphisms, the period of the continuation orbit γ_Y may not be equal to the period of γ.

For any hyperbolic closed orbit γ, the sets $W^s(\gamma) = \{x \in M : X_t(x) \to \gamma \text{ as } t \to \infty\}$ and $W^u(\gamma) = \{x \in M : X_t(x) \to \gamma \text{ as } t \to -\infty\}$ are said to be the *stable manifold* and *unstable manifold* of γ, respectively. We say that the dimension of the stable manifold $W^s(\gamma)$ of γ is the *index* of γ, and denoted by $ind(\gamma)$.

Denote by $C_X(\gamma)$ the chain component of X which contains the hyperbolic closed orbit γ. The *homoclinic class* of X_t associated to γ, denoted by $H_X(\gamma)$, is defined as the closure of the transversal intersection of the stable and unstable manifolds of γ, that is;

$$H_X(\gamma) = \overline{W^s(\gamma) \pitchfork W^u(\gamma)},$$

where $W^s(\gamma)$ is the stable manifold of γ and $W^u(\gamma)$ is the unstable manifold of γ. By definition, we easily see that the set is closed and X_t-invariant. Moreover $H_X(\gamma) \subset C_X(\gamma)$, but the converse is not true in general. For two hyperbolic closed orbits γ_1 and γ_2 of X_t, we say γ_1 and γ_2 are *homoclinically related*, denoted by $\gamma_1 \sim \gamma_2$, if $W^s(\gamma_1) \pitchfork W^u(\gamma_2) \neq \emptyset$ and $W^s(\gamma_2) \pitchfork W^u(\gamma_1) \neq \emptyset$. When γ_1 and γ_2 are homoclinically related, their indices must be the same. By Smale's Theorem, it is well known that

$$H_X(\gamma) = \overline{\{\gamma' : \gamma' \sim \gamma\}}.$$

A point $x \in M$ is called *nonwandering* if for any neighborhood U of x, there is $t \ge 1$ such that $X_t(U) \cap U \neq \emptyset$. The set of all nonwandering points of X_t is called the *nonwandering set* of X_t, denoted by $\Omega(X_t)$. Let $Sing(X)$ be the set of all singularities of X, and let $PO(X_t)$ be the set of all closed orbits (which are not singularities) of X_t. Clearly, $Sing(X) \cup PO(X_t) \subset \Omega(X_t)$. We say that X satisfies *Axiom A* if $PO(X_t)$ is dense in $\Omega(X_t) \setminus Sing(X)$, and $\Omega(X_t)$ is hyperbolic for X_t.

Now we give the definitions of robust shadowability for various invariant sets of vector fields.

Definition 1.1. Let $X \in \mathcal{X}^1(M)$, and γ be a hyperbolic closed orbit of X_t. The chain component $C_X(\gamma)$ containing γ is said to be C^1 *robustly shadowable* if there is a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that for any $Y \in \mathcal{U}$, $C_Y(\gamma_Y)$ is shadowable for Y_t, where γ_Y is the continuation of γ.

Moreover we say that the chain recurrent set $CR(X_t)$ is C^1 *robustly shadowable* if there is a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that for any $Y \in \mathcal{U}$, $CR(Y_t)$ is shadowable for Y_t, where $CR(Y_t)$ is the chain recurrent set of Y_t.

Similarly, we can introduce the notions of C^1 robust shadowability of homoclinic class $H_X(\gamma)$ and nonwandering set $\Omega(X_t)$ as follows: $H_X(\gamma)$ is C^1 *robustly shadowable* if there is a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that for any $Y \in \mathcal{U}$, $H_Y(\gamma_Y)$ is shadowable for Y_t; $\Omega(X_t)$ is C^1 *robustly shadowable* if there is a neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that for any $Y \in \mathcal{U}$, $\Omega(Y_t)$ is shadowable for Y_t.

In this paper, we prove the following main theorem.

Main Theorem. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. If the chain component $C_X(\gamma)$ is C^1 robustly shadowable, then it does not contain a hyperbolic singularity. Moreover it is hyperbolic if $C_X(\gamma)$ does not contain a non-hyperbolic singularity.*

From the robustness of hyperbolic sets, it is easy to see if $C_X(\gamma)$ is hyperbolic, then it is C^1 robustly shadowable. In the above theorem, the non-existence of non-hyperbolic singularity in the chain component is a technical condition that leaves an open question on possibility of removing the condition. The authors have no example of C^1 robustly shadowable chain component which contains a non-hyperbolic singularity at present time.

The paper is organized as follows. Section 2 is devoted to Poincaré map and linear Poincaré flow (LPF) including two perturbation lemmas which are necessary to prove our theorems. In particular, we introduce a theorem which is crucial to get the hyperbolicity of invariant subsets for vector fields.

In Section 3, we show that the C^1 robustly shadowable chain component $C_X(\gamma)$ admits a dominated splitting for the linear Poincaré flow Ψ_t of X if $C_X(\gamma)$ does not contain a non-hyperbolic singularity. To prove this, we first show that there is no hyperbolic singularity contained in $C_X(\gamma)$ (Proposition 3.1). Then we claim the chain component $C_X(\gamma)$ and the homoclinic class $H_X(\gamma)$ coincide (Lemma 3.5). Moreover, we prove that there is a lower bound for the angles between tangent spaces of stable manifolds and unstable manifolds of hyperbolic closed orbits on normal sections (Proposition 3.7). Finally we show that $C_X(\gamma)$ admits a Ψ_t-dominated splitting $N_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ with $\dim \Delta^s = ind(\gamma)$ (Proposition 3.9).

In Section 4, we state and prove a proposition (Proposition 4.1) which is important to prove our Main Theorem. The proposition originally comes from the classical results by Mañé [14, Proposition II.1].

In Section 5, we prove that the Ψ_t-dominated splitting $N_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ obtained in Section 3 is in fact a hyperbolic splitting with respect to the linear Poincaré flow Ψ_t. The proof is completed by showing that if Δ^s is not contracting for Ψ_t, then we can find a "good" hyperbolic periodic point of a hyperbolic closed orbit γ' (γ' is homoclinically related to γ) which contradicts to the property (2) of Proposition 4.1.

2. Poincaré map and linear Poincaré flow

Hereafter we assume that the exponential map $\exp_p : T_pM(1) \to M$ is well defined for all $p \in M$, where $T_pM(r)$ denotes the r-ball $\{v \in T_pM : \|v\| \le r\}$ in T_pM.

For any regular point $x \in M$ (i.e., $X(x) \ne 0$), we let $N_x = (\text{span } X(x))^\perp \subset T_xM$ and $N_x(r)$ the r-ball in N_x. Let $\hat{N}_{x,r} = \exp_x(N_x(r))$. Given any regular point $x \in M$ and $t \in \mathbb{R}$, we can take a constant $r > 0$ and a C^1 map $\tau : \hat{N}_{x,r} \to \mathbb{R}$ such that $\tau(x) = t$ and $X_{\tau(y)}(y) \in \hat{N}_{X_t(x),1}$ for any $y \in \hat{N}_{x,r}$. Now we define the *Poincaré map*

$$f_{x,t} : \hat{N}_{x,r} \to \hat{N}_{X_t(x),1}, \; f_{x,t}(y) = X_{\tau(y)}(y)$$

for $y \in \hat{N}_{x,r}$. Let $M_X = \{x \in M : X(x) \ne 0\}$. Then it is easy to check that for any fixed t there exists a continuous map $r_0 : M_X \to (0,1)$ such that for any $x \in M_X$, the Poincaré map $f_{x,t} : \hat{N}_{x,r_0(x)} \to \hat{N}_{X_t(x),1}$ is well defined and the respective time function $\tau(y)$ satisfies $2t/3 < \tau(y) < 4t/3$ for $y \in \hat{N}_{x,r_0(x)}$.

Let t_0 be fixed. At each $x \in M_X$, one can consider a *flow box chart* $(\hat{U}_{x,t_0,\delta}, F_{x,t_0})$ at x such that

$$\hat{U}_{x,t_0,\delta} = \{tX(x) + y : 0 \le t \le t_0, y \in N_x(\delta)\} \subset T_xM,$$

where $F_{x,t_0} : \hat{U}_{x,t_0,\delta} \to M$ is defined by $F_{x,t_0}(tX(x) + y) = X_t(\exp_x y)$. Then it is well known that if $X_t(x) \ne x$ for any $t \in (0, t_0]$, then there is $\delta > 0$ such that $F_{x,t_0} : \hat{U}_{x,t_0,\delta} \to M$ is an embedding.

Lemma 2.1. *Let $t_0 > 0$ be given. Then there are a constant $K > 1$ and a continuous function $r_1 : M_X \to (0,1)$ such that for any $x \in M_X$, if we let $\hat{U}_x = \{tX(x) + y : 0 \le t \le t_0, y \in N_x(r_1(x))\}$ and $F_x = F_{x,t_0}|_{\hat{U}_x}$, then $K^{-1} < \|D_uF_x\|$ and $m(D_uF_x) < K$ for any $u \in \hat{U}_x$, where $m(D_uF_x)$ denotes the minimum norm of D_uF_x.*

Proof. See [16, pages 290–291]. □

For $\varepsilon > 0$ and $r > 0$, let $\mathcal{N}_\varepsilon(\hat{N}_{x,r})$ be the set of all diffeomorphisms $\phi : \hat{N}_{x,r} \to \hat{N}_{x,r}$ such that $supp(\phi) \subset \hat{N}_{x,r/2}$ and $d_{C^1}(\phi, id) < \varepsilon$. Here d_{C^1} is the usual C^1 metric, id denotes the identity map and the $supp(\phi)$ is the closure of the set of points where it differs from id.

Proposition 2.2. *Let $X \in \mathcal{X}^1(M)$, and let $\mathcal{U} \subset \mathcal{X}^1(M)$ be a neighborhood of X. For any constant $t_0 > 0$, there are a constant $\varepsilon > 0$ and a neighborhood $\mathcal{V}$ of X such that for any $Y \in \mathcal{V}$, there exists a continuous map $r : M_Y \to (0,1)$ satisfying the following property: for any $x \in M_Y$ satisfying $Y_t(x) \neq x$ for $0 < t \leq 2t_0$ and any $\phi \in \mathcal{N}_\varepsilon(\hat{N}_{x,r(x)})$, there is $Z \in \mathcal{U}$ such that $Y(z) = Z(z)$ for all $z \in M \backslash F_x(\hat{U}_x)$ and $Z_t(y) = Y_t(\phi(y))$ for any $y \in \hat{N}_{x,r(x)}$ and $2t_0/3 < t < 4t_0/3$, where $F_x(\hat{U}_x)$ is the flow box of Y at x.*

Proof. See [16, pages 293–295]. □

Remark 2.3. In Proposition 2.2, it is easy to see that if $\phi(x) = x$, then $f_{x,t_0} \circ \phi$ is the Poincaré map of Z, where $f_{x,t_0} : \hat{N}_{x,r(x)} \to \hat{N}_{X_{t_0}(x),1}$ is the Poincaré map of Y.

To study the Stability Conjecture (now it is proved; for more details, see [7]) posed by Palis and Smale, Liao [10] introduced the notion of linear Poincaré flow for a C^1 vector field as follows. Let $\mathcal{N} = \bigcup_{x \in M_X} N_x$ be the normal bundle based on M_X. Then we can introduce a flow (which is called a *linear Poincaré flow* for X)

$$\Psi_t : \mathcal{N} \to \mathcal{N}, \ \Psi_t|_{N_x} = \pi_{N_x} \circ D_x X_t|_{N_x},$$

where $\pi_{N_x} : T_x M \to N_x$ is the natural projection along the direction of $X(x)$, and $D_x X_t$ is the derivative map of X_t. Then we can see that

$$\Psi_t|_{N_x} = D_x f_{x,t} \quad \text{and} \quad f_{x,t} \circ \exp_x = \exp_{X_t(x)} \circ \Psi_t.$$

Using Proposition 2.2, we can prove the following lemma which has the same philosophy as that of the Franks' Lemma for diffeomorphisms. One can find another proof for the lemma in [2].

Lemma 2.4. *Let $\mathcal{U}$ be a C^1 neighborhood of $X \in \mathcal{X}^1(M)$. For any $T > 0$, there exists a constant $\eta > 0$ such that for any tubular neighborhood U of an orbit arc $\gamma = X_{[0,T]}(x)$ of X_t and for any η-perturbation $\mathcal{F}$ of the linear Poincaré flow $\Psi_T|_{N_x}$, there exists a vector field $Y \in \mathcal{U}$ such that the linear Poincaré flow $\tilde{\Psi}_T|_{N_x}$ associated to Y coincides with $\mathcal{F}$, and Y coincides with X outside U and along $X_{[-t_1,t_2]}(x)$, where $t_1 = \min\{t > 0, X_{-t}(x) \in \partial U\}$ and $t_2 = \min\{t > 0, X_t(x) \in \partial U\}$.*

We define the notions of hyperbolic splitting and dominated splitting for linear Poincaré flows as follows.

Definition 2.5. Let Λ be an invariant set of X_t which contains no singularity. We call a Ψ_t-invariant splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$ is an l-dominated splitting if

$$\left\| \Psi_t|_{\Delta^s(x)} \right\| \cdot \left\| \Psi_{-t}|_{\Delta^u(X_t(x))} \right\| \leq \frac{1}{2}$$

for any $x \in \Lambda$ and any $t \geq l$, where $l > 0$ is a constant. Moreover, if $\dim(\Delta^s_x)$ is constant for all $x \in \Lambda$, then we say that the splitting is a homogeneous dominated splitting.

We say a Ψ_t-invariant splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$ is a hyperbolic splitting if there exist $C > 0$ and $\lambda \in (0,1)$ such that

$$\left\|\Psi_t|_{\Delta^s(x)}\right\| \leq C\lambda^t \quad \text{and} \quad \left\|\Psi_{-t}|_{\Delta^u(x)}\right\| \leq C\lambda^t$$

for any $x \in \Lambda$ and $t > 0$.

The geometric interpretation of the dominated splitting is that for any one-dimensional subspace $L \subset \Delta^s_x \oplus \Delta^u_x$ not contained in Δ^s_x, $x \in \Lambda$, the angle (see the definition of angle in Section 3) between $\Psi_t(L)$ and $\Delta^u(X_t(x))$ converges exponentially to zero as $t \to \infty$.

The following theorem which is crucial to get the hyperbolicity of compact invariant sets for vector fields was proved by Liao and Doering. For a detailed proof, see Proposition 1.1 in [3].

Theorem 2.6. *Let $\Lambda \subset M$ be a compact invariant set of X_t such that $\Lambda \cap Sing(X) = \emptyset$. Then Λ is hyperbolic for X_t if and only if the linear Poincaré flow Ψ_t restricted on Λ has a hyperbolic splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$.*

3. Chain component, homoclinic class and dominated splitting

In this section we show that if the chain component $C_X(\gamma)$ of X containing a hyperbolic closed orbit γ is C^1 robustly shadowable, then there is no hyperbolic singularity contained in $C_X(\gamma)$, and $C_X(\gamma)$ equals to the homoclinic class $H_X(\gamma)$. Finally we prove that $C_X(\gamma)$ admits a Ψ_t-dominated splitting $\mathcal{N}_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ with $\dim(\Delta^s) = ind(\gamma)$ if it does not contain a non-hyperbolic singularity.

