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Development of a magnetic field calculation program for air-core solenoids which can control the precision of a magnetic field

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Abstract

A numerical method of magnetic field calculation for the air-core solenoid is presented in this paper. In application of the Biot-Savart law, the magnetic field induced from the source current can be obtained by a double integration ormula. The numerical method named composite Simpson's rule for the integration is applied to the program and the adaptive quadrature method is used to adjust the step size in the calculation according to the precision we need. When the target point is in the solenoid and the integrand's denominator may be zero in the process of calculation, the method still can provide an appropriate result. We have developed a program which calculates the magnetic field with at least 1ppm precision and named it as rzBI() to implement this method. The method has been used in the design of an MRI magnet, and the result show it is very flexible and convenient.

Keywords: adaptive quadrature method; magnetic field calculation; MRI; significant figure; solenoids

1. INTRODUCTION

An air-core solenoid is widely used in industry products. According the Biot-Savart law, an electric current can generate a magnetic field, and the magnitude and direction of the magnetic field will be changed by the electric current. It is very easy to generate a controllable magnetic field to satisfy some special demands, such as Magnetic Resonance Imaging (MRI), and so on. For the solenoid, the magnetic flux density (**B**) is a fundamental parameter in the design or application. The calculation of the magnetic field generated by solenoid is interesting and several methods are presented in this respect. Based on the definition of magnetic vector potential, the magnetic field can be computed by applying the Biot-Savart law directly. The calculation contains the various power series expansions or integrations [1-3]. Even though there are some tables that can be used in calculation, it is hard to calculate the exact value of the magnetic field. With digital computers, the Finite Element Method (FEM) and Boundary Element Method (BEM) are routinely used for magnetostatic problems. However, the accuracy relies on the density of the elements that are used, especially regarding sharp surface discontinuities. And using a large number of elements, it will take a longer computation time [4-5]. Considering in some applications, people are more concerned the accuracy of the calculation in order to obtain a high homogeneous magnetic field. The slight disturbance in the calculation can lead a significant change to the homogeneity of the field. The purpose of this article is to develop a magnetic field calculation program, which can control the accuracy of the result. The formulas used by program are derived from the Biot-Savart law. In order to calculate the integration, some numerical methods are used in the program, such as Simpson's rules and adaptive quadrature. Applying the Simpson's rules, the integral can be approximated by a series of polynomials. And the adaptive quadrature method automatically adjusts these polynomials to achieve a desired accuracy.

In order to simplify the analysis, the air-core solenoid is assumed to have the thin insulation layer on the wire and the current uniformly distributed over the whole cross section of the winding, which is rectangular.

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2. BASIC FORMULA

According to the Biot-Savart law, at any point of P, the magnetic field $\boldsymbol{B}$ generated by the current can be written as

$$\boldsymbol{B} = \oint \frac{\mu_0 I d\boldsymbol{L} \times \boldsymbol{a}_R}{4\pi R^2} \quad (1)$$

Where μ_0 is the permeability of free space, $d\boldsymbol{L}$ is the differential vector length of the current I carrying filament, R is the distance from the differential element to the point P [6].

Fig.1 shows an air-core solenoid, which has the inner diameter $2a_1$, outer diameter $2a_2$, and length $2b$, is located in the cylindrical coordinate system centered at the origin, and carries a current I.

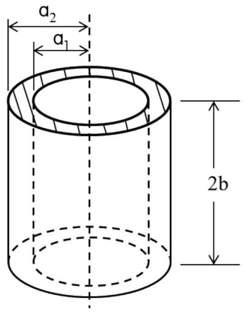


Fig. 1. Solenoid geometry.

The component of magnetic field $\boldsymbol{B}$ at point $P(\rho,\phi,z)$ in Fig.1 can be given by the double integral of Eq. (1) on the cross section of the solenoid,

$$B_\rho = \int_{a_1}^{a_2}\int_{-b}^{b}\frac{\mu_0 I}{2\pi}\frac{(z-b)}{\rho\sqrt{(a+\rho)^2+(z-b)^2}}\left[-K+\frac{a^2+\rho^2+(z-b)^2}{(a-\rho)^2+(z-b)^2}E\right]dadb \quad (2)$$

$$B_z = \int_{a_1}^{a_2}\int_{-b}^{b}\frac{\mu_0 I}{2\pi}\frac{1}{\rho\sqrt{(a+\rho)^2+(z-b)^2}}\left[K+\frac{a^2-\rho^2-(z-b)^2}{(a-\rho)^2+(z-b)^2}E\right]dadb \quad (3)$$

$$B_\phi = 0 \quad (4)$$

where K and E are complete elliptic integrals of the first and second kinds. Eq. (2)~(4) are the basic formulas which are derived from the Biot-Savart law [7].

