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On A New Framework of Autoregressive Fuzzy Time Series Models

Qiang Song*
Atlanta Forecasting Systems, Alpharetta, GA, USA

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ABSTRACT

Since its birth in 1993, fuzzy time series have seen different classes of models designed and applied, such as fuzzy logic relation and rule-based models. These models have both advantages and disadvantages. The major drawbacks with these two classes of models are the difficulties encountered in identification and analysis of the model. Therefore, there is a strong need to explore new alternatives and this is the objective of this paper. By transforming a fuzzy number to a real number via integrating the inverse of the membership function, new autoregressive models can be developed to fit the observation values of a fuzzy time series. With the new models, the issues of model identification and parameter estimation can be addressed; and trends, seasonalities and multivariate fuzzy time series could also be modeled with ease. In addition, asymptotic behaviors of fuzzy time series can be inspected by means of characteristic equations.

Keywords: Fuzzy Time Series, Auto-Regressive Fuzzy Time Series Models

* Corresponding Author, E-mail: qsong3@hotmail.com

1. INTRODUCTION

A fuzzy time series is a sequence of observations each of which is a fuzzy set or a fuzzy number (Song and Chissom, 1993a). The concept of fuzzy time series was proposed to model dynamic processes with fuzzy observations that are hard if not impossible to be modeled by the conventional time series. For instance, it is perhaps more appropriate to model human being's feelings or moods using fuzzy sets than using a random variable because feelings or moods are of a subjective matter. As a result, if one records one's own overall mood on a daily basis, one will have a sequence of fuzzy sets observed over time. Such a sequence of fuzzy sets describes, if one admits, the overall feelings a person may experience each day, and forms a fuzzy time series. Abundant examples of fuzzy time series can be found.

Since the publication of the first paper on fuzzy time series, fuzzy time series as a framework of modeling technique have found numerous applications in forecast, experienced various enhancements both in modeling methods and applications (Chen, 1996, 2002; Chen and Chung, 2006; Chen, and Hwang, 2000; Chen and Hsu, 2004, 2008; Hwang *et al*., 1998; Huarng, 2001a, 2001b; Huarng and Yu, 2004, 2006; Lee and Chou, 2004; Lee *et al*., 2006; Own and Yu, 2005; Sah and Degtiarev, 2005; Song and Chissom, 1993b, 1994), and have also encountered, in my opinion, at least three major difficulties or obstacles in the course of their development.

The first difficulty is to select a proper model for a given fuzzy time series. In applications of fuzzy time series, the practice has been to choose a model according to experience or for convenience, say, choose a first order model, and then determine the parameters or details of the model using historical data. The selected model is then used in forecasting. To find the best model, different candidate models with different parameters have to be created and used in this fashion, and the modeling or forecast errors are used as the criterion for choosing the best model. This heuristically sound approach, although practical and applicable, has a number of drawbacks. The most striking one is the lack of a systematic way to build a fuzzy time series model, the process called model identification. For this reason, modeling fuzzy time series has been mainly a trial and error process. However, recently a method has been proposed to identify the order of the fuzzy time series before constructing the model (Song, 2003). The main idea is to estimate the sample autocorrelation functions of the fuzzy observations and then use the estimated autocorrelation functions as guidance in selecting a proper model. This method, albeit still in its rudimental stage, provides a systematic approach to model selections for fuzzy time series, and also provides a potential tool for model analysis.

The second difficulty in fuzzy time series applications is the construction of high order models. It is quite evident that very often high order models are better suitable to a given fuzzy time series than a first-order model. Although high order models were proposed in terms of fuzzy relations (Song and Chissom, 1993a), there are a certain number of issues and difficulties in implementation of such models. It seems to me that the fuzzy relations might not be good candidates for high order models of fuzzy time series if the operations are solely 'max-min' composition operator. Unless different and more efficient forms can be found to express fuzzy relations of high orders, fuzzy relations may not be the right choice for fuzzy time series of high orders. In the literature, a rule-based model has been proposed to implement high-order models, and the results are very impressive (Chen, 2002; Chen and Chung, 2006; Chen and Hsu; 2006; Tsai and Wu, 1999). In the rule-based models, instead of the max-min composition operator being used, temporal information hidden in the fuzzy time series is identified in the form of rules to describe the relationship among different fuzzy observations, and thus the model determines the output according the rules identified in such a way using historical data. Such rules can be created to describe the temporal relationships among any numbers of fuzzy observations in the past. As long as a resolution protocol is provided for any potentially conflicting rules, such models can be used well for high-order fuzzy time series. However, such models are not friendly to analysis and the order of the model, i.e., the number of inputs of the rules, must be determined in advance either by experience or by heuristics, and hence more efforts are still needed to enhance this type of models.