Proposition 3.1. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. If $C_X(\gamma)$ is C^1 robustly shadowable, then it does not contain a hyperbolic singularity.*

Proof. Assume there is a hyperbolic singularity σ contained in $C_X(\gamma)$. We know that there is a constant $\eta > 0$ such that

$$\begin{aligned}
W^s_\eta(\sigma) &= \{x \in M : d(X_t(x), \sigma) \leq \eta \ \text{ for all } t > 0\},\\
W^u_\eta(\sigma) &= \{x \in M : d(X_t(x), \sigma) \leq \eta \ \text{ for all } t < 0\},\\
W^s_\eta(\gamma) &= \{x \in M : d(X_t(x), \gamma) \leq \eta \ \text{ for all } t > 0\},\\
W^u_\eta(\gamma) &= \{x \in M : d(X_t(x), \gamma) \leq \eta \ \text{ for all } t < 0\}
\end{aligned}$$

are embedded submanifolds of M, and satisfy $W^s_\eta(\sigma) \subset W^s(\sigma)$, $W^u_\eta(\sigma) \subset W^u(\sigma)$, $W^s_\eta(\gamma) \subset W^s(\gamma)$ and $W^u_\eta(\gamma) \subset W^u(\gamma)$.

Since $C_X(\gamma)$ is shadowable for X_t, there is $\delta > 0$ such that every δ-pseudo orbit in $C_X(\gamma)$ can be η-shadowed. Since $\sigma \in C_X(\gamma)$, we can construct a δ-pseudo orbit $\xi = \{(x_i, t_i)\}_{i=0}^n$ such that all $x_i \in C_X(\gamma)$ and $x_0 = \sigma$ and

$x_n(=p) \in \gamma$. We extend ξ to $\{(x_i,t_i)\}_{i=-\infty}^{\infty}$ by defining $x_{-k}=\sigma, t_{-k}=1$ for all $k>0$ and $x_{n+k}=X_k(p), t_{n+k}=1$ for all $k \geq 0$. Let y be a point in M which η shadows the extended pseudo orbit. Then we can easily check that $y \in W^u(\sigma) \cap W^s(\gamma)$. Similarly we can find a point y' satisfying $y' \in W^u(\gamma) \cap W^s(\sigma)$. Since $\dim W^u(\sigma) + \dim W^s(\sigma) = \dim M$ and $\dim W^u(\gamma) + \dim W^s(\gamma) = \dim M - 1$, we get either

$$T_yW^u(\sigma) + T_yW^s(\gamma) \neq T_yM \quad \text{or} \quad T_{y'}W^s(\sigma) + T_{y'}W^u(\gamma) \neq T_{y'}M.$$

In the following steps, we will prove that $T_yW^u(\sigma) + T_yW^s(\gamma) \neq T_yM$ leads to a contradiction by applying the C^1 robust shadowability of $C_X(\gamma)$. Similarly we can show that $T_{y'}W^s(\sigma) + T_{y'}W^u(\gamma) \neq T_{y'}M$ leads to a contradiction.

Without loss of generality, we assume that there is $p \in \gamma$ such that $y, y' \in \hat{N}_{p,r}$. Let $f : \hat{N}_{p,r} \to \hat{N}_{p,1}$ be the Poincaré map associated to the period of p. For any $\delta > 0$, by applying Proposition 3.1 of [6], one can construct a diffeomorphism $f' : \hat{N}_{p,r} \to \hat{N}_{p,1}$ such that $d_{C^1}(f,f') < \delta$, f' keeps the fixed point p, just differs from f in a δ-neighborhood of p, and f' satisfies the following properties:

(a) $y \in W^s(p,f')$, $y' \in W^u(p,f')$,
(b) there is $r' > 0$ such that $\exp_p^{-1} \circ f' \circ \exp_p(v) = L(v)$ whenever $|v| \leq r'$, where $L : N_p \to N_p$ is a linear map close to D_pf and all eigenvalues of L is are multiplicity 1,
(c) $T_yW^s(p,f') = T_yW^s(p,f)$.

Then we get the following lemma using Proposition 2.2.

Lemma 3.2. *For any neighborhood $\mathcal{U}$ of X, there exists $Y_1 \in \mathcal{U}$ such that Y_1 keeps the orbit of γ unchanged, just differs from X in a small neighborhood of γ, and satisfies the following properties*:

(1) *$y \in W^u(\sigma,Y_1) \cap W^s(\gamma,Y_1)$ and $y' \in W^s(\sigma,Y_1) \cap W^u(\gamma,Y_1)$,*
(2) *if $f' : \hat{N}_{p,r} \to \hat{N}_{p,1}$ is the Poincaré map associated to Y_1 with the period of γ, then there exist $r' > 0$ and a linear map A such that any eigenvalue of A is of multiplicity 1 satisfying $f'|_{\hat{N}_{p,r'}} = \exp_p \circ A \circ \exp_p^{-1}|_{\hat{N}_{p,r'}}$,*
(3) *$T_yW^u(\sigma,Y_1) + T_yW^s(\gamma,Y_1) \neq T_yM$.*

It is easy to check that σ, y and y' are elements of the chain component of Y_1 that contains γ. Let $N_p = \Delta^s \oplus \Delta^u$ be the hyperbolic splitting associated to f'. Since f' is locally linear at p, there is $\eta' > 0$ such that

$$W^s_{\eta'}(\gamma,Y_1) \cap \hat{N}_{p,r} \subset \exp_p(\Delta^s) \quad \text{and} \quad W^u_{\eta'}(\gamma,Y_1) \cap \hat{N}_{p,r} \subset \exp_p(\Delta^u).$$

Without loss of generality, we can assume $y \in W^s_{\eta'}(\gamma,Y_1) \cap \hat{N}_{p,r}$. Let $V^u(y)$ be the connected component of $W^u(\sigma,Y_1) \cap \hat{N}_{p,r}$ containing y. Then

$$\exp_p^{-1}(V^u(y) \subset N_p \quad \text{and} \quad T_yV^u(y) = T_y(W^u(\sigma,Y_1) \cap N_p).$$

Let $L \subset N_p$ be an affine space tangent to $\exp_p^{-1}(V^u(y))$ at $\exp_p^{-1}(y)$. Denote by $\pi : N_p \to \Delta^u$ the natural projection parallel to Δ^s. Then the non-transversality of $V^u(y)$ and $W^s(p, f')$ at y means $\dim \pi(L) < \dim \Delta^u$.

For any $\varepsilon > 0$, let $L_\varepsilon = \{v \in L : \|v - \exp_p^{-1}(y)\| < \varepsilon\}$. Choose $t_1 > 0$ such that $Y1_{[-t_1,-t_1+1]}(y) \cap B_{r'}(\gamma) = \emptyset$. Using Proposition 2.2, one can construct a vector Y arbitrarily close to Y_1 and just differs from Y_1 in an arbitrarily small neighborhood of the arc $Y1_{[-t_1,-t_1+1]}(y)$ such that there is $\varepsilon > 0$ such that $\exp_p(L_\varepsilon)$ is contained in the connected component of $W^u(\sigma, Y) \cap \hat{N}_{p,r}$ containing y. Now we will prove the $C_Y(\gamma)$ is not shadowable for Y_t.

For any $\varepsilon > 0$, we denote by $C^u_\varepsilon(y)$ the set of all points $x \in \hat{N}_{p,r}$ which has the following property: there exists an increasing continuous function $h : (-\infty, 0] \to \mathbb{R}$ such that $d(Y_{h(t)}(x), Y_t(y)) < \varepsilon$ for all $t \in (-\infty, 0]$. Then we have the following lemma.

Lemma 3.3. *There exists $\varepsilon_0 > 0$ such that $C^u_\varepsilon(y) \subset \exp_p(L_\varepsilon)$ for any $0 < \varepsilon \le \varepsilon_0$.*

Proof. Let $\eta > 0$ be a constant such that the local unstable manifold $W^u_\eta(\sigma, Y)$ is well defined. Choose a constant $t_0 > 0$ satisfying $Y_{-t_0}(y) \in W^u_{\eta/2}(\sigma, Y)$. By the Tubular Flow Theorem, we know that there is $\varepsilon_1 > 0$ such that if a point $x \in M$ and an increasing continuous function $h : \mathbb{R} \to \mathbb{R}$ satisfy $d(Y_{h(t)}(x), Y_t(y)) < \varepsilon_1$ for all $t \in [-t_0, 0]$, then

$$\frac{t_0}{2} < h(0) - h(-t_0) < \frac{3t_0}{2}.$$

Choose $\varepsilon_2 > 0$ such that $B_{\varepsilon_2}(y) \cap Y_{3t_0/2}(W^u_\eta(\sigma, Y)) \subset \exp_p L_\varepsilon$. Then the constant $\varepsilon = \min\{\eta/2, \varepsilon_1, \varepsilon_2\}$ satisfies the lemma. □

Denote by $g : \hat{N}_{p,r} \to \hat{N}_{p,r}$ the Poincaré map associated to Y and the hyperbolic closed orbit γ. Let τ be the period of γ. We can assume that r is small enough so that for any $x \in \hat{N}_{p,r}$, the first return time $\tau(x)$ of y satisfies $3\tau/4 < \tau(x) < 5\tau/4$ for all $x \in \hat{N}_{p,r}$. From Lemma 3.2, there is $r' > 0$ such that

$$g|_{\hat{N}_{p,r'}} = \exp_p \circ A \circ \exp_p^{-1}|_{\hat{N}_{p,r'}}.$$

From the Tubular Flow Theorem, one can get the following lemma.

Lemma 3.4. *For any $\varepsilon' > 0$, there exists $\varepsilon > 0$ satisfying the following properties: for any $x \in \hat{N}_{p,r'}$, if a point $x' \in M$ and an increasing continuous map $h : [0, +\infty) \to [0, +\infty)$ satisfies $d(Y_t(x), Y_{h(t)}(x')) < \varepsilon$ for any $0 \le t \le \tau(x)$, then there exists $a, b \in (-\tau/10, \tau/10)$ such that*

1. *$Y_{h(0)-a}(x') \in \hat{N}_{p,r}$,*
2. *$|\exp_p^{-1}(Y_{h(0)-a}(x')) - \exp_p^{-1}(x)| < \varepsilon'$,*
3. *$Y_{h(\tau(x))+b}(x') \in \hat{N}_{p,r}$,*
4. *$|\exp_p^{-1}(Y_{h(\tau(x))+b}(x')) - \exp_p^{-1}(Y_{\tau(x)}(x))| < \varepsilon'$, and*
5. *$h(\tau(x)) - h(0) + b + a$ is just the first return time of $Y_{h(0)-a}(x')$.*

Let λ_1 be the eigenvalue of A which satisfies

$$|\lambda_1| = \min\{|\lambda| : \lambda \text{ is an eigenvalue of } A \text{ with } |\lambda| > 1\}.$$

By Lemma 3.2, λ_1 has multiplicity 1. Let E_1^u be the eigenspace corresponding to the eigenvalue λ_1, and Δ^{uu} be the eigenspace associated to the other unstable eigenvalues. Now we have two possible cases: λ_1 is real or λ_1 is complex.

Case 1: λ_1 *is real.* In the construction of Y, we can assume $\pi(L) \cap E_1^u = \{0\}$ (since $\dim \pi(L) < \dim \Delta^u$). We can also assume that y' (which is contained in the intersection of $W^s(\sigma, Y)$ and $W^u(\sigma, \gamma)$) is very close to p, but it is not contained in $\exp_p \Delta^{uu}$. Now we find $\varepsilon > 0$ such that for any $\delta > 0$, there exists a δ-pseudo orbit which can not be ε-shadowed. In the following steps, we will take an equivalent norm $\|\cdot\|$ defined on $N_p = \Delta^s \oplus E_1^u \oplus \Delta^{uu}$ defined by:

$$\|v_1 + v_2 + v_3\| = \max\{\|v_1\|, \|v_2\|, \|v_3\|\}$$

for any $v_1 \in \Delta^s$, $v_2 \in E_1^u$, and $v_3 \in \Delta^{uu}$.

Choose $\varepsilon_1 > 0$ such that $B(2\varepsilon_1) = \{\exp_p(v) : v \in N_p, \|v\| \le 2\varepsilon_1\} \subset \hat{N}_{p,r'}$. Without loss of generality, we can assume $\{g^n(y) : n \in \mathbb{Z}^+\} \subset B(\varepsilon_1)$ and $\exp_p(L_\varepsilon)$ cross the ball $B(2\varepsilon_1)$ (otherwise we just choose a large l and use $g^l(y)$ instead of y). We can also assume $y' \in B(2\varepsilon_1)$. Choose $\varepsilon' \in (0, \varepsilon_0)$ such that $d(\exp_p^{-1} y', \Delta^{uu}) > 2\varepsilon'$. Take a constant $0 < \varepsilon < \varepsilon_0$ (ε_0 is given in Lemma 3.3) which satisfies Lemma 3.4 associated to $\varepsilon'/2$. We will show that for any $\delta > 0$, we can construct a δ-pseudo orbit of Y_t which can not be ε-shadowed.

Since E_1^u is the eigenspace corresponding to the weakest expanding direction, we have $A^k\pi(L) \to \Delta^{uu}$ as $k \to \infty$. Hence, there is $K > 0$ such that for any $k \ge K$,

$$d(y', A^k(\pi(L))) > \varepsilon'/2.$$

Let $y_m = g^m(y)$. Then it is easy to check that $d(p, y_m) \to 0$ as $m \to +\infty$. If we let $y'_{-n} = g^{-n}y'$, we have $d(p, y'_{-n}) \to 0$ as $n \to +\infty$. We construct a pseudo orbit $\xi = \{(x_i, t_i)\}_{i=-\infty}^{+\infty}$ by defining $x_i = X_i(y_m)$ for $i < 0$, $x_i = X_i(y'_{-n})$ for $i \ge 0$, and all $t_i = 1$. If there is $z_{m,n} \in M$ which ε-shadows the pseudo orbit ξ, then there is an increasing continuous map $h : \mathbb{R} \to \mathbb{R}$ such that

$$d(X_{h(t)}(z_{m,n}), X_t(y_m)) < \varepsilon \quad \text{for} \quad t < 0, \quad \text{and}$$
$$d(X_{h(t)}(z_{m,n}), X_t(y'_{-n})) < \varepsilon \quad \text{for} \quad t > 0.$$

Let τ_i be the first return time of $g^i(y'_{-n})$ and $T_i = \sum_{i=0}^{n} t_i$ for $i = 0, 1, \ldots, n$. By Lemma 3.4, there is $a_i \in [-\tau/10, \tau/10]$ such that

$$Y_{h(T_i)-a_i}(z_{m,n}) \in \hat{N}_{p,r'} \quad \text{and} \quad d(X_{h(T_i)-a_i}(z_{m,n}), g^i(y'_{-n})) < \varepsilon'/2$$

for $i = 0, 1, \ldots, m$. Let $x_i = X_{h(T_i)-a_i}(z_{m,n})$. By Lemma 3.4, we get

$$x_{i+1} = g(x_i) \quad \text{and} \quad \|\exp_p^{-1}(g^i(y'_{-n})) - \exp_p^{-1}(x_i)\| < \varepsilon'/2$$

for $i = 0, 1, \ldots, n$. Similarly, we can take $x_{-i} = g^{-i}(x_0)$ which satisfies

$$\|\exp_p^{-1}(g^{-i}(y'_m)) - \exp_p^{-1}(x_i)\| < \varepsilon'/2$$

for $0 \le i \le m$. From Lemma 3.3, we have $x_{-m} \in \exp_p(L_\varepsilon)$ and hence $x_{-m+k} \in g^k(\exp_p(L_\varepsilon))$. It means that

$$\exp_p^{-1}(x_{-m+k}) \in A^k(\exp_p^{-1}(y)) + A^k(\pi(L))$$

for $i = 0, \dots, m+n$. If $m + n > K$, then

$$\| \exp_p^{-1}(g^n(y'_{-n})) - \exp_p^{-1}(x_n)\| = \|y' - \exp_p^{-1}(x_m)\| > \varepsilon'/2,$$

and so we get a contradiction.