It is very difficult to obtain the analytic expression of Eq. (2) and (3), except the symmetric axis line of the solenoid. The magnetic field at the axis line only has the z component, and

$$B_z = \frac{\mu_0 J}{2}\left[(b-z)\cdot ln\frac{a_2+\sqrt{a_2^2+(b-z)^2}}{a_1+\sqrt{a_1^2+(b-z)^2}}+(b+z)\cdot ln\frac{a_2+\sqrt{a_2^2+(b+z)^2}}{a_1+\sqrt{a_1^2+(b+z)^2}}\right] \quad (5)$$

3. NUMERICAL METHOD OF CALCULATION

In Eq. (2) and (3), we need to compute the double integral on the cross section of the solenoid. The adaptive quadrature method of iteration will be employed to compute the answers. And the precision of the answers is controlled by the significant figures we wanted.

3.1. Adaptive Quadrature

The adaptive quadrature method is designed to obtain the desired precise value of Eq. (2) and (3). Consider the double integral

$$I = \iint_R f(x,y)dxdy \quad (6)$$

where $R = \{(x,y)|a \le x \le b, c \le y \le d\}$.

Simpson's rule [9] is applied to evaluate the double integral by using a higher-order polynomial. The polynomial is created by the three points on each dimension and given by

$$I = \frac{hk}{9}[f(a,c)+f(a,d)+f(b,c)+f(b,d)+4(f(h,c)+f(h,d)+ fa,k+fb,k+16fh,k+E \quad (7)$$

and the truncation error E,

$$E = -\frac{(b-a)(d-c)}{180}\left[h^4\frac{\partial^4}{\partial x^4}f(\xi_1,\eta_1)+k^4\frac{\partial^4}{\partial y^4}f(\xi_2,\eta_2)\right] \quad (8)$$

where $h=\frac{b-a}{2}$, $k=\frac{d-c}{2}$, $\xi_1,\xi_2 \in [a,b]$, and $\eta_1,\eta_2 \in [c,d]$.

Rather than being proportional to the third derivative, the error is proportional to the fourth derivative. Consequently, Eq. (8) is third-order accurate. Refine the region of the integration, and adjust the region size by dividing it into small four equal pieces. Applying the composite Simpson's rule, the polynomial is given by

$$I = \frac{hk}{9}\sum_{i=0}^{1}\sum_{j=0}^{1}[f(a+2ih,c+2jk)+f(a+2ih,c+2jk+2k)+ fa+2ih+2h,c+2k+fa+2ih+2h,c+2jk+2k+4fa+2ih+h,c+2jk+fa +2ih+h,c+2jk+2k+fa+2ih,c+2jk+k+fa+2ih+2h,c+2jk+k+16f a+2ih+h,c+2jk+k+E' \quad (9)$$

and the truncation error becomes

$$E' = -\frac{(b-a)(d-c)}{180}\left[h^4\frac{\partial^4}{\partial x^4}f(\xi_1,\eta_1)+k^4\frac{\partial^4}{\partial y^4}f(\xi_2,\eta_2)\right] \quad (10)$$

where $h=\frac{b-a}{4}$, $k=\frac{d-c}{4}$, $\xi_1,\xi_2 \in [a,b]$, and $\eta_1,\eta_2 \in [c,d]$.

Comparison between Eq. (8) and (10) indicates that a more accurate estimate can be obtained by refining the region of integration. Because both calculations before and after refinement are estimates of the same integral, their difference provides a measure of the error E_ε. The region with the maximum value of E_ε is chosen to refine and the new approximation can be calculated in the same way.

Thus, in the calculation, this process is performed iteratively to successively compute better and better approximations. The stopping criterion is given to indicate that there is no region to refine and this may be represented as follows:

$$|\varepsilon_a| < (0.5\times 10^{2-n})\% \quad (11)$$

where ε_a is the percent relative error, n is the minimum significant figure. If the stopping criterion is satisfied, the quadrature result is correct regarding at least n significant figures [10].