And the third difficulty in fuzzy time series application is the lack of a powerful tool to analyze the properties of a model and in turn analyze the properties of the fuzzy time series. Such a tool is in an urgent need as it may provide insights into the fuzzy time series by analyzing the model itself. Without such a tool, fuzzy time series as a research field will have very limited potentials and future. To the best of my knowledge, no progress in this direction has been reported so far in the literature. We can anticipate that these three aspects are complementary to one another, and the solution of one may affect the others.

The motivation of this paper is to propose a new modeling framework for fuzzy time series. Specifically, we address the issue of modeling fuzzy time series of fuzzy numbers. Collectively, we model the observation values of a fuzzy time series by means of α-level points at different times, and the forms of the models take exactly those of autoregressive, moving average, or autoregressive and moving average. It is believed that such models will describe collectively the behavior of a fuzzy time series, and by analyzing the models, properties of the fuzzy time series can be determined. Also, it is expected that such models have the advantages of easy implementation, even with high-order models, and being easy to analyze. The remainder of this paper is organized as follows. In Section 2, we briefly review the definitions of fuzzy time series, the models, and then the sample autocorrelation functions of fuzzy time series, and how to use sample autocorrelation functions to assist model selection. After that, we propose the new modeling framework for fuzzy time series in Section 3, which hopefully provides one tentative solution to the first two problems encountered in fuzzy time series modeling and applications, namely, the model order identification and higher order modeling problems; we will show that fuzzy time series with trends and seasonalities can be modeled easily under the new framework, and multivariate models are possible as well; characteristic equations of fuzzy time series are defined and relationships between properties of the roots and stability will be given. Conclusions and discussions are found in Section 4.

2. LITERATURE REVIEW

2.1 Fuzzy Time Series

First, we review the definition of fuzzy time series and its models.

Let $Y(t)$ ($t = \cdots, 0, 1, 2, \cdots$), a subset of R^1, be the universe of discourse on which fuzzy sets $f_i(t)$ are defined where $i = 1, 2, 3, \cdots$, and $F(t)$ is a collection of $f_i(t)$ where $i = 1, 2, 3, \cdots$ Then, $F(t)$ is called a fuzzy time series on $Y(t)$ (Song and Chissom, 1993). This definition allows a distribution of fuzzy sets to be defined at a given time instant. A special case is where the fuzzy time series has a degenerate distribution at each time instant so that only one fuzzy set is defined. The latter case is more often seen in practice. In some literatures, fuzzy time series is given in the sense of the latter.

A model of a fuzzy time series describes how the observations in the past are related to the current or the future ones. The main approach has been using fuzzy relations as the model of fuzzy time series, either in an equation or in a rule based form. When in a fuzzy relational equation form, a model of the first order can be expressed as follows,

$$F(t) = F(t-1) \circ R(t,\ t-1) \tag{1}$$

where '○' is the 'max-min' operator, $R(t,\ t-1)$ is called a fuzzy relation between $F(t-1)$ and $F(t)$, and is calculated using the following formula,

$$R(t, t-1) = \cup_{i,j}\left(f_i(t-1)\times f_j(t)\right) \tag{2}$$

where '×' is the Cartesian product, '∪' is the union operator, $f_i(t-1)$ and $f_j(t)$ are two fuzzy observations at t–1 and t, respectively. Higher order models of fuzzy relations can be given below,

$$F(t) = \left(F(t-1)\times F(t-2)\times\cdots\times F(t-m)\right)\circ R(t,\ t-m) \tag{3}$$

which reflects the relationship between the observations in the following rule-based form

$$\left(F(t-1)\cap F(t-2)\cap\cdots\cap F(t-m)\right)\to F(t) \tag{4}$$

or

$$F(t) = \left(F(t-1)\cup F(t-2)\cup\cdots\cup F(t-m)\right)\circ R(t,\ t-m) \tag{5}$$

which can be expressed equivalently in the following rule-based form

$$\left(F(t-1)\cup F(t-2)\cup\cdots\cup F(t-m)\right)\to F(t) \tag{6}$$

All of the above three models can be written using the IF-THEN form, and it is this form that is used in many publications as the high-order models of fuzzy time series in applications (Chen, 2002). Depending upon how the membership functions of the outputs are calculated from the membership functions of the inputs in each of the above models, different models may be obtained.