Case 2: *λ_1 is complex.* Since $\dim \pi(L) < \dim(E_1^u \oplus \Delta^{uu})$, we can assume $\pi(L) \cap E_1^u \ne E_1^u$. Let $\lambda_1 = |\lambda_1| e^{i\theta}$. Without loss of generality, we may assume θ is a rational angle. Then $\{A^k(\pi(L) \cap E_1^u) : k \in \mathbb{Z}\}$ consists of finite lines. After an arbitrarily small perturbation, we can also assume that $\pi_1(y')$ is not contained in $\{A^k \pi(L) \cap E_1^u : k \in \mathbb{Z}\}$, where π_1 denotes the projection from Δ^u to E_1^u. Similarly we can choose a small constant $\varepsilon' > 0$ such that

$$2\varepsilon' < d\left(y', \{A^k(\pi(L) \cap E_1^u) : k \in \mathbb{Z}\}\right).$$

If k is large enough, we have $d(y', A^k(\pi(L))) > \varepsilon'$. Let $y_m = g^m(y)$ and $y'_{-n} = g^{-n}y'$. It is easy to check that $d(y_m, y'_{-n}) \to 0$ as $m, n \to +\infty$. As in the previous case, we can construct a pseudo orbit $\xi = \{(x_i, t_i)\}_{i=-\infty}^{+\infty}$ by defining $x_i = X_i(y_m)$ for $i < 0$, $x_i = X_i(y'_{-n})$ for $i \ge 0$, and all $t_i = 1$. Let ε_0 be the constant given in Lemma 3.3, and take a constant $\varepsilon \in (0, \varepsilon_0)$ corresponding to $\varepsilon'/2$ in Lemma 3.4. Then we can see that the pseudo orbit ξ can not be ε-shadowed when $m+n$ is large enough. This completes the proof of Proposition 3.1. □

We know that every point in a shadowable chain component can be approximated by homoclinic points as we see in the following lemma.

Lemma 3.5. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. If $C_X(\gamma)$ is shadowable, then for any point $x \in C_X(\gamma)$ and $\varepsilon > 0$, there exists $y \in W^s(\gamma) \cap W^u(\gamma)$ satisfying $d(x, y) < \varepsilon$; that is, $C_X(\gamma) = H_X(\gamma)$.*

Proof. For any $x \in C_X(\gamma)$ and any $\delta > 0$, we can construct a δ-pseudo orbit $\{(x_i, t_i)\}_{i=-k}^{n}$ in $C_X(\gamma)$ such that $x_0 = x$. Then we can extend the finite pseudo orbit to an infinite pseudo orbit by defining; $x_{n+i} = X_i(x_n)$ and $t_{n+i} = 1$ for $i \ge 0$, and $x_{-(k+i)} = X_{-i}(x_{-k})$ and $t_{-(k+i+1)} = 1$ for $i \ge 1$. Fix $\eta > 0$ small enough so that

$$W_\eta^s(\gamma) = \{x \in M : d(X_t(x), \gamma) \le \eta \ \text{ for all } t > 0\}$$

and

$$W_\eta^u(\gamma) = \{x \in M : d(X_t(x), \gamma) \le \eta \ \text{ for all } t < 0\}$$

are embedded submanifolds of M, and satisfy $W_\eta^s(\gamma) \subset W^s(\gamma)$ and $W_\eta^u(\gamma) \subset W^u(\gamma)$. For any $0 < \varepsilon < \eta$, if y ε-shadows ξ then we can see $y \in W^s(\gamma) \cap W^u(\gamma)$ and $d(x, y) < \varepsilon$. This completes the proof of Lemma 3.5. □

As we can see in the following proposition, the C^1 robust shadowability prevents the existence of non-transverse homoclinic points.

Proposition 3.6. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. If $C_X(\gamma)$ is C^1 robustly shadowable, then there exists a neighborhood $\mathcal{U}$ of X such that for any $Y \in \mathcal{U}$ and any $y \in W^s(\gamma_Y, Y) \cap W^u(\gamma_Y, Y)$, we have*

$$T_yM = T_yW^s(\gamma_Y, Y) \oplus T_yW^u(\gamma_Y, Y).$$

Proof. The proof is similar to the proof of Proposition 3.1. In fact, we can get a contradiction if we assume $T_yW^s(\gamma_Y, Y) + T_yW^u(\gamma_Y, Y) \neq T_yM$. We just note here that we can take $X_{-T}(y)$ for a large T to replace the point y' in the proof of Proposition 3.1. We omit the details here. □

Now we show that if $C_X(\gamma)$ is C^1 robustly shadowable and does not contain a non-hyperbolic singularity, then $C_X(\gamma)$ admits a Ψ_t-dominated splitting $\mathcal{N}_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ with $\dim(\Delta^s) = ind(\gamma)$. For this, let us first introduce the notion of angle between two subspaces of R^d. Let E and F be two subspaces of $\mathbb{R}^d$ with $E \oplus F = \mathbb{R}^d$. Hence $\dim(F) = \dim(E^\perp)$ and F is the graph of the linear map $L : E^\perp \to E$ defined as follows: given $v \in E^\perp$, there exists a unique pair (u, v), $u \in E, w \in F$ such that $v + u = w$; define $L(v) = u$ so that L is linear and $graph(L) = F$. We define the *angle* $\angle(E, F)$ between E and F by

$$\angle(E, F) = \frac{1}{\|L\|}$$

(for more details, see [14]).

From Proposition 3.6, we have the following proposition that gives a lower bound for the angles between tangent spaces of stable manifolds and unstable manifolds of hyperbolic closed orbits on normal sections.

Proposition 3.7. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. If $C_X(\gamma)$ is C^1 robustly shadowable, then there exist a neighborhood $\mathcal{V}_1$ of X and a positive constant α such that for any $Y \in \mathcal{V}_1$ and any $y \in W^s(\gamma_Y, Y) \cap W^u(\gamma_Y, Y)$, we have*

$$\angle\left(T_yW^s(\gamma_Y, Y) \cap N_y, T_yW^u(\gamma_Y, Y) \cap N_y\right) \geq \alpha.$$

Proof. Let $\mathcal{U}$ be a neighborhood of X given in Proposition 3.6, and let $\mathcal{V}_1$ and ε be given in Proposition 2.2 associated to $\mathcal{U}$ and $t_0 = 1$. Let $\alpha = \varepsilon/10$. We will prove that $\mathcal{V}_1$ and α satisfy the conclusion of Proposition 3.7.

For $Y \in \mathcal{V}_1$ and $y \in W^s(\gamma_Y, Y) \cap W^u(\gamma_Y, Y)$, let

$$\Delta^s = T_yW^s(\gamma_Y, Y) \cap N_y \quad \text{and} \quad \Delta^u = T_yW^u(\gamma_Y, Y) \cap N_y.$$

If $\angle(\Delta^s, \Delta^u) < \alpha$, then there exist $v \in \Delta^u$ and $w \in (\Delta^u)^\perp$ such that $|v| = 1$, $|w| < \alpha$ and $v + w \in \Delta^s$. Select a linear map $L : \Delta^u \to (\Delta^u)^\perp$ such that $\|L\| = |w|$ and $L(v) = w$. Let $T : N_y \to N_y$ be the linear map given by

$$\begin{pmatrix} id & 0 \\ L & id \end{pmatrix}$$

with respect to the decomposition $N_y = \Delta^u \oplus (\Delta^u)^\perp$. Then one can see that

$$\|T - id\| < \alpha \quad \text{and} \quad T(\Delta^u) \cap \Delta^s \neq \{0\}.$$

Let $\beta : [0,1] \to [0,1]$ be a bump function satisfying the following properties:

$$\beta|_{[0,1/3]} = 1, \quad \beta|_{[2/3,1]} = 0 \quad \text{and} \quad 0 \leq |\beta'(x)| < 4$$

for any $x \in [0,1]$. For $r > 0$ small enough, one can define a map $g : \hat{N}_{y,r} \to \hat{N}_{y,1}$ by

$$g(z) = \exp_p \left(\exp_p^{-1}(z) + \beta \left(\frac{|\exp_p^{-1}(z)|}{r} \right) (T - id) \exp_p^{-1}(z) \right)$$

for any $z \in \hat{N}_{y,r}$. Then we see that if r is small enough, then $d_{C^1}(g, id) < \varepsilon$.

By applying Proposition 2.2, we can choose $Z \in \mathcal{U}$ such that Z keeps the orbit of y unchanged, just differs with Y in $Y_{[0,1]}(\hat{N_{y,r}})$, and $Z_t(x) = Y_t(g(x))$ for any $x \in \hat{N}_{y,r}$. If r is small enough, then the connected component of $W^u(\gamma_Y, Y) \cap \hat{N}_{y,r}$ containing y is contained in $W^u(\gamma_Y, Z)$ and the connected component of $W^s(\gamma_Y, Y) \cap \hat{N}_{Y_t(y),r}$ containing $Y_t(y)$ is also contained in $W^s(\gamma_Y, Z)$. By the construction of g, we have

$$N_{Y_t(y)} \neq \left(T_{Y_t(y)} W^s(\gamma, Z) \cap N_{Y_t(y)} \right) \oplus \left(T_{Y_t(y)} W^u(\gamma, Z) \cap N_{Y_t(y)} \right).$$

This contradicts to Proposition 3.6, and so completes the proof. □

In the following lemma, we show that the set $W^s(\gamma, Y) \cap W^u(\gamma, Y)$ has the homogeneous dominated splitting structure if $C_X(\gamma)$ is C^1 robustly shadowable.

Lemma 3.8. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. If $C_X(\gamma)$ is C^1 robustly shadowable, then there exist a neighborhood $\mathcal{V}$ of X and a constant $T > 0$ such that for any $Y \in \mathcal{V}$, $x \in W^s(\gamma_Y, Y) \cap W^u(\gamma_Y, Y)$, and $t \geq T$,*

$$\log \left\| \Psi_{t,Y}|_{\Delta^s(x,Y)} \right\| - \log m \left(\Psi_{t,Y}|_{\Delta^u(x,Y)} \right) < -1.$$

Moreover, $\dim(\Delta^s(x,Y)) = ind(\gamma)$ *for all $x \in W^s(\gamma_Y, Y) \cap W^u(\gamma_Y, Y)$.*

Proof. Let $\mathcal{V}_1$ and α be given in Proposition 3.7. Let $\mathcal{V}$ and ε be given by Proposition 2.2 associated to $\mathcal{V}_1$ and time $t_0 = 1$. Let $\delta = \frac{\alpha\varepsilon}{10(1+\alpha)}$. One can check that for any subspaces $E, F \subset \mathbb{R}^d$ with $E \oplus F = \mathbb{R}^d$, if $\angle(E,F) > \alpha$ and a linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\|T|_E - id\| < \delta \quad \text{and} \quad T|_F = id,$$

then T satisfies $\|T - id\| < \varepsilon/10$ (for more details, see Lemma II.10.b in [14]). Choose $m > 0$ such that $\delta(1+\delta)^m > e(\alpha^{-1} + 1)$ and $T = 2m + 2$.

Assume there exist $Y \in \mathcal{V}$, $x \in W^s(\gamma_Y, Y) \cap W^u(\gamma_Y, Y)$, and $t_x > T$ such that

$$\log \left\| \Psi_{t_x,Y}|_{\Delta^s(x,Y)} \right\| - \log m \left(\Psi_{t_x,Y}|_{\Delta^u(x,Y)} \right) \geq -1.$$

Then there exist $u \in \Delta^s(x,Y)$ and $v \in \Delta^u(x,Y)$ with $|u| = |v| = 1$ such that

$$\frac{|\Psi_{t_x,Y}(u)|}{|\Psi_{t_x,Y}(v)|} > e^{-1}.$$

By Hahn-Banach Theorem, we can take a linear map $L : \Delta^u(x,Y) \to \Delta^s(x,Y)$ such that

$$L(v) = \delta u \quad \text{and} \quad \|L\| = \delta.$$

Let $k = [t_x/2]$, $t_i = 2i$ for all $0 \le i \le k-1$, and $t_k = t_x$. Let $T_0 : N_x \to N_x$ be a linear map such that

$$T_0|_{\Delta^s(x,Y)} = id \quad \text{and} \quad T_0|_{\Delta^u(x,Y)} = id + L.$$

Let $T_i : N_{Y_{t_i}(x)} \to N_{Y_{t_i}(x)}$ be a linear map such that

$$T_i|_{\Delta^s(Y_{t_i}(x),Y)} = (1+\delta)id \quad \text{and} \quad T_i|_{\Delta^u(Y_{t_i}(x),Y)} = id$$

for all $1 \le i \le k-1$. Then we can see that T_i satisfies $\|T_i - id\| < \varepsilon/10$. If $r > 0$ is sufficiently small, by Franks' Lemma, there are $g_i : \hat{N}_{Y_{t_i}(x),r} \to \hat{N}_{Y_{t_i}(x),r}, 0 \le i \le k-1$ such that

$$g_i(Y_{t_i}(x)) = Y_{t_i}(x), \quad D_{Y_{t_i}(x)}g_i = T_i, \quad \text{and} \quad d_{C^1}(g_i, id) < \varepsilon.$$

By applying Proposition 2.2, we get a vector field $Z \in \mathcal{V}_1$ such that Z keeps the orbit of x unchanged, and just differs from Y in a neighborhood of $Y_{[0,t_x]}(x)$. If r is small enough, then the connected component of $W^u(\gamma_Y, Y) \cap N_x$ containing t_x is also contained in $W^u(\gamma_Y, Z)$ and the connected component of $W^s(\gamma_Y, Y) \cap N_{Y_{t_x}(x)}$ containing $Y_{t_x}(x)$ is also contained in $W^s(\gamma_Y, Z)$. Hence,

$$v \in T_x W^u(\gamma_Y, Z) \cap N_x \quad \text{and} \quad \Psi_{t_x,Y}(u) \in T_{Y_{t_x}(x)} W^s(\gamma_Y, Z) \cap N_{Y_{t_x}(x)}.$$

By the construction of Z, we know that

$$\Psi_{2,Y}(v + \delta u) \in T_{Y_2(x)} W^u(\gamma, Z) \cap N_{Y_2(x)}$$

and

$$\Psi_{t_i,Y}(v) + \delta(1+\delta)^{i-1}\Psi_{t_i,Y}(u) \in T_{Y_{t_i}(x)} W^u(\gamma, Z) \cap N_{Y_{t_i}(x)}$$

for all $1 \le i \le k$. Since

$$\frac{|\Psi_{t_x,Y}(u)|}{|\Psi_{t_x,Y}(v)|} > e^{-1},$$

we have

$$\frac{|\Psi_{t_x,Y}(v)|}{|\delta(1+\delta)^{k-1}\Psi_{t_x,Y}(u)|} < e\delta^{-1}(1+\delta)^{-k+1} < \frac{\alpha}{1+\alpha}.$$

This implies that

$$\angle\left(T_{Y_{t_i}(x)} W^u(\gamma, Z) \cap N_{Y_{t_i}(x)}, T_{Y_{t_i}(x)} W^u(\gamma, Z) \cap N_{Y_{t_i}(x)}\right) < \alpha.$$

The contradiction completes the proof. $\square$

From Proposition 3.1, we know that if $C_X(\gamma)$ is C^1 robustly shadowable, then there is no hyperbolic singularity contained in $C_X(\gamma)$. Therefore, if $C_X(\gamma)$ has no non-hyperbolic singularity, then $C_X(\gamma)$ has no singularity. It is well known that if an invariant set Λ admits a dominated splitting for the linear Poincaré flow and the closure, $\overline{\Lambda}$, of Λ has no singularity, then the dominated splitting over Λ can be extended to $\overline{\Lambda}$ (for more details, see [9]). Hence, Lemma 3.8 implies that if $C_X(\gamma)$ does not have a non-hyperbolic singularity, then $C_X(\gamma)$ admits a dominated splitting. In fact, we have the following proposition.