When the point P appears in the coil area, the denominators of the integrand in Eq. (2) and (3) may have zero value in the calculation process. The Newton-Cotes open integration formula is used to avoid this situation. In consideration of the number of calculations and the accuracy, the midpoint method formula is applied to replace the previous formula. The polynomial is given by

$$I = (b-a)(d-c)f(h,k) \quad (12)$$

where $h=\frac{b-a}{2}$, $k=\frac{d-c}{2}$.

3.2. Program

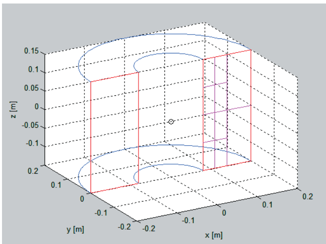
A program, named as rzBI(), is developed to implement this adaptive quadrature method. The program is designed as

$$function\ [Br, Bz] = rzBI(J, x1, x2, y1, y2, rho, z, n)$$

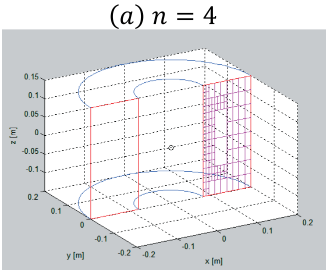
where J: the density of the current in the solenoid;
$x1, x2, y1, y2$: The dimensions of the solenoid;
rho, z: The target point's coordinate in the cylindrical system;
n: The significant figure;
Br: The radial component of a magnetic field;
Bz: The axial component of a magnetic field.

The input parameters are very easy to obtain. And the adaptive method is applied to refine the integration region. According to the desired significant figure, the program divides the region into a certain number of pieces. The Fig.2 shows the different refinement level in which the significant figures are 4, 6, and 8. The magnetic field we calculated was obtained by the output parameters, and the azimuthal component of the magnetic field is equal to 0 as shown in Eq. (4).

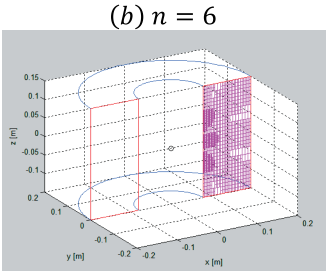
As shown in Fig.2, when the number of significant figures becomes larger, the computation time will increase rapidly due to the refinement. In other words, in order to improve the accuracy, the more computation time is needed. In some cases, the high accuracy is very important to the calculation of the magnetic field, such as the MRI magnet design and so on. In contrast, the consumption time in the calculation is not so urgent for the design. In fact, the computation time is acceptable in the example when the number of significant figures is 6. And the magnetic field at the arbitrary point can be calculated in this way.



$(a)\ n = 4$



$(b)\ n = 6$



$(c)\ n = 8$

Fig. 2. Refinement with different significant figures.

4. NUMERICAL EXAMPLE

In an MRI magnet design, the major specification of the static fields is to have the intensity of the generated magnetic field within strict limits and with a high degree of homogeneity. In this example, the magnet has the homogeneity of 100ppm within 10cm DSV. The superconducting solenoid was used to make the main magnet in the design, which had the parameters as shown in Fig.3 and Table.1.

As for the magnetic field at the center point in the solenoid, there is only one component B_z in the magnetic field in cylindrical coordinate system. Using Eq. (5), it is easy to obtain the exact value of the magnetic field at the center line of the solenoid, which is shown in Fig.4. The

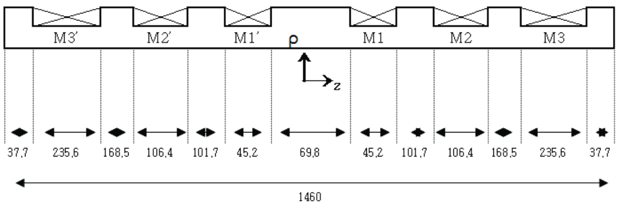


Fig. 3. Position of main magnet(unit = mm).

TABLE I
THE PARAMETERS FOR MAIN MAGNET.