An obvious question can be asked before applying these models; how do we determine, in a systematic fashion, the value of m in the above models, i.e., how do we determine the order of the models systematically? This question falls in the realm of model identification, and applies equally to both the fuzzy relational equation models and the rule-based models. The fuzzy relation model $R(t,\ t-1)$ is a 2-dimensional matrix for a first-order fuzzy time series, and $R(t,\ t-m)$ is an $(m+1)$-dimensional matrix for an m-order fuzzy time series. This not only requires a large amount of memory in calculations but also creates difficulty in illustrations. Therefore, it is necessary to look for different schemes to model the relationships among observations of a fuzzy time series.

2.2 Sample Autocorrelation Functions of Fuzzy Time Series

To address the issue of model identification for fuzzy time series, sample autocorrelation functions of fuzzy time series are proposed in Song (2003). Fuzzy sets are defined on intervals of R^1. To investigate the correlation between fuzzy sets, we can study the correlation between the intervals upon which the corresponding fuzzy sets are defined. Therefore, the main idea is to treat each fuzzy observation as an interval value, and the estimated correlation between the intervals may provide information about how the intervals are correlated to each other. Correlation between intervals can be defined as a collective correlation between all different points in each interval, or can be defined as correlation between some specific points in each interval. Evidently, as the cordiality of such data points approaches infinity, correlation measures given by such points could be the true indication of correlation between intervals. But, this is impractical computationally. Hence, it is more realistic to consider correlation between a finite number of different points with certain properties in each interval. For this reason, three different approximate measures are considered in Song (2003), which we will briefly introduce below.

2.2.1 Using defuzzified values to calculate autocorrelation functions

The simplest case is to calculate the correlation between the defuzzified values of each fuzzy set. Suppose we have a set of observed fuzzy sets $f_1,\ f_2,\cdots,\ f_n$ which are from a fuzzy time series. First, let us calculate the defuzzified value for each fuzzy set as $DF(f_k)$ where k = 1 to n and $DF(\ )$ is a properly defined defuzzification operator. Then, we calculate the mean value of the defuzzified values as follows

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} DF(f_i) \tag{7}$$

The sample auto-covariance function can be estimated as follows,

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-|h|}\left(DF(f_{t+|h|})-\bar{x}\right)\left(DF(f_t)-\bar{x}\right) \tag{8}$$

where h is the lag satisfying $-n < h < n$, and the sample autocorrelation function is estimated as follows,

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \tag{9}$$

where $-n < h < n$. The drawback of this approach is that the autocorrelation coefficient does not reflect the widths of each interval.

2.2.2 α-Contour average autocorrelation functions

For an α value satisfying $0 < \alpha \le 1$, we are able to find a real number on the universe of discourse of the fuzzy set so that $\mu(x)=\alpha$. Denote such a real number as x_α. In the case that multiple values x_α can be found, pick an arbitrary one. Now, suppose $f_1, f_2, \cdots, f_n, \cdots$ are the fuzzy sets each of which is an observation of a fuzzy time series at a time instant. Let x_i^α be a real number from the universe of discourse of f_i such that $\mu_{f_i}(x_i^\alpha)=\alpha$, and suppose $\bar{x}^\alpha = \frac{1}{n}\sum_{i=1}^{n} x_i^\alpha$. Then, the auto-correlation function at the level α can be calculated with (9) and the auto-covariance function can be estimated as follows,

$$\hat{\gamma}^\alpha(h) = \frac{1}{n}\sum_{t=1}^{n-|h|}\left(x_{t+|h|}^\alpha - \bar{x}^\alpha\right)\left(x_t^\alpha - \bar{x}^\alpha\right) \tag{10}$$

Suppose different α values are chosen and the corresponding autocorrelation functions are calculated. Then, the average of the autocorrelation function values at the same lag is called α-contour average autocorrelation function. It can be seen that the width of the universe of discourse can be reflected if there are enough different α values considered in (10).

2.2.3 Randomized average autocorrelation functions

We may randomly pick a data point from each interval and thus form a time series. Then, we calculate the autocorrelation function of such a time series. If we repeat this process for N times, and calculate the average of the autocorrelation function values at each lag, then we will obtain a randomized-average autocorrelation function. Such an average autocorrelation function can reflect the widths of each interval as the data points are selected from the entire interval.