Proposition 3.9. *If $C_X(\gamma)$ is C^1 robustly shadowable and does not contain a non-hyperbolic singularity, then $C_X(\gamma)$ admits a Ψ_t-dominated splitting $\mathcal{N}_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ with $\dim(\Delta^s) = ind(\gamma)$.*

4. The main proposition

In this section, we prove the following proposition.

Proposition 4.1. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. If $C_X(\gamma)$ is C^1 robustly shadowable and contains no non-hyperbolic singularity, then there exist constants $T \geq 1$, $\eta > 0$, and $\tilde{T} > 0$ such that for any $\gamma' \sim \gamma$, if the period τ of γ' is greater than $\tilde{T}$, then the following properties are satisfied:*

(1) *for any $x \in \gamma'$ and $t \geq T$,*
$$\frac{1}{t}\left(\log \left\|\Psi_t|_{\Delta^s(x)}\right\| - \log m\left(\Psi_t|_{\Delta^u(x)}\right)\right) < -2\eta;$$
(2) *for any $x \in \gamma'$ and any partition $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ with $T \leq T_i - T_{i-1} < 2T$ for $i = 1, 2, \ldots, \iota$, we have*
$$\frac{1}{\tau}\sum_{i=1}^{\iota} \log \left\|\Psi_{T_i - T_{i-1}}|_{\Delta^s\left(X_{T_{i-1}}(x)\right)}\right\| < -\eta,$$
$$\frac{1}{\tau}\sum_{i=1}^{\iota} \log m\left(\Psi_{T_i - T_{i-1}}|_{\Delta^u\left(X_{T_{i-1}}(x)\right)}\right) > \eta.$$

The above proposition is in sprit extracted from Proposition II.1 and Lemma II.3 in [14]. The inequality in (1) in the above proposition just comes from Lemma 3.8. In fact, for any periodic point x, if $orb(x) \sim \gamma$, then one can find a sequence $x_k \in W^s(\gamma, X) \pitchfork W^u(\gamma, X)$ such that $x_k \to x$ as $k \to +\infty$. By taking T as in Lemma 3.8 and $\eta < \frac{1}{4T}$, one can verify the inequality in (1).

Now we prove the inequalities in (2) of the above proposition. The first inequality in (2) expresses so-called *contracting in period property* (see Figure 2). To prove (2), let us recall a well known fact proved by Mañé in ([14], Lemma II.5).

Lemma 4.2. *Given any $k \in \mathbb{N}$, $K > 0$ and $\varepsilon > 0$, there exist $0 < \lambda < 1$, $m > 0$ and $N > 0$ such that for any sequence $\mathcal{A} = \{A_0, A_1, \ldots, A_{n-1}\} \subset GL(k)$ with $n \geq N$ and every $\|A_i\| \leq K$, we have*

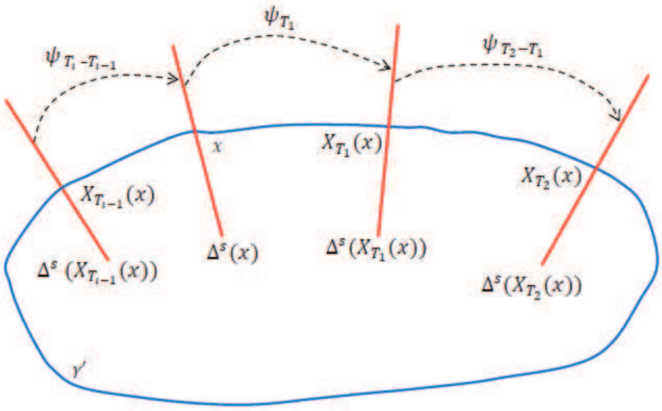

FIGURE 2. Contracting in period property

(1) *either* $\prod_{i=0}^{[n/m]-1} \|A_{im+m-1} \circ A_{im+m-2} \circ \cdots \circ A_{im}\| < \lambda^{[n/m]}$, *or*
(2) *there exists* $\mathcal{B} = \{B_0, B_1, \ldots, B_{n-1}\}$ *with every* $\|B_i - A_i\| < \varepsilon$ *such that* $B_{n-1} \circ B_{n-2} \circ \cdots \circ B_0$ *has an eigenvalue* μ *with* $|\mu| \geq 1$.

Since $C_X(\gamma)$ contains no singularity, there exists $K > 0$ such that $\|\Psi_t|_{N_x}\| < K/2$ for any $x \in C_X(\gamma)$ and $1 \leq t \leq 2$. Then there exists $r_0 > 0$ such that for any $x \in C_X(\gamma)$, one has $\|D_y f_{x,t}\| < K$ for any $y \in \hat{N}_{x,r_0}$ and $1 \leq t \leq 2$. Let $\mathcal{U}$ be a neighborhood of X such that $C_Y(\gamma_Y)$ is shadowable for $Y \in \mathcal{U}$. Let ε and $r(x)$ be chosen as in Proposition 2.2 associated to $\mathcal{U}$ and $t_0 = 1$. Then there exist $\varepsilon_1 > 0$ and $r_1 > 0$ such that for any $x \in C_X(\gamma)$ and $t \in [1, 2]$, if a map $g : \hat{N}_{x,r_0} \to \hat{N}_{X_t(x),1}$ just differs from $f_{x,t}$ in the neighborhood $\hat{N}_{x,r_1}$ and $d_{C^1}(f, g) < \varepsilon_1$, then $f_{x,t}^{-1} \circ g \in \mathcal{N}_\varepsilon(\hat{N}_{x,r_1(x)})$.

Now we take $K > 0$ as in the previous paragraph, k equals to the dimension of $E^s(\gamma)$ and $\varepsilon = \frac{\alpha\varepsilon_1}{10(1+\alpha)}$ (where α is in Lemma 3.7). Let λ, $m > 0$ and $N > 0$ be given in Lemma 4.2. Now we take $T = m$, $\tilde{T} = 2N$ and $\eta = -\frac{\log \lambda}{2m}$. We will prove that the constants $T, \tilde{T}$ and η satisfy the first inequality in (2) of Proposition 4.1.

Let γ' be a closed orbit which is homoclinically related to γ, and let $\tau > \tilde{T}$ be the period of γ'. Let $x \in \gamma'$, and let $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ be a partition of $[0, \tau]$ with $T \leq T_{i+1} - T_i < 2T$ for $i = 0, 1, \ldots, \iota - 1$. Let us divide $[T_i, T_{i+1}]$ into m parts as

$$T_i < T_i + \frac{T_{i+1} - T_i}{m} < T_i + 2\frac{T_{i+1} - T_i}{m} < \cdots < T_{i+1}.$$

Then we get a subdivision

$$0 = t_0 < t_1 < t_2 < \cdots < t_{\iota m} = \tau$$

which satisfies the following properties:

(a) $t_{\iota m} = T_i$ for $i = 0, 1, \dots, \iota$,
(b) $t_{i+1} - t_i \in [1, 2)$ for $i = 0, 1, \dots, \iota$,
(c) $\iota m > N$.

Let $A_i = \Psi_{t_{i+1}-t_i}|_{\Delta^s\left(X_{t_i}(x)\right)}$ for $i = 0, 1, \dots, \iota m - 1$, and let $\mathcal{A} = \{A_0, A_1, \dots, A_{\iota m-1}\}$. From Lemma 4.2, we can see that if the inequality

$$\frac{1}{\tau}\sum_{i=1}^{\iota} \log \left\| \Psi_{T_i - T_{i-1}}|_{\Delta^s\left(X_{T_{i-1}}(x)\right)} \right\| < -\eta$$

does not hold, then there exists $\mathcal{B} = \{B_0, B_1, \dots, B_{\iota m-1}\}$ such that $\|B_i - A_i\| < \varepsilon$ and $B_{\iota m-1} \circ B_{\iota m-2} \circ \cdots \circ B_0$ has an eigenvalue μ with $|\mu| > 1$. Since $A_{\iota m-1} \circ A_{\iota m-1} \circ \cdots \circ A_0$ is contracting, we can assume that $\mathcal{B}$ satisfies the following properties:

(a) $B_{\iota m-1} \circ B_{\iota m-2} \circ \cdots \circ B_0$ has all eigenvalue less than or equal 1 in modulus,
(b) for each $a \in [0, 1)$,

$$(aB_{\iota m-1} + (1-a)A_{\iota m-1}) \circ (aB_{\iota m-2} + (1-a)A_{\iota m-2}) \circ \cdots \circ (aB_0 + (1-a)A_0)$$

is contracting, i.e., the eigenvalues are less than 1 in modulus.

Denote by $d(\mathcal{A}, \mathcal{B}) = \sup\{\|A_i - B_i\| : i = 0, 1, \dots, \iota m - 1\}$. Given any $a \in [0, 1)$, we have the following lemma.

Lemma 4.3. *For any $\rho > 0$, there exists a sequence of diffeomorphisms $g_i : \mathbb{R}^k \to \mathbb{R}^k$, $i = 0, 1, \dots, \iota m - 1$, such that*

(1) *g_i just differs with A_i in $B(\rho)$,*
(2) *$\|D_y g_i - A_i\| < 6d(\mathcal{A}, \mathcal{B})$ for $y \in \mathbb{R}^k$,*
(3) *there exists $\rho' > 0$ such that*

$$g_i|_{B(\rho')} = aB_i + (1-a)A_i,$$

(4) *by denoting $g = g_{\iota m-1} \circ g_{\iota m-2} \circ \cdots \circ g_0$, we have $g^n(y) \to 0$ as $n \to \infty$ for any $y \in \mathbb{R}^k$.*

Proof. Let $\beta : [0, +\infty) \to [0, 1]$ be a bump function which satisfies

$$\beta|_{[0,1/3]} = 0, \quad \beta|_{[2/3,+\infty)} = 1, \quad \text{and} \quad \beta'(x) < 4$$

for any $x \in [0, +\infty)$. For any $c \in [0, 1]$, denote by

$$\mathcal{C}^c = \{C_i^c : C_i^c = cB_i + (1-c)A_i,\ i = 0, 1, \dots, \iota m - 1\}$$

and

$$M_c = C^c_{\iota m-1} \circ C^c_{\iota m-2} \circ \cdots \circ C^c_0.$$

For any $c_1, c_2 \in [0, 1]$, define

$$d(\mathcal{C}^{c_1}, \mathcal{C}^{c_2}) := \sup\left\{\|C_i^{c_1} - C_i^{c_2}\| : 0 \le i < \iota m\right\}.$$

Since M_c is contracting for any $c \in [0,1)$, there exists $\delta(c) > 0$ satisfying the following property: for any $\mathcal{C} = \{C_0, C_1, \dots, C_{\iota m-1}\}$ with every $\|C_i - C_i^c\| < \delta(c)$, $i = 0, 1, \dots, \iota m - 1$, if we let $h_i : \mathbb{R}^k \to \mathbb{R}^k$ be the map defined by

$$h_i(x) = \beta(|x|)C_i^c(x) + (1 - \beta(|x|))C_i(x),$$

then the differentiable map

$$H = h_{\iota m-1} \circ h_{\iota m-2} \circ \cdots \circ h_0$$

satisfies $H^n(x) \to 0$ as $n \to +\infty$ for any $x \in \mathbb{R}^k$. Since $[0, a]$ is closed, we can find a finite sequence $0 = c_0 < c_1 < \cdots < c_s = a$ such that $d(\mathcal{C}^{c_j}, \mathcal{C}^{c_{j+1}}) < \delta(c_i)$ for any $0 \le j < s$. Now we construct $\{g_i\}$. Let h_i^j be defined as in the last paragraph corresponding $\mathcal{C}^{c_j}$ and $\mathcal{C}^{c_{j+1}}$, i.e.,

$$h_i^j = \beta(|x|)C_i^{c_j}(x) + (1 - \beta(|x|))C_i^{c_{j+1}}(x)$$

for any $0 \le i < \iota m$ and $0 \le j < s$. One can check that $\|D_y h_i^j - C_i^{c_j}\| < 5\delta(c_j)$ for $0 \le i < \iota m$, $0 \le j < s$, and $y \in \mathbb{R}^k$. Let

$$H_j = h^j_{\iota m-1} \circ h^j_{\iota m-2} \circ \cdots \circ h^j_0.$$

Then we know that $H_j^n(x) \to 0$ as $n \to +\infty$ for any $x \in \mathbb{R}^k$.

Let $\lambda_0 = \rho$ and $\tilde{h}_i^0 = \lambda_0 \circ h_i^0 \circ \lambda_0^{-1} id$ for $0 \le i < \iota m$. Let

$$\tilde{H}_0 = \tilde{h}^0_{\iota m-1} \circ \tilde{h}^0_{\iota m-2} \circ \cdots \circ \tilde{h}^0_0.$$

Then we can see that the following properties are satisfied:

(a) $\|D_y \tilde{h}_j^0 - C_i^{c_0}\| < 5\delta(c_0)$ for $0 \le i < \iota m$ and $y \in \mathbb{R}^k$,

(b) $H_0^n(x) \to 0$ as $n \to +\infty$ for $x \in \mathbb{R}^k$.

Let $B(r) = \{x \in \mathbb{R}^k : |x| \le r\}$. Then we can take $\lambda_1 \in (0, \lambda_0/3)$ such that

$$(\tilde{h}^0_{i+j} \circ \tilde{h}^0_{i-1+j} \circ \cdots \circ h^0_j)(B(\lambda_1)) \subset B(\lambda_0/3)$$

for $0 \le i, j < \iota m$, where $\tilde{h}_j^0 = \tilde{h}^0_{j-\iota m}$ if $j \ge \iota m$. Let $\tilde{h}_i^1 = \lambda_1 \circ h_i^1 \circ \lambda_1^{-1} id$ for $0 \le i < \iota m$, and let

$$\tilde{H}_1 = \tilde{h}^1_{\iota m-1} \circ \tilde{h}^1_{\iota m-2} \circ \cdots \circ \tilde{h}^1_0.$$

One can see that $\{\tilde{h}_j^1\}$ and $\tilde{H}_1$ have the similar property with $\{\tilde{h}_j^0\}$ and $\tilde{H}_0$, respectively. Next we take λ_2, $\{\tilde{h}_i^2 : 0 \le i < \iota m\}$ and $\tilde{H}_2$ inductively, i.e., take $\lambda_2 \in (0, \lambda_1/3)$ such that

$$(\tilde{h}^0_{i+j} \circ \tilde{h}^0_{i-1+j} \circ \cdots \circ h^0_j)(B(\lambda_1)) \subset B(\lambda_0/3)$$

for $0 \le i, j < \iota m$, and then take

$$\tilde{h}_i^2 = \lambda_2 \circ h_i^2 \circ \lambda_2^{-1} id \quad \text{and} \quad \tilde{H}_2 = \tilde{h}^2_{\iota m-1} \circ \tilde{h}^2_{\iota m-2} \circ \cdots \tilde{h}^2_0.$$

Let λ_j, $\{\tilde{h}_i^j : 0 \le i < \iota m\}$ and $\tilde{H}_j$ be defined inductively as above. Let g_i be defined by

$$g_i|_{\mathbb{R}^k \setminus B(\lambda_1)} = \tilde{h}_i^0; \quad g_i|_{B(\lambda_j) \setminus B(\lambda_{j+1})} = \tilde{h}_i^j$$

for $j = 1, 2\dots, s-2$ and $g_i|_{B(\lambda_{s-1})} = \tilde{h}^s_j$. It is obvious that g_i just differs from A_i in $B(\rho)$, and there exists $\rho' = \lambda_{s-1}/3$ such that $g_i|_{B(\rho')} = aB_i + (1-a)A_i$. To verify the second condition of Lemma 4.3, one can check that

· $D_y g_i = A_i$ if $y \in \mathbb{R}^k \backslash B(\lambda_0)$,

· $\|D_y g_i - A_i\| \le \|D_y g_i - C_i^{c_j}\| + \|C_i^{c_j} - A_i\| < 6d(\mathcal{A}, \mathcal{B})$ if $y \in B(\lambda_j)\backslash B(\lambda_{j+1})$,

· $\|D_y g_i - A_i\| < 6d(\mathcal{A}, \mathcal{B})$ if $y \in B(\lambda_{s-1})$.