Coil	Width (mm)	Thickness (mm)	Current Density (A/mm^2)
M1	45.2	21	225
M2	106.4	17	225
M3	235.6	21	225
M1'	45.2	21	225
M2'	106.4	17	225
M3'	235.6	21	225

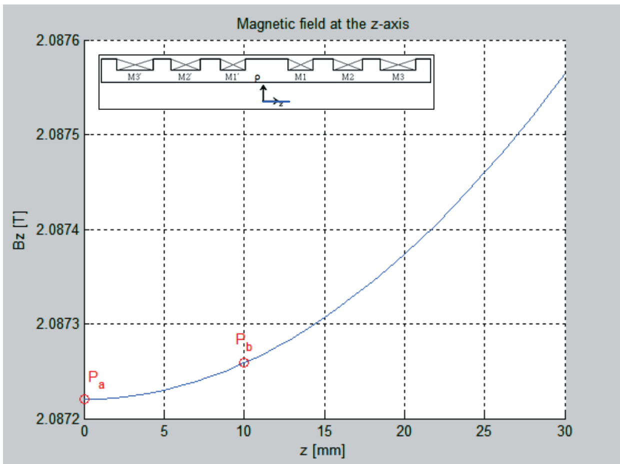


Fig. 4. Exact value of magnetic field Bz at z-axis.

magnetic fields at point $P_a(0,0,0)$ and $P_b(0,0,10)$ can be found in Fig.4 and the values are equal to 2.087220166760109T and 2.087258575810820T, respectively. Table 2 shows the calculation of the magnetic field at the point $P_a(0,0,0)$ and $P_b(0,0,10)$ using the program rzBI().

TABLE Ⅱ
RESULT OF CALCULATION USING RZBI()

Desired Significant Figure (n)	Calculation Result B_z(T)			
	Point $P_a(0,0,0)$	Actual Significant Figure	Point $P_b(0,0,10)$	Actual Significant Figure
3	2.087237404363371	5	2.087275852623942	5
4	2.087217796512010	5	2.087307500645487	4
5	2.087220044057111	7	2.087258546361567	8
6	2.087220144086130	8	2.087258533257109	8
7	2.087220162282067	9	2.087258580220246	8
8	2.087220155099187	8	2.087258575509581	10
9	2.087220166723806	11	2.087258575760321	10

From the table, we can see that the actual significant figure is more than the desired significant figure. And the method can adjust the number of iteration automatically when the different accuracy is needed. When significant figure is set to 6, the magnetic field result has an accuracy of 1ppm.

Using the MagNet software, the magnetic field can be obtained by the FEM method. The accuracy of the FEM method depends on the mesh grids in the calculation region. It is very hard to control the accuracy in the magnetic field calculation. However, in order to obtain a high homogeneity field, the value of field is required to have a sufficient number of significant figures. This is also the reason why the FEM method is rarely used in high homogeneity field design. The magnetic field B_z on the ρ axis are calculated by the FEM method and rzBI(), and the results are shown in Fig.5. The value of fields calculated by rzBI() have 6 significant figures, thus we can find that there are some slight differences between the results of FEM method and rzBI().

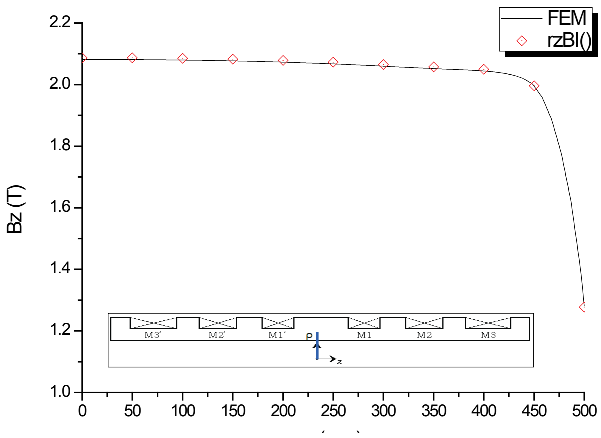


Fig. 5. Comparison of magnetic field at ρ-axis in two methods.

5. CONCLUSION

The adaptive quadrature method of magnetic field calculation was presented in this paper. This method can control the precision of the result by the significant figure. And using this method, the magnetic field in the whole space can be calculated, including the interior of the solenoid. In some applications which required a high accuracy, this method is superior to other approximation methods. A program rzBI() is developed to implement this method, and the results show that rzBI() can easily obtain the magnetic field with a desire precision. This program can help us to find the magnetic field of the solenoid with desired precision, and this can be widely used in the application of NMR, MRI, and so on. The results in the numerical example show the method is flexible and convenient.

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