Once the autocorrelation information of a fuzzy time series is obtained from any of the aforementioned three measures, a plot of the autocorrelation function can be created and characteristics of the plot will reveal rich information about the order and type of the model. This will be addressed in Section 3.2.1.B.

3. MAIN RESULTS

In this section, we present a new class of fuzzy time series models, address how to identify the model and estimate the model parameters, discuss how to forecast using the new models, consider the asymptotic behavior of fuzzy time series, explore multivariate models, and exhibit how to model trends and seasonality using the new models.

3.1 New Models of Fuzzy Time Series

We propose three different models in this section. These are the autoregressive, moving average, and autoregressive-moving average models. These models take the same formulation as those in conventional time series. But, the assumptions on these models are quite different.

3.1.1 Autoregressive models

The idea of calculating autocorrelation functions of sampled data points on the universe of discourses of a fuzzy time series in Section 2 can be extended. When calculating sample autocorrelation functions of a fuzzy time series, only two data points at two time instants are considered. If multiple time instants are involved, we may be able to develop a new model. To proceed, let's assume that the observations of the fuzzy time series are fuzzy number with a unique mode. Without loss of generality, suppose a fuzzy time series is given as $\{F_t\}$ and the universe of discourse is $\{X_t\}$ each of which is an interval in R^1 for $t=\cdots,1,2,3,\cdots$. Let $\mu_{f_t}(x)$, a unimodal and continuous function, be the membership function of the fuzzy number f_t, and let α be a real number so that $0<\alpha\le 1$. Let $x_t^\alpha \in X_t$ be a real number such that $\mu_{f_t}(x_t^\alpha)=\alpha$. Now, consider the time series formed by $\{x_t^\alpha\}$. Suppose this time series is stationary and can be modeled with an AR(p) model (Box *et al.*, 1994). Then, we have the following difference equation,

$$x_t^\alpha = a_1 x_{t-1}^\alpha + a_2 x_{t-2}^\alpha + \cdots + a_p x_{t-p}^\alpha + \varepsilon_t^\alpha \tag{11}$$

where $\{\varepsilon_t^\alpha\}$ is a modeling error process with certain properties. For a given α value, there are two different real numbers x_t^α satisfying the condition that $\mu_{f_t}(x_t^\alpha)=\alpha$ and this creates ambiguity. To eliminate this ambiguity when solving for x_t^α from the equation $\mu_{f_t}(x_t^\alpha)=\alpha$, let's decompose any fuzzy number f_t into two parts: a left part and a right part, denoted respectively as f_t^L and f_t^R. Accordingly, the membership function can be denoted as $\mu_t^L(x)$ and $\mu_t^R(x)$, respectively. The left part has the property that the membership function is increasing and the maximum value is 1, and the right part has the property that it is decreasing and has a maximum value of 1. With such decomposition, the membership functions $\mu_{f_t}^L(x_t)$ and $\mu_{f_t}^R(x_t)$ have inverse functions which we denote as $x_t^L(\alpha)$ and $x_t^R(\alpha)$, respectively. Figure 1 below demonstrates an example of $\mu_{f_t}^L(x_t)$ and $\mu_{f_t}^R(x_t)$ for a given α level,

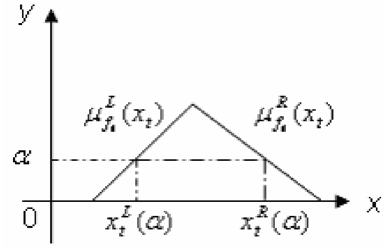

Figure 1. Illustration of a fuzzy membership function decomposed into two parts.

where the membership function is piecewise linear although the membership functions could be piecewise nonlinear. For simplicity, we will ignore the superscript L and R in these notations by assuming that in the model all membership functions or their inverse are of the same part, and all the models should be understood accordingly. For simplicity, denote $x_t(\alpha) = x_t^{\alpha}$. In addition, it is assumed that $x_t(\alpha)$ is integrable with respect to α for all t. Then, (11) can be rewritten as

$$x_t(\alpha) = a_1 x_{t-1}(\alpha) + a_2 x_{t-2}(\alpha) + \cdots + a_p x_{t-p}(\alpha) + \varepsilon_t(\alpha) \quad (12)$$