Hence we get $\|D_y g_i - A_i\| < 6d(\mathcal{A}, \mathcal{B})$ for any $y \in \mathbb{R}^k$.

For any $x \in \mathbb{R}^k$, one can see that if $|x| > \lambda_1$, then the orbit of x is operated by $\{\tilde{h}^0_i\}$, and there exists n_0 such that the n_0-iterate of x go into the ball $B(\lambda_1)$. Furthermore, it will be operated by $\{\tilde{h}^1_i\}$ till it goes into the ball $B(\lambda_2)$, and also it will be operated by $\{\tilde{h}^2_i\}$ till it goes into $B(\lambda_3)$, etc. Finally, we can see that after some time, the orbit of x will be operated by $\{\tilde{h}^{s-1}_i\}$, and then it will go to 0. This completes the proof of Lemma 4.3. $\square$

Now we will take a perturbation Y of X such that $C_Y(\gamma_Y)$ is not shadowable for Y_t. By applying Proposition 2.2, we can choose an arbitrarily small perturbation Y of X such that Y keeps the orbit γ' unchanged and its Poincaré map is given by

$$\tilde{f}_{Y_{t_i}(x), t_{i+1}-t_i}|_{\hat{N}_{Y_{t_i}(x), r_2}} = \exp_{Y_{t_{i+1}}(x)} \circ \Psi_{t_{i+1}-t_i} \circ \exp^{-1}_{Y_{t_i}(x)}|_{N_x(r_2)}$$

for some $r_2 > 0$. Since the perturbation can be arbitrarily small, we can keep the relation $\gamma' \sim \gamma$. To simplify the notations, we still use X to denote the perturbation Y.

Let $y_1 \in W^s(\gamma') \pitchfork W^u(\gamma)$ and $y_2 \in W^u(\gamma') \pitchfork W^s(\gamma)$. Without loss of generality, we can assume

$$y_1, y_2 \in \hat{N}_{x, r_2}, f_{x, t_i}(y_1) \in \hat{N}_{X_{t_i}(x), r_2} \text{ and } f^{-1}_{X_{t_{\iota m - i}}(x), t_{\iota m} - t_{\iota m - i}} \in \hat{N}_{X_{t_{\iota m - i}}(x), r_2}$$

for any $0 \le i \le \iota m$. Take a close to 1, let $D_i = aA_i + (1-a)B_i$ for all $0 \le i < \iota m$. Then we know that $D_{\iota m-1} \circ D_{\iota m - 2} \circ \cdots \circ D_0$ has an eigenvalue λ with modulus close to 1. Without loss of generality, we can assume λ has multiplicity 1 and the other eigenvalues have modulus less than or equal to $|\lambda|$ (otherwise, we take an arbitrarily small perturbation whose weakest eigenvalue has multiplicity 1). By Lemma 4.3, we can construct a sequence g_i which just differs from A_i in a small ball $B(\rho)$ of 0, and $g_i = D_i$ in a ball $B(\rho')$. If we take ρ small enough, then $g^n(\exp_x(y_1)) \to 0$ as $n \to +\infty$.

Take another perturbation of g_i in the ball $B(\rho')$. We take $\rho'' < \rho'$ such that

$$(g_{i+j} \circ g_{i+j-1} \circ \cdots \circ g_i)(B(\lambda^{-1}\rho'')) \subset B(\rho')$$

for all $0 \le i, j < \iota m$, where $g_i = g_{i - \iota m}$ if $i \ge \iota m$. Then we take

$$\tilde{g}_0(v) = \beta\left(\frac{|v|}{\rho''}\right) g_0(v) + \left(1 - \beta\left(\frac{|v|}{\rho''}\right)\right) \frac{D_0}{|\lambda|}(v)$$

instead of g_0. One can easily check that

$$\|D_y \tilde{g}_0 - D_y g_0\| < 5(|\lambda|^{-1} - 1)\|D_0\|$$

for all $y \in \mathbb{R}^k$. Hence if we take a close to 1 enough, we have

$$\|D_y\tilde{g}_0 - D_y g_0\| < 3d(\mathcal{A}, \mathcal{B})$$

for all $y \in \mathbb{R}^k$. Let $\tilde{g}_i = g_i$ for $1 \leq i < \iota m$, and $\tilde{g} = \tilde{g}_{\iota m-1} \circ \tilde{g}_{\iota m-2} \circ \cdots \circ \tilde{g}_0$. Then we know that $\tilde{g}|_{B(\rho''/3)} = |\lambda|^{-1} g|_{B(\rho''/3)}$. If λ is real, then $\tilde{g}$ has an arc I located in the eigenspace of λ such $\tilde{g}^2|_I$ is the identity map. If λ is complex, then $\tilde{g}$ has a disc D located in the two-dimensional eigenspace of λ such that $\tilde{g}|_D$ is a rotation. Since the other eigenvalues except $\lambda, \overline{\lambda}$ have modulus less than $|\lambda|$, we can get that $\tilde{g}^n(\exp_x^{-1}(y_1))$ tend to a point in I (if λ is real) or D (if λ is complex) as $n \to \infty$.

Under the decomposition $N_{X_{t_i}(x)} = \Delta^s(X_{t_i}(x)) \oplus \Delta^u(X_{t_i}(x))$, $i = 0, 1, \ldots, \iota m - 1$, the linear Poincaré flow Ψ_t has the following form

$$\begin{aligned}\Psi_{t_{i+1}-t_i}|_{N_{X_{t_i}(x)}} &= \exp^{-1}_{X_{t_{i+1}}(x)} \circ f_{X_{t_i}(x), t_{i+1}-t_i} \circ \exp_{X_{t_i}(x)} |_{N_{X_{t_i}(x), r_2}} \\ &= \begin{pmatrix} A_i & 0 \\ 0 & \Psi_{t_{i+1}-t_i}|_{\Delta^u(X_{t_i}(x))} \end{pmatrix}.\end{aligned}$$

Take

$$\tilde{f}_i = \begin{pmatrix} \tilde{g}_i & 0 \\ 0 & \Psi_{t_{i+1}-t_i}|_{\Delta^u(X_{t_i}(x))} \end{pmatrix}.$$

From Proposition 3.7, we have $\angle\left(\Delta^s(X_{t_i}(x)), \Delta^u(X_{t_i}(x))\right) > \alpha$. By the construction of $\tilde{g}_i$, we know that

$$\|D_y\tilde{g}_i - A_i\| < 9d(\mathcal{A}, \mathcal{B}) < 9\varepsilon = \frac{9\alpha\varepsilon_1}{10(1+\alpha)}$$

for all $y \in \mathbb{R}^k$. Finally we obtain

$$\|D_z\tilde{f}_i - D_z f_i\| < 9\varepsilon_1/10$$

for all $z \in \hat{N}_{X_{t_i}(x), r_2}$. If ρ is small enough, we have $d_{C^1}(f_i, \tilde{f}_i) < \varepsilon_1$. Then we take a perturbation Y by Proposition 2.2 such that the orbit γ' is unchanged, and

$$\exp_{X_{t_{i+1}}(x)} \circ \tilde{f}_i \circ \exp_{X_{t_i}(x)} : \hat{N}_{Y_{t_i}(x), r_2} \to \hat{N}_{Y_{t_{i+1}}(x), 1}$$

is just the Poincaré map associated to Y. Let $F : \hat{N}_x \to \hat{N}_x$ be the Poincaré first return map associated to the periodic orbit γ' and the vector filed Y. Then there exists an arc I (or a disc D) such that $F^2|_I$ is the identity map (or $F|_D$ is a rotation map). Moreover we know that

$$Y_{-t}(y_1) \to \gamma \text{ and } Y_t(y_2) \to \gamma \text{ as } t \to \infty;$$

$$F^n(y_1) \to I \text{ (or } D) \text{ and } F^n(y_1) \to I \text{ (or } D) \text{ as } n \to \infty.$$

Hence I (or D) is a subset of $C_Y(\gamma)$. By the standard argument, we can find a constant ε_0 such that for any δ, there exists a δ-pseudo orbit which can not be

ε_0-shadowed. One can find such kind of construction in [13] (for more details, see [13]). This contradicts to the robust shadowability of $C_Y(\gamma)$. Hence we get

$$\frac{1}{\tau}\sum_{i=1}^{\iota}\log\left\|\Psi_{T_i-T_{i-1}}|_{\Delta^s(X_{T_{i-1}}(x))}\right\| < -\eta.$$

Note that the multiples of m also satisfy Lemma 4.2, and we can take m and N uniformly for both the stable and unstable directions. Similarly, we can get the second inequality of (2) in Proposition 4.1. This completes the proof of Proposition 4.1.

5. The proof of the main theorem

In Section 3, we have proved that if $C_X(\gamma)$ is C^1 robustly shadowable and does not contain a non-hyperbolic singularity, then $C_X(\gamma)$ admits a Ψ_t-dominated splitting $\mathcal{N}_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ with $\dim(\Delta^s) = ind(\gamma)$. We will prove that this invariant splitting is indeed a hyperbolic splitting with respect to the linear Poincaré flow Ψ_t. First we recall some results for the dominated splitting. By the standard argument of dominated splitting, we can get the following proposition which says that the dominated splitting of a closed invariant set can be extended to its neighborhood.

Proposition 5.1. *Let Λ be a closed invariant set of the flow X_t. If Λ contains no singularity and the linear Poincaré flow Ψ_t admits a dominated splitting over Λ, then there exists a neighborhood U of Λ such that the dominated splitting can be extended to the set $\overline{\bigcap_{t\in\mathbb{R}} X_t(U)}$.*

By the standard argument of dominated splitting, we may assume that the subbundles Δ^s and Δ^u are continuous. As usual, let $\mathcal{N}$ be the normal bundle over Λ. Denote by $Emb^1_\Lambda(\Delta^s,\mathcal{N})$ the set of all bundle maps $\sigma : \Delta^s \to \mathcal{N}$ satisfying:

(a) $\sigma_x : \Delta^s(x) \to N_x$ is a C^1 map for any $x \in \Lambda$,

(b) $D_{v_n}\sigma_{x_n} \to D_v\sigma_x$ for any sequence $v_n(\in \Delta^s(x_n)) \mapsto v \in (\Delta^s(x))$.

Similarly, we can define $Emb^1_\Lambda(\Delta^u,\mathcal{N})$. Let $\Delta^s(x,\delta)$ be the δ-ball in $\Delta^s(x)$, and let $\Delta^u(x,\delta)$ be the δ-ball in $\Delta^u(x)$.

Proposition 5.2. *Let Λ be a closed invariant set of the flow X_t. If Λ admits a l-dominated splitting $\Delta^s \oplus \Delta^u$ with respect to the linear Poincaré flow Ψ_t, then there exist a bundle map $\sigma^s \in Emb^1_\Lambda(\Delta^s,\mathcal{N})$ and a bundle map $\sigma^u \in Emb^1_\Lambda(\Delta^u,\mathcal{N})$ such that for any $x \in \Lambda$,*

(1) *$T_x \exp_x \sigma^s_x(\Delta^s(x)) = \Delta^s(x)$ and $T_x \exp_x \sigma^u_x(\Delta^u(x)) = \Delta^u(x)$;*

(2) *for any $0 < \delta_0 < 1$, there exists $0 < \delta_1 < \delta_0$ such that*

$$f_{x,t}(W^{cs}_{\delta_1}(x)) \subset W^{cs}_{\delta_0}(X_t(x)) \text{ and } f^{-1}_{x,t}(W^{cu}_{\delta_1}(X_t(x))) \subset W^{cu}_{\delta_0}(x),$$

for $0 < t \leq l$, where

$$W^{cs}_\delta(x) = \exp_x(\sigma^s_x(\Delta^s(x,\delta))) \text{ and } W^{cu}_\delta(x) = \exp_x(\sigma^u_x(\Delta^u(x,\delta))).$$

We call $W^{cs}_\delta(x)$ and $W^{cu}_\delta(x)$ the *local center stable* and *local center unstable manifolds* of x, respectively.

Lemma 5.3. *Let Λ be a closed invariant set of the flow X_t, and assume Λ admits a l-dominated splitting for the linear Poincaré flow Ψ_t. Then for any $\eta > 0$, there exists $r > 0$ satisfies the following properties:*

(1) *if a point $x \in \Lambda$ and a sequence $0 = t_0 < t_1 < \cdots$ with the property $t_{i+1} - t_i \in [l, 2l)$ for $i \geq 0$ satisfy*

$$\frac{1}{t_k}\sum_{i=0}^{k-1}\log\left\|\Psi_{t_{i+1}-t_i}|_{\Delta^s\left(X_{t_i}(x)\right)}\right\| < -\eta,$$

for any $k > 0$, then

$$d(X_t(y), orb(x)) \to 0 \quad as \quad t \to +\infty$$

for any $y \in W^{cs}_r(x)$, or equivalently, $W^{cs}_r(x) \subset W^s(orb(x))$.

(2) *if a point $x \in \Lambda$ and a sequence $0 = t_0 > t_1 > \cdots$ with the property $t_i - t_{i+1} \in [l, 2l)$ for $i \geq 0$ satisfy*

$$\frac{1}{-t_k}\sum_{i=0}^{k-1}\log\left\|\Psi_{t_{i+1}-t_i}|_{\Delta^u\left(X_{t_i}(x)\right)}\right\| < -\eta,$$

for any $k > 0$, then

$$d(X_t(y), orb(x)) \to 0 \quad as \quad t \to -\infty$$

for any $y \in W^{cu}_r(x)$, or equivalently, $W^{cu}_r(x) \subset W^u(orb(x))$.

Proof. We just prove (1). By reversing the vector field X, (2) can be obtained from (1). Let $\lambda = e^{-l\eta}$, and take a constant $\lambda_1 \in (\lambda, 1)$. Fix a constant $0 < \delta_0 < 1$ such that

$$\|D_y\sigma^s_x\| < \sqrt{\lambda_1/\lambda} \quad \text{and} \quad \|D_y(\sigma^s_x)^{-1}\| < \sqrt{\lambda_1/\lambda}$$

for any $x \in \Lambda$ and $y \in \Delta^s(x, \delta_0)$. By Proposition 5.2, there exists $0 < \delta_1 < \delta_0$ such that

$$f_{x,t}(W^{cs}_{\delta_1}(x)) \subset W^{cs}_{\delta_0}(X_t(x))$$

for $x \in \Lambda$ and $0 \leq t \leq 2l$. Since Λ is compact, we can choose $0 < r < \delta_1$ such that

$$\|D_y f_{x,t} - \Psi_t|_{N_x}\| < (\sqrt{\lambda_1/\lambda} - 1)\|\Psi_t|_{N_x}\|$$

for any $t \in [l, 2l]$, $x \in \Lambda$ and $y \in \hat{N}_{x,r}$. Let $x \in \Lambda$, and let $0 = t_0 < t_1 < \cdots$ be a sequence given in (1). Denote by $x_n = X_{t_n}(x)$ for all $n \geq 0$. Then $x_{n+1} = X_{t_{n+1}-t_n}(x_n)$. Given $y \in W^{cs}_r(x)$, let $y_0 = y$ and $y_{n+1} = f_{x_n, t_{n+1}-t_n}(y_n)$. To prove (1), it suffices to show that $d(x_n, y_n) \to 0$ as $n \to +\infty$.

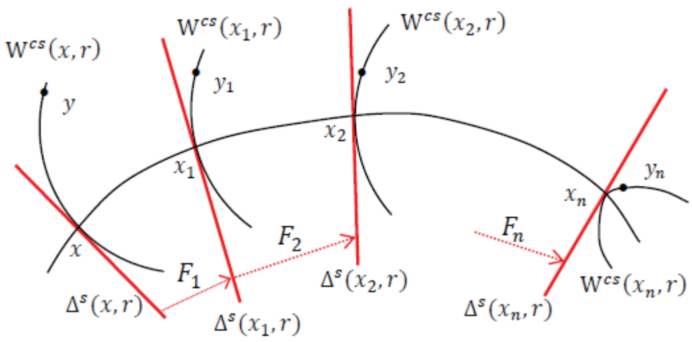

FIGURE 3. Contracting on Δ^s implies contracting on W^{cs}

For each $n \geq 0$, consider a map $F_n : \Delta^s(x_n, r) \to \Delta^s(x_{n+1})$ which is defined via the following diagram

$$\begin{array}{ccc} \Delta^s(x_n, r) & \xrightarrow{\sigma^s_{x_n}} & N_{x_n} \\ \Big\downarrow F_n & & \Big\downarrow D_{x_n} f_{x_n, t_{n+1}-t_n} \\ \Delta^s(x_{n+1}, r) & \xrightarrow{(\sigma^s_{x_{n+1}})^{-1}} & N_{x_{n+1}} \end{array}$$

or

$$F_n = \left(\sigma^s_{x_{n+1}}|_{\Delta^s(x_{n+1},\delta_0)}\right)^{-1} \circ D_{x_n} f_{x_n, t_{n+1}-t_n} \circ \sigma^s_{x_n}|_{\Delta^s(x_n,r)}.$$

What we are going to prove is that the contracting property on Δ^s implies the contracting property on W^{cs} (see Figure 3). Since $r < \delta_1$, each F_n is well defined. Then we can see that

$$\|D_y F_0\| < (\lambda_1/\lambda) \cdot \|\Psi_{t_1 - t_0}|_{\Delta^s(x_0)}\| < \lambda_1$$

for all $y \in \Delta^s(x_0, r)$. Hence we have

$$F_0(\Delta^s(x_0, r)) \subset \Delta^s(x_1, \lambda_1 r).$$

Because $y \in W^{cs}_r(x)$, there is $v \in \Delta^s(x, r)$ such that $y = \exp_x(\sigma^s_x(v))$. Thus we get

$$\begin{aligned} y_1 &= f_{x,t_1}(y) = f_{x,t_1}(\exp_x(\sigma^s_x(v))) \\ &= \exp_{x_1}(\Psi_{t_1}(\sigma^s_x(v))) = \exp_{x_1}(\sigma^s_{x_1}(F_0(v))) \\ &\in \exp_{x_1}(\sigma^s_{x_1}(F_0(\Delta^s(x, r))) \subset \exp_{x_1}(\sigma^s_{x_1}(\Delta^s(x_1, \lambda_1 r))). \end{aligned}$$

This means that $y_1 \in W^{cs}_{\lambda_1 r}(x)$. Similarly, we can check that

$$\|D_y(F_1 \circ F_0)\| < (\lambda_1/\lambda)^2 \|\Psi_{t_1 - t_0}|_{\Delta^s(x_0)}\| \cdot \|\Psi_{t_2 - t_1}|_{\Delta^s(x_1)}\| < \lambda_1^2$$

for all $y \in \Delta^s(x_0, r)$. This implies

$$(F_1 \circ F_0)(\Delta^s(x, r) \subset \Delta^s(x_2, \lambda_1^2 r)$$

and so $y_2 \in W^{cs}_{\lambda_1^2 r}(x_2)$. Inductively, we can prove that

$$\|D_y(F_n \circ F_{n-1} \circ \cdots \circ F_0)\| < \lambda_1^n$$

and

$$(F_n \circ F_{n-1} \circ \cdots \circ F_0)\,(\Delta^s(x, r)) \subset \Delta^s(x_n, \lambda_1^n r)$$

for any $y \in \Delta^s(x, r)$. This means that $y_n \in W^{cs}_{\lambda_1^n r}(x_n)$, and so $d(x_n, y_n) \to 0$ as $n \to +\infty$. This completes the proof of Lemma 5.3. □

If the dominated splitting $\mathcal{N}_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ is not a hyperbolic splitting for Ψ_t, then either Δ^s is not Ψ_t-contracting or Δ^u is not Ψ_t-expanding. We assume that Δ^s is not Ψ_t-contracting.

First of all, we have the following lemma.

Lemma 5.4. *Let Λ be a compact invariant set of the flow X_t, and assume Λ admits a dominated splitting $\mathcal{N} = \Delta^s \oplus \Delta^u$ for the linear Poincaré flow Ψ_t. If the subbundle Δ^s is not contracting for Ψ_t, then there exists a point $b \in \Lambda$ (which is called a "bad" point) such that*

$$\left\|\Psi_t|_{\Delta^s(b)}\right\| \geq 1$$

for all $t \geq 0$.

Proof. Suppose not. Then for any $x \in \Lambda$, there is $t_x > 0$ such that $\|\Psi_{t_x}|_{\Delta^s(x)}\| = \lambda_x < 1$. For each $x \in \Lambda$, take δ_x such that if $y \in B(x, \delta_x) \cap \Lambda$, then $\|\Psi_{t_x}|_{\Delta^s(y)}\| < \sqrt{\lambda_x}$. Since Λ is compact, we can choose a finite set $\{x_1, x_2, \ldots, x_k\}$ such that $\Lambda \subset \bigcap_{i=1}^k B(x_i, \delta_{x_i})$. Let $K = \max\{t_{x_i} : 1 \leq i \leq k\}$, $\lambda = (\max\{\lambda_{x_i} : 1 \leq i \leq k\})^{1/2K}$, and $C = \sup\{\|\Psi_t|_{\Delta^s(x)}\| : 0 \leq t \leq K, x \in \Lambda\} \cdot \lambda^{-K}$. Given any $x \in \Lambda$ and $t > 0$, take $0 < t_1 \leq K$ such that

$$\|\Psi_{t_1}|_{\Delta^s(x)}\| < \lambda^K \leq \lambda^{t_1}.$$

Then we can choose t_2 such that $0 < t_2 - t_1 \leq K$ and

$$\left\|\Psi_{t_2 - t_1}|_{\Delta^s\left(X_{t_1}(x)\right)}\right\| < \lambda^K \leq \lambda^{t_2 - t_1}.$$

Hence we have $\|\Psi_{t_2}|_{\Delta^s(x)}\| < \lambda^{t_2}$. Similarly we can take $0 < t_1 < t_2 < \cdots < t_l < t$ with $t - t_l < K$ such that

$$\left\|\Psi_{t_i - t_{i-1}}|_{\Delta^s\left(X_{t_{i-1}}(x)\right)}\right\| < \lambda^k < \lambda^{t_i - t_{i-1}}.$$

Finally we can show that $\|\Psi_t|_{\Delta^s(x)}\| < C\lambda^t$. This contradicts to the assumption which Δ^s is not Ψ_t-contracting. □

Lemma 5.5. *Let γ' be a hyperbolic closed orbit with period τ, and let T and η be positive constants. Suppose*

$$\frac{1}{t}\left(\log\left\|\Psi_t|_{\Delta^s(x)}\right\| - \log m\left(\Psi_t|_{\Delta^u(x)}\right)\right) < -2\eta$$

for $x \in \gamma'$ and $t \geq T$. Then for any $x \in \gamma'$ and a partition $0 = T_0 < T_1 < \cdots < T_\iota = \tau$ of $[0,\tau]$ with $T \leq T_i - T_{i-1} < 2T$ for $i = 1, 2, \ldots, \iota$, if an extended partition $\{T_i\}_{i\in\mathbb{Z}}$ of $\{T_i\}_{0\leq i\leq\iota}$ with $T_{i+k\iota} = T_i + k\tau$ for any integer i, k satisfy

$$\frac{1}{\tau}\sum_{i=1}^{\iota}\log\left\|\Psi_{T_i - T_{i-1}}|_{\Delta^s\left(X_{T_{i-1}}(x)\right)}\right\| < -\eta$$

and

$$\frac{1}{\tau}\sum_{i=1}^{\iota}\log m\left(\Psi_{T_i - T_{i-1}}|_{\Delta^u\left(X_{T_{i-1}}(x)\right)}\right) > \eta,$$

then there exists an integer $0 \leq i_0 < \iota$ such that

$$\frac{1}{T_{i_0+k} - T_{i_0}}\sum_{j=1}^{k}\log\left\|\Psi_{T_{i_0+j} - T_{i_0+j-1}}|_{\Delta^s\left(X_{T_{i_0+j-1}}(x)\right)}\right\| \leq -\eta$$

and

$$\frac{1}{T_{i_0} - T_{i_0-k}}\sum_{j=1}^{k}\log m\left(\Psi_{T_{i_0-j+1} - T_{i_0-j}}|_{\Delta^u\left(X_{T_{i_0-j}}(x)\right)}\right) \geq \eta$$

for any $k \geq 1$.

Proof. For each $k \in \mathbb{Z}$, define $S(T_k)$ by

$$S(T_k) = \begin{cases} \sum\limits_{j=0}^{k-1}\log\left\|\Psi_{T_{j+1}-T_j}|_{\Delta^s\left(X_{T_j}(x)\right)}\right\| & \text{if} \quad k > 0, \\ -\sum\limits_{j=-k}^{-1}\log\left\|\Psi_{T_{j+1}-T_j}|_{\Delta^s\left(X_{T_j}(x)\right)}\right\| & \text{if} \quad k < 0, \\ 0 & \text{if} \quad k = 0. \end{cases}$$

Then we know that

$$S(T_\iota) < -\eta T_\iota \quad \text{and} \quad S(T_{l+i}) = S(T_l) + S(T_i)$$

for any $i \in \mathbb{Z}$. Hence we get

$$S(T_{n\iota+i}) - S(T_i) < -\eta(T_{n\iota+i} - T_i)$$

for any $i \in \mathbb{Z}$. First we prove that the set

$$A := \{j_0 \in \mathbb{Z} : S(T_{j_0+k}) - S(T_{j_0}) < -\eta(T_{j_0+k} - T_{j_0}) \text{ for } k \geq 0\}$$

is not empty. Suppose not. Then for $j \geq 0$, there exists $k_j > 0$ such that

$$S(T_{j+k_j}) - S(T_j) \geq -\eta(T_{j+k_j} - T_j).$$

Let $0 < k_0 < k_1 < \cdots$ be a sequence such that

$$S(T_{k_i}) - S(T_{k_{i-1}}) \geq -\eta(T_{k_i} - T_{k_{i-1}})$$

for any $i > 0$. Then there exists $0 < j < i$ such that $k_i - k_j$ is a multiple of ι. The choice of the sequence means that

$$S(T_{k_i}) - S(T_{k_j}) \geq -\eta(T_{k_i} - T_{k_j}).$$

This contradicts to our assumption.

Similarly we can define $\tilde{S}(T_k)$ by

$$\tilde{S}(T_k) = -\sum_{j=0}^{k-1} \log m\left(\Psi_{T_i - T_{i-1}}|_{\Delta^u\left(X_{T_{i-1}}(x)\right)}\right),$$

$$\tilde{S}(T_{-k}) = \sum_{j=-k+1}^{0} \log m\left(\Psi_{T_i - T_{i-1}}|_{\Delta^u\left(X_{T_{i-1}}(x)\right)}\right)$$

for any $k > 0$. Then we can see that

$$\tilde{S}(T_\iota) < -\eta T_\iota \quad \text{and} \quad \tilde{S}(T_{l+i}) = \tilde{S}(T_l) + \tilde{S}(T_i)$$

for any integer i. Also we can prove that the set

$$B := \{j_0 \in \mathbb{Z} : \tilde{S}(T_{j_0}) - \tilde{S}(T_{j_0-k}) < -\eta(T_{j_0} - T_{j_0-k}) \text{ for } k \geq 0\}$$

is not empty. It is clear that if $a \in A$, then $\pm\iota + a \in A$, and if $b \in B$, then $\pm\iota + b \in B$.

Now we prove that $A \cap B \cap [0, \iota)$ is not empty. Suppose $A \cap B \cap [0, \iota) = \emptyset$. Then there are $a \in A$ and $b \in B$ such that $b < a$ and $(b, a) \cap (A \cup B) = \emptyset$. Since $a - 1 \notin A$, we can see that

$$S(T_a) - S(T_{a-1}) \geq -\eta(T_a - T_{a-1}).$$

From

$$\frac{1}{T_a - T_{a-1}}\left(\log\left\|\Psi_t|_{\Delta^s\left(X_{T_{a-1}}(x)\right)}\right\| - \log m\left(\Psi_t|_{\Delta^u\left(X_{T_{a-1}}(x)\right)}\right)\right) < -2\eta,$$

we get

$$\tilde{S}(T_a) - \tilde{S}(T_{a-1}) < -\eta(T_a - T_{a-1}).$$

Similarly if $a - 2 \notin A$, we get

$$S(T_a) - S(T_{a-2}) \geq -\eta(T_a - T_{a-2}).$$

If it does not hold, then we have

$$S(T_a) - S(T_{a-2}) < -\eta(T_a - T_{a-2})$$

and

$$S(T_{a-1}) - S(T_{a-2}) < -\eta(T_{a-1} - T_{a-2}).$$

These two inequalities and $a \in A$ imply $a - 2 \in A$. From

$$\frac{\log\left\|\Psi_{T_a - T_{a-1}}|_{\Delta^s\left(X_{T_{a-1}}(x)\right)}\right\| - \log m\left(\Psi_{T_a - T_{a-1}}|_{\Delta^u\left(X_{T_{a-1}(x)}\right)}\right)}{T_a - T_{a-1}} < -2\eta$$

and

$$\frac{\log \left\| \Psi_{T_{a-1}-T_{a-2}}|_{\Delta^s\left(X_{T_{a-2}}(x)\right)} \right\| - \log m\left(\Psi_{T_{a-1}-T_{a-2}}|_{\Delta^u\left(X_{T_{a-1}(x)}\right)}\right)}{T_{a-1}-T_{a-2}} < -2\eta,$$

we have

$$\tilde{S}(T_a) - \tilde{S}(T_{a-2}) < -\eta(T_a - T_{a-2}).$$

Inductively we can show that for any $i \in [b, a)$, we have

$$\tilde{S}(T_a) - \tilde{S}(T_i) < -\eta(T_a - T_i).$$

These two inequalities and the fact $b \in B$ imply $a \in B$, which lead to a contradiction to the assumption $A \cap B = \emptyset$. Consequently we can choose an integer $i_0 \in A \cap B \cap [0, \iota)$ so that the integer i_0 satisfies the conclusion of Lemma 5.5. This completes the proof of Lemma 5.5. □

We will say that the point $X_{T_{i_0}}(x)$ obtained in Lemma 5.5 is a *"good"* hyperbolic periodic point contained in the hyperbolic closed orbit γ'. If $\delta(:= r)$ is a constant given by Lemma 5.3 with respect to the dominated splitting $\Delta^s \oplus \Delta^u$ over $C_X(\gamma)$, then we can see that

$$W^{cs}_\delta(X_{T_{i_0}}(x)) \subset W^s(\gamma') \text{ and } W^{cu}_\delta(X_{T_{i_0}}(x)) \subset W^u(\gamma').$$

Now we glue the orbit of "bad" point b obtained in Lemma 5.4 and the hyperbolic closed orbit γ' to get a "quasi hyperbolic" pseudo orbit. Let T, η, and $\tilde{T}$ be constants as in Proposition 4.1. Let $x_0 = X_{T_0}(x)$. Then we have the following lemma.

Lemma 5.6. *Let $X \in \mathcal{X}^1(M)$, and let γ be a hyperbolic closed orbit of X_t. Assume $C_X(\gamma)$ is C^1 robustly shadowable and has no hyperbolic singularity. Let $\mathcal{N}_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ be the dominated splitting of the linear Poincaré flow Ψ_t with $\dim \Delta^s = ind(\gamma)$. If the subbundle Δ^s is not Ψ_t-contracting, then for any constants $\delta > 0$ and $0 < \eta_1 < \eta_2 < \eta$, there exists a δ-pseudo orbit $\{(x_i, t_i)\}_{i=0}^{n-1}$ in $C_X(\gamma)$ such that*

(1) *x_0 is a "good" hyperbolic periodic point of a hyperbolic closed orbit γ' which is homoclinically related to γ,*
(2) *$5T/4 < t_i < 7T/4$ for $0 \le i \le n-1$,*
(3) *$X_{t_{n-1}}(x_0) = x_0$,*
(4) *by letting $T_k = t_0 + \cdots + t_{k-1}$ for all $1 \le k \le n$, we have*

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \left\| \Psi_{t_i}|_{\Delta^s(x_i)} \right\| < -\eta_1,$$

$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k} \log m\left(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})}\right) > \eta_1,$$

and

$$\frac{1}{T_n}\sum_{i=0}^{n-1}\log\left\|\Psi_{t_i}|_{\Delta^s(x_i)}\right\| > -\eta_2.$$

where $1 \le k \le n$.

Proof. Since Δ^s is not contracting for Ψ_t, by Lemma 5.4, there exists a "bad" point $b \in C_X(\gamma)$ such that $\|\Psi_t|_{\Delta^s(b)}\| \ge 1$ for all $t \ge 0$. Take and fix constants $\delta > 0$ and $0 < \eta_1 < \eta_2 < \eta$, and choose a constant $0 < \varepsilon < \delta$ so that $d(x,y) < \varepsilon$ implies $d(X_t(x), X_t(y)) < \delta$ for any $0 \le t \le 2T$. Since $C_X(\gamma)$ equals to the homoclinic class of γ, there exists a hyperbolic closed orbit $\gamma'(\sim\gamma)$ with arbitrarily large period τ (here we just consider the non-trivial case) such that $C_X(\gamma) \subset B(\gamma', \varepsilon)$. We can assume the period τ is big enough so that $\frac{5}{4}T < \frac{\tau}{\iota} < \frac{7}{4}T$, where ι is the integer part of $\frac{\tau}{3T/2}$. Now we take $x \in \gamma$ such that $d(x,b) < \varepsilon < \delta$, and divide $[0,\tau]$ into ι part

$$0 < \tau/\iota < 2\tau/\iota < \cdots < (\iota-1)\tau/\iota < \tau.$$

From Lemma 5.5, we can get $0 \le i_0 < \tau$ such that

$$\frac{1}{k\tau/\iota}\sum_{j=0}^{k-1}\log\left\|\Psi_{\tau/\iota}|_{\Delta^s\left(X_{(j+i_0)\tau/\iota}(x)\right)}\right\| \le -\eta$$

and

$$\frac{1}{k\tau/\iota}\sum_{j=1}^{k}\log m\left(\Psi_{\tau/\iota}|_{\Delta^u\left(X_{(i_0-j)\tau/\iota}(x)\right)}\right) \ge \eta$$

for any $k \ge 1$. Choose an integer $s > 0$ (we will fix s in the future), and take $\overline{x}_i = X_{(i_0+i)\tau/k}(x)$ and $\overline{t}_i = \tau/\iota$ for $0 \le i < s\iota - i_0$. We know that $\overline{x}_0$ is a "good" periodic point of γ' and $X_{\tau/\iota}(\overline{x}_{s\iota-i_0-1}) = x$. Now for $j \ge 0$, let

$$\overline{x}_{s\iota-i_0+j} = X_{3jT/2}(b) \quad \text{and} \quad \overline{t}_{s\iota-i_0+j} = 3T/2.$$

By the property of the "bad" point b, we can choose $L > 0$ such that

$$\frac{1}{\overline{t}_0 + \cdots + \overline{t}_{s\iota-i_0+L-1}}\sum_{i=0}^{s\iota-i_0+L-1}\log\left\|\Psi_{\overline{t}_i}|_{\Delta^s(\overline{x}_i)}\right\| \ge -\frac{\eta_1+\eta_2}{2}$$

and

$$\frac{1}{\overline{t}_0 + \cdots + \overline{t}_{s\iota-i_0+l-1}}\sum_{i=0}^{s\iota-i_0+l-1}\log\left\|\Psi_{\overline{t}_i}|_{\Delta^s(\overline{x}_i)}\right\| < -\frac{\eta_1+\eta_2}{2}$$

for all $0 \le l < L$. Then we can see that L is increasing as s is increasing.

Since $C_X(\gamma) \subset B(\gamma', \varepsilon)$, we can find $0 \le j_0 < \iota$ and a point $t' \in [j_0\tau/\iota, (j_0+1)\tau/\iota)$ such that

$$d\left(X_{3T/2}(\overline{x}_{s\iota-i_0+L-1}), X_{t'}(x)\right) < \varepsilon.$$

Since $(j_0+1)\tau/\iota - t' < 2T$, we have

$$d(X_{3T/2+(j_0+1)\tau/\iota-t'}(\overline{x}_{s\iota-i_0+L-1}), X_{(j_0+1)\tau/\iota}(x)) < \delta.$$

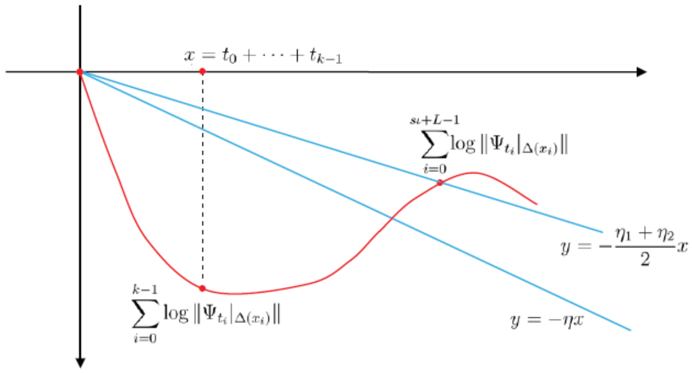

FIGURE 4. The figure of the sum $\sum_{i=0}^{k-1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\|$

To construct a δ-pseudo orbit, we still let $\overline{x}_{s\iota-i_0+L-1} = X_{3(L-1)T/2}(b)$, but change $\overline{t}_{s\iota-i_0+L-1}$ to $3T/2 + (j_0+1)\tau/\iota - t'$. Moreover we let $\overline{x}_{s\iota-i_0+L+i} = X_{(j_0+1+i)\tau/\iota}(x))$ and $\overline{t}_{s\iota-i_0+L} = \tau/\iota$ for any $0 \leq i < 2\iota - j_0 + i_0$. Put

$$n = s\iota - i_0 + L + 2\iota - j_0 + i_0 = (s+2)\iota + L - j_0.$$

Then we can see that $\{(\overline{x}_i, \overline{t}_i)\}_{i=0}^{n-1}$ is a δ-pseudo orbit in $C_X(\gamma)$ and $X_{\overline{t}_{n-1}}(\overline{x}_{n-1}) = x_0$. Note that the constant $\overline{t}_{s\iota-i_0+L-1}$ may not belong to $(5T/4, 7T/4)$. We modify the δ-pseudo orbit $\{(\overline{x}_i, \overline{t}_i)\}_{i=0}^{n-1}$ so that the constant $\overline{t}_{s\iota-i_0+L-1}$ belongs to $(5T/4, 7T/4)$. We let $x_i = \overline{x}_i$ and $t_i = \overline{t}_i$ for $0 \leq i < s\iota - i_0 + L - 7$ and $s\iota - i_0 + L \leq i < n$. By letting

$$\begin{aligned}
x_{s\iota-i_0+L-7} &= \overline{x}_{s\iota-i_0+L-7}, \\
t_{s\iota-i_0+L-7} &= 3T/2 + ((j_0+1)\tau/\iota - t')/7, \\
x_{s\iota-i_0+L-7+i} &= X_{i(3T/2+((j_0+1)\tau/\iota-t')/7)}(x_{s\iota-i_0+L-7}), \\
t_{s\iota-i_0+L-7+i} &= 3T/2 + ((j_0+1)\tau/\iota - t')/7,
\end{aligned}$$

for $1 \leq i \leq 6$, we can see that $t_i \in (5T/4, 7T/4)$ for every $0 \leq i < n$.

Now we will check that if τ and s are large enough, then $\{(x_i, t_i)\}_{i=0}^{n-1}$ is a desired δ-pseudo orbit in $C_X(\gamma)$.

Let $K = \sup\{\|\Psi_t|_{N_x}\| : x \in C_X(\gamma), -2T \leq t \leq 2T\}$. To simplify the notation, we change $\overline{t}_{s\iota-i_0+L-1}$ back to $3T/2$. Then we know

$$\left| \log \frac{\left\| \Psi_{\overline{t}_{s\iota-i_0+L-8+i}}|_{\Delta^s(\overline{x}_{s\iota-i_0+L-8+i})} \right\|}{\left\| \Psi_{t_{s\iota-i_0+L-8+i}}|_{\Delta^s(x_{s\iota-i_0+L-8+i})} \right\|} \right| \leq 2 \log K$$

and

$$\left|\log \frac{m\left(\Psi_{\overline{t}_{s\iota-i_0+L-8+i}}|_{\Delta^u(\overline{x}_{s\iota-i_0+L-8+i})}\right)}{m\left(\Psi_{t_{s\iota-i_0+L-8+i}}|_{\Delta^u(x_{s\iota-i_0+L-8+i})}\right)}\right| \le 2\log K$$

for all $1 \le i \le 7$. If we let $n_1 = s\iota - i_0 + L - 7$, by the choice of L, we have

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\| < -\frac{\eta_1+\eta_2}{2}$$

for $0 < k \le n_1$. Moreover we get

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \left\|\Psi_{t_i}|_{\Delta^s(x_i)}\right\| < \frac{-T_{n_1}\cdot \frac{\eta_1+\eta_2}{2} + (k-n_1)\log K}{T_k}$$

for $n_1 < k < n$. Since $n - n_1 = 2\iota - j_0 + i_0 + 7 < 3\iota + 8$ we obtain

$$T_k - T_{n_1} < (3\iota+8)2T \quad \text{and} \quad (k-n_1)\log K < (3\iota+8)\log K$$

for any $n_1 < k < n$. So if s is large enough, then we have

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \|\Psi_{t_i}|_{\Delta^s(x_i)}\| < -\eta_1.$$

By the choice of L, we get

$$\frac{1}{\overline{t}_0+\cdots+\overline{t}_{n_1+6}}\sum_{i=0}^{n_1+6} \log \left\|\Psi_{\overline{t}_i}|_{\Delta^s(\overline{x}_i)}\right\| \ge -\frac{\eta_1+\eta_2}{2},$$

and so we obtain

$$\begin{aligned}\sum_{i=0}^{n_1} \log \left\|\Psi_{t_i}|_{\Delta^s(x_i)}\right\| &\ge -\frac{\eta_1+\eta_2}{2}(\overline{t}_0+\cdots+\overline{t}_{n_1+6}) - 7\log K \\ &> -\frac{\eta_1+\eta_2}{2}T_{n_1} - 7\log K.\end{aligned}$$

Then we have

$$\frac{1}{T_n}\sum_{i=0}^{n_1} \log \left\|\Psi_{t_i}|_{\Delta^s(x_i)}\right\| > \frac{-\frac{\eta_1+\eta_2}{2}T_{n_1} - 7\log K - (n-n_1)\log K}{T_n}.$$

If s is large enough, we know

$$\frac{1}{T_n}\sum_{i=0}^{n-1} \log \left\|\Psi_{t_i}|_{\Delta^s(x_i)}\right\| > -\eta_2.$$

Let $n_2 = s\iota - i_0 + L$. By the choice of x_0 and x_{n-1}, we have

$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k} \log m(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})}) > \eta > \eta_1$$

for any $1 \le k \le n - n_2$. By the choice of L, we get

$$\frac{1}{\bar{t}_{n_2-k} + \cdots + \bar{t}_{n_2-1}} \sum_{i=1}^{k} \log \left\| \Psi_{\bar{t}_{n_2-i}}|_{\Delta^s(\overline{x}_{n_2-i})} \right\| > -\frac{\eta_1 + \eta_2}{2},$$

and so

$$\frac{1}{\bar{t}_{n_2-k} + \cdots + \bar{t}_{n_2-1}} \sum_{i=1}^{k} \log m \left(\Psi_{\bar{t}_{n_2-i}}|_{\Delta^u(\overline{x}_{n_2-i})} \right) > 2\eta - \frac{\eta_1 + \eta_2}{2} > \eta$$

for all $1 \le k \le n_2$. Hence we have

$$\begin{aligned}\sum_{i=1}^{k} \log m \left(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})} \right) &> \eta(T_n - T_{n_2}) + \eta(\bar{t}_{n-k} + \cdots + \bar{t}_{n_2-1}) - 14 \log K \\ &> \eta(T_n - T_{n-k} - 2T) - 14 \log K\end{aligned}$$

for all $n - n_2 < k < n$. Consequently we obtain

$$\frac{1}{T_n - T_{n-k}} \sum_{i=1}^{k} \log m \left(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})} \right) > \frac{\eta(T_n - T_{n-k} - 2T) - 14 \log K}{T_n - T_{n-k}}$$

for all $n - n_2 < k < n$. We know that $n - n_2 > \iota$ and $T_n - T_{n-k} > \iota T$ if $k > n - n_2$. If we choose τ large enough, then we get

$$\frac{1}{T_n - T_{n-k}} \sum_{i=1}^{k} \log m \left(\Psi_{t_{n-i}}|_{\Delta^u(x_{n-i})} \right) > \eta_1$$

for all $n - n_2 < k < n$. This completes the proof of Lemma 5.6. □

By the shadowability of the above δ-pseudo orbit $\{(x_i, t_i)\}$, we can take a point $z \in M$ and an increasing continuous map $h : \mathbb{R} \to \mathbb{R}$ satisfying

$$d(X_{t-T_i}(x_i), X_{h(t)}(z)) < \varepsilon$$

for all $t \in [T_i, T_{i+1})$. Lemma 5.8 tells us that if ε is small enough, we may assume $9T/8 < h(T_{i+1}) - h(T_i) < 15T/8$. The idea of the proofs of the following two lemmas comes from [8].

Lemma 5.7. *Let X be a C^1 vector filed, and let Λ be a closed invariant set containing no singularity. Then there exist a neighborhood U of Λ and a constant $T_0 > 0$ which satisfy the following properties*:

(1) *for $0 < \varepsilon < T_0$, there is $\delta > 0$ such that for any $x \in U$ and $0 \le s, t \le T_0$, $d(X_s(x), X_t(x)) < \delta$ implies $|s - t| < \varepsilon$,*
(2) *for $T \in (0, T_0)$, there is $\varepsilon > 0$ such that $X_{[0,t]}(x) \subset B(x, \varepsilon)$ implies $t \in [0, T]$ for any $x \in U$.*

Proof. Take a neighborhood U_0 of Λ such that the closure of U_0 contains no singularity. Take $T_0 > 0$ such that $X_t(x) \ne x$ for any $0 < t \le T_0$. Take a neighborhood U of Λ such that $X_t(x) \in U_0$ for any $x \in \overline{U}$ and $0 < t \le T_0$. We will prove that U and T_0 satisfy the above lemma.

If (1) is false, then we can take $\varepsilon \in (0, T_0)$, $x_n \in U$, and $t_n, s_n \in [0, T_0]$ such that

$$d(X_{s_n}(x_n), X_{t_n}(x_n)) < 1/n \quad \text{and} \quad |s_n - t_n| \geq \varepsilon$$

for each $n \geq 0$. Without loss of generality, we can assume $x_n \to x \in \overline{U}$, $s_n \to s \in [0, T_0]$ and $t_n \to t \in [0, T_0]$. Then we have $X_s(x) = X_t(x)$ and $\varepsilon < |s - t| \leq T_0$. This is a contradiction to the choice of T_0.

If (2) is false, then there is $T \in (0, T_0)$ such that for any $n > 0$, there exist $x_n \in U$ and $t_n > T$ satisfying $X_{[0,t_n]}(x_n) \subset B(x_n, 1/n)$. Without loss of generality, we can assume $x_n \to x$. Then we see that $X_T(x) = x$, which contradicts to the choice of T_0. □

Lemma 5.8. *Let X be a C^1 vector filed, and let Λ be a closed invariant set containing no singularity. Let U and T_0 be constants given in the above lemma, and take a neighborhood V of Λ such that $X_t(x) \in U$ for any $x \in \overline{V}$ and $0 \leq t \leq T_0$. Then for any $\varepsilon \in (0, 1)$ and $T_1 \in (0, T_0]$, there is $\varepsilon' > 0$ such that for any $x, y \in V$, if an increasing continuous map $g : [0, \tau] \to \mathbb{R}$ satisfies $g(0) = 0$ and $d(X_t(x), X_{g(t)}(y)) \leq \varepsilon'$ for all $t \in [0, T_1]$, then $|g(T_1) - T_1| \leq \varepsilon T_1$.*

Proof. Let $\varepsilon > 0$ and $T_1 \in (0, T_0]$ be given. Then there is $\gamma > 0$ such that $X_{[0,t]}(x) \subset B(x, \gamma)$ implies $t \leq \varepsilon T_1$ for any $x \in U$. Take $\eta \in (0, \varepsilon T_1)$ such that $d(x, X_t(x)) < \gamma/3$ for all $x \in \overline{U}$ and $t \in [0, \eta]$. Choose $\lambda > 0$ such that for any $x \in U$, $d(X_s(x), X_t(x)) < \lambda$ implies $|s - t| < \eta$. Take $\varepsilon' > 0$ such that $d(x, y) \leq \varepsilon'$ implies $d(X_t(x), X_t(y)) \leq \min\{\lambda/2, \gamma/3\}$ for any $x, y \in U$ and $t \in [0, T_1]$.

Now we prove that ε' satisfies the lemma. Consider $x, y \in V$ and an increasing continuous map $g : [0, T_1] \to \mathbb{R}$ such that $g(0) = 0$ and $d(X_t(x), X_{g(t)}(y)) \leq \varepsilon'$ for all $t \in [0, T_1]$. Since $d(x, y) \leq \varepsilon'$, we have $d(X_{T_1}(x), X_{T_1}(y)) \leq \gamma/3$. If $g(T_1) < T_1$, we get

$$d(X_{T_1}(y'), X_{g(T_1)}(y)) \leq d(X_{T_1}(x), X_{g(T_1)}(y)) + d(X_{T_1}(x), X_{T_1}(y)) \leq \gamma.$$

Hence $T_1 - g(T_1) \leq \varepsilon T_1$. If $g(T_1) > T_1$, then, by letting $S = g^{-1}(T_1)$, we have $S < T_1$. Therefore we obtain

$$d(X_{T_1}(x), X_S(x)) \leq d(X_{T_1}(x), X_{T_1}(y)) + d(X_S(x), X_T(y)) \leq \lambda/2 + \varepsilon' \leq \lambda.$$

So we have $T_1 - S < \eta$ and $d(X_{g^{-1}(t)}(x), X_{T_1}(x)) < \gamma/3$ for all $T_1 < t < g(T_1)$. Moreover we get

$$\begin{aligned} d(X_{g(T_1)}(y), X_t(y)) &\leq d(X_{g(T_1)}(y), X_{T_1}(x)) + d(X_{T_1}(x), X_{g^{-1}(t)}(x)) \\ &\quad + d(X_{g^{-1}(t)}(x), X_t(y)) < \gamma/3 + \gamma/3 + \gamma/3 = \gamma \end{aligned}$$

for all $t \in [T_1, g(T_1)]$. Consequently we have $g(T_1) - T_1 < \varepsilon T_1$, and so completes the proof. □

Let γ be a hyperbolic closed orbit. Assume $C_X(\gamma)$ is C^1 robustly shadowable and has no non-hyperbolic singularity. Let $\mathcal{N}_{C_X(\gamma)} = \Delta^s \oplus \Delta^u$ be the dominated splitting of Ψ_t which is obtained from Proposition 3.8. By Proposition 5.1, we know that the dominated splitting $\Delta^s \oplus \Delta^u$ can be extended to a neighborhood U_0 of $C_X(\gamma)$. Now we let $\Lambda = \overline{\bigcap_{t\in\mathbb{R}} X_t(U_0)}$. To simplify notations, we also use $\Delta^s \oplus \Delta^u$ to denote the extended dominated splitting, and assume that the inequality

$$\frac{1}{t}\left(\log \left\|\Psi_t|_{\Delta^s(x)}\right\| - \log m\left(\Psi_t|_{\Delta^u(x)}\right)\right) < -2\eta$$

is true for any $x \in \Lambda$ and $t \geq T$. By the robust shadowability of $C_X(\gamma)$, we get the following lemma from Lemma 5.6 and Lemma 5.8.

Lemma 5.9. *If the subbundle Δ^s is not contracting for Ψ_t, then for any $\varepsilon > 0$, any neighborhood U of $C_X(\gamma)$ and any constants $0 < \eta_1 < \eta_2 < \eta$, there exists a point $z \in U$ such that*

(1) *$orb(z) \subset U$,*
(2) *there exist a hyperbolic closed orbit $\gamma'(\sim \gamma)$ and a "good" hyperbolic periodic point $q \in \gamma'$ such that $d(q, z) < \varepsilon$,*
(3) *there exist positive numbers $0 = T_0 < T_1 < \cdots < T_n$ with every $9T/8 < T_{i+1} - T_i < 15T/8$ such that*

$$\frac{1}{T_k}\sum_{i=0}^{k-1} \log \left\|\Psi_{T_{i+1}-T_i}|_{\Delta^s\left(X_{T_i}(x)\right)}\right\| < -\eta_1,$$

$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k} \log m\left(\Psi_{T_{n-i+1}-T_{n-i}}|_{\Delta^u\left(X_{T_{n-i}}(x)\right)}\right) > \eta_1,$$

$$\frac{1}{T_n}\sum_{i=0}^{n-1} \log \left\|\Psi_{T_{i+1}-T_i}|_{\Delta^s\left(X_{T_i}(x)\right)}\right\| > -\eta_2$$

for all $1 \leq k \leq n$,
(4) *$d(z, X_{T_n}(z)) < \varepsilon$.*

Let $\Lambda \subset M_X$ be a closed invariant set of X_t that has a continuous Ψ_t-invariant splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$ with $\dim \Delta^s = p$, $1 \leq p \leq \dim M - 2$. For any two constants $T > 0$ and $\eta > 0$, an orbit arc $(x, t) = X_{[0,t]}(x)$ is said to be an (η, T, p)-*quasi hyperbolic orbit arc* of X_t with respect to the splitting $\Delta^s \oplus \Delta^u$ if $[0, t]$ has a partition $0 = T_0 < T_1 < \cdots < T_l$ with $T < T_i - T_{i-1} < 2T$ satisfying:

- $\frac{1}{T_k}\sum_{j=1}^{k} \log \left\|\Psi_{T_j - T_{j-1}}|_{\Delta^s\left(X_{T_{j-1}}(x)\right)}\right\| \leq -\eta$,
- $\frac{1}{T_l - T_{k-1}}\sum_{j=k}^{l} \log m\left(\Psi_{T_j - T_{j-1}}|_{\Delta^u\left(X_{T_{j-1}}(x)\right)}\right) \geq \eta$,
- $\log \left\|\Psi_{T_k - T_{k-1}}|_{\Delta^s(X_{T_{k-1}}(x))}\right\| - \log m\left(\Psi_{T_k - T_{k-1}}|_{\Delta^u\left(X_{T_{k-1}}(x)\right)}\right) \leq -2\eta$

  for $k = 1, 2, \ldots, l$.

The most important property of quasi hyperbolic orbit arc is that it can be shadowed by a hyperbolic periodic point if two end points of that quasi hyperbolic orbit arc are sufficiently close. The proof of the following proposition can be found in [10].

Proposition 5.10. *Let X be a C^1 vector field, and let Λ be a closed invariant set containing no singularity. Assume there exists a continuous invariant splitting $\mathcal{N}_\Lambda = \Delta^s \oplus \Delta^u$ with $\dim \Delta^s = p$, $1 \le p \le \dim M - 2$. Then for any $\eta > 0$, $T > 0$ and $\varepsilon > 0$, there exists $\zeta > 0$ such that if (x, τ) is a (η, T, p)-quasi hyperbolic orbit arc of X_t with respect to the splitting $\Delta^s \oplus \Delta^u$ and $d(X_\tau(x), x) < \zeta$, then there exist a hyperbolic periodic point $y \in M$ and an orientation-preserving homeomorphism $g : [0, \tau] \to \mathbb{R}$ with $g(0) = 0$ such that $d(X_{g(t)}(y), X_t(x)) < \varepsilon$ for any $t \in [0, \tau]$ and $X_{g(\tau)}(y) = y$.*

By applying the above proposition to the (η_1, T)-quasi hyperbolic orbit arc obtained in Lemma 5.9, we get the following proposition.

Proposition 5.11. *If the subbundle Δ^s is not contracting for Ψ_t, then for any $\varepsilon > 0$, any neighborhood U of $C_X(\gamma)$ and any constants $0 < \eta_1 < \eta_2 < \eta$, there exists a hyperbolic periodic point $z \in U$ such that*

(1) *$orb(z) \subset U$,*
(2) *there exist a hyperbolic closed orbit $\gamma'(\sim \gamma)$ and a "good" hyperbolic periodic point $p \in \gamma'$ such that $d(z, p) < \varepsilon$,*
(3) *there are positive numbers $0 = T_0 < T_1 < \cdots < T_n$ with $T < T_{i+1} - T_i < 2T$ such that*
$$\frac{1}{T_k}\sum_{i=0}^{k-1}\log\left\|\Psi_{T_{i+1}-T_i}|_{\Delta^s(X_{T_i}(z))}\right\| < -\eta_1,$$
$$\frac{1}{T_n - T_{n-k}}\sum_{i=1}^{k}\log m\left(\Psi_{T_{n-i+1}-T_{n-i}}|_{\Delta^u(X_{T_{n-i}}(z))}\right) > \eta_1,$$
$$\frac{1}{T_n}\sum_{i=0}^{n-1}\log\left\|\Psi_{T_{i+1}-T_i}|_{\Delta^s(X_{T_i}(z))}\right\| > -\eta_2$$
for all $1 \le k \le n$,
(4) *$X_{T_n}(z) = z$.*

It is easy to see that z is also a "good" hyperbolic periodic point associated to the constants η_1 and T.

Now are going to complete the proof of our main theorem.

End of the proof of Main Theorem

Fix constants $0 < \eta_1 < \eta_2 < \eta$, and let $W^{cs}_\delta(x)$ and $W^{cu}_\delta(x)$ be the local center stable and local center unstable manifolds of x associated to the dominated splitting $\Delta^s \oplus \Delta^u$ (on the extended set Λ), respectively. Let $r > 0$ be the constant given in Lemma 5.3 associated to η_1 and T. As in the case of Poincaré map, we can define a holonomy map $h_{y,x} : \hat{N}_{y,r} \to \hat{N}_{x,1}$ such that $h_{y,x}(z)$ is the unique point of the intersection $orb(z) \cap \hat{N}_{x,1}$ if $x, y \in \Lambda$ are close enough. For the concept of holonomy maps, see Section 2.2 in [1]. Note that the holonomy map acts on the manifold as the linear Poincaré flow acts on the normal bundle. In particular, $h_{y,x}(z)$ and z are in the same orbit of X_t. Moreover, if $y \in orb(x)$ with $X_t(x) = y$, then $h_{y,x} = f_{y,t}$, where $f_{y,t}$ is the Poincaré map mentioned in Section 2. It is obvious that if $y \to x$, then

$$h_{y,x}W_r^{cs}(y) \to W_r^{cs}(x) \quad \text{and} \quad h_{y,x}W_r^{cu}(y) \to W_r^{cu}(x).$$

Since Λ is compact, we can take a constant $\varepsilon > 0$ such that if $d(x, y) < \varepsilon$, then

$$h_{y,x}W_r^{cs}(y) \cap W_r^{cu}(x) \neq \emptyset \quad \text{and} \quad h_{y,x}W_r^{cu}(y) \cap W_r^{cs}(x) \neq \emptyset.$$

By Proposition 5.11, we can take a hyperbolic closed orbit $\gamma'(\sim \gamma)$, a "good" hyperbolic periodic point $p \in \gamma'$ and another hyperbolic "good" periodic point x which is ε close to p. By applying Lemma 5.3, we know

$$W^s(orb(x)) \cap W^u(\gamma') \neq \emptyset \quad \text{and} \quad W^u(orb(x)) \cap W^s(\gamma') \neq \emptyset.$$

Hence we have $\gamma' \sim orb(x) \sim \gamma$, and so $orb(x) \subset H_X(\gamma) = C_X(\gamma)$. In the construction of the pseudo orbit in Lemma 5.6, we can take L to be arbitrarily large. Hence we may assume the period of x is greater than $\tilde{T}$. However the following inequality

$$\frac{1}{T_n}\sum_{i=0}^{n-1} \log \left\| \Psi_{T_{i+1}-T_i}|_{\Delta^s\left(X_{T_i}(x)\right)} \right\| > -\eta_2.$$

contradicts to Proposition 4.1. This completes the proof of the main theorem.

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