where $\varepsilon_t(\alpha) = x_t(\alpha) - \sum_{i=1}^{p} a_i x_{t-i}(\alpha)$. Let E_t be a fuzzy set whose membership function is given by $\mu_{E_t}(\varepsilon_t(\alpha)) = \alpha$. Then, E_t can be taken as the modeling error. It can be seen that model (12) describes the relationship among $x_t(\alpha)$, a real number that has a membership function value α at t, and $x_{t-1}(\alpha), x_{t-2}(\alpha), \cdots, x_{t-p}(\alpha)$ all real numbers with the same membership function values at t-1, t-2, $\cdots$, t-p. Thus, we have derived a time series of real numbers from a fuzzy time series. All the data in the derived time series have the same fuzzy membership value α. As $\{x_t(\alpha)\}$ is an ordinary time series, we can identify and build an AR model for it. Such a model can be seen as a partial model of the fuzzy time series because this model only deals with a subset of points in the universe of discourse at different t. We hope to develop a model which can describe the fuzzy time series collectively. This model should contain $x_t(\alpha)$ for all different α values, and different α values should have different contributions in the model. To this end, let's multiply α to both sides of (12) to get a weighted version of the model, and integrate both sides with respect to α over [0, 1]. Suppose such an integral exists. Then, we have the following equation,

$$\int_0^1 x_t(\alpha)\alpha\, d\alpha = \int_0^1 a_1 x_{t-1}(\alpha)\alpha\, d\alpha + \cdots + \int_0^1 a_p x_{t-p}(\alpha)\alpha\, d\alpha + \int_0^1 \varepsilon_t(\alpha)\alpha\, d\alpha \quad (13)$$

It can be seen that $\varepsilon_t(\alpha)$ is also a function of α from (12). Therefore, Eq. (13) is well defined. As each a_i (i = 1, 2, $\cdots$, p) is simply a constant, (13) can be written equivalently as

$$\tilde{x}_t = a_1 \tilde{x}_{t-1} + a_2 \tilde{x}_{t-2} + \cdots + a_p \tilde{x}_{t-p} + \tilde{\varepsilon}_t(\alpha) \quad (14)$$

where $\tilde{x}_t = \int_0^1 x_t(\alpha)\alpha\, d\alpha$ for each t, and $\tilde{\varepsilon}_t = \int_0^1 \varepsilon_t(\alpha)\alpha\, d\alpha$, which are transformations from fuzzy numbers to real numbers. $\tilde{x}_t$ can be seen as a type of representative of a fuzzy set or fuzzy number in R^1. Thus, (14) models the fuzzy time series in the sense that it models the behavior of the transformed fuzzy time series via integration of the weighted inverse of the membership functions. As each $\tilde{x}_t$ is a real number, $\{\tilde{x}_t\}$ forms an ordinary time series, and thus a proper model can be identified and built for it. However it must be pointed out that such a model is for the entire universe of discourse of the observations of a fuzzy time series. Therefore, it is a type of models of fuzzy time series. In applications, fuzzy sets are sometimes defined on a discrete subset of R^1. To calculate $\{\tilde{x}_t\}$, we may select a finite number of values for α, and calculate the weighted sum of $x_t(\alpha)$ for different t where α is the weight, as follows,

$$\tilde{x}_t = \sum_{i=1}^{K} \alpha_i x_t(\alpha_i) \quad (15)$$

It can be seen that in the formula x_t with large α values will have large influence on the value of $\tilde{x}_t$ and will in turn have large influence on the determination of the model parameters a_i in (12). The above models are evidently the autoregressive type or AR for short.

3.1.2 Moving average models

Now, let us consider the moving average models for a fuzzy time series. The key is to construct a modeling error fuzzy time series and use this time series in the model. Suppose $\{E_t\}$ is a fuzzy time series, and that $\{F_t\}$ can be expressed by $\{E_t\}$ in the following manner provided that b_i are known for $i = 1, 2, \cdots, q$,

$$x_t(\alpha) = e_t(\alpha) + b_1 e_{t-1}(\alpha) + b_2 e_{t-2}(\alpha) + \cdots + b_q e_{t-q}(\alpha) \quad (16)$$

To construct the time series $\{E_t\}$, let us define $e_t(\alpha) = 0$ for t = 1, 2, $\cdots$, q-1, $e_q(\alpha) = x_q(\alpha)$, and

$$e_t(\alpha) = x_t(\alpha) - \sum_{j=1}^{q} b_j e_{t-j}(\alpha), \forall \alpha, \ \forall t > q \quad (17)$$

Thus, $e_t(\alpha)$ is uniquely determined for each α and the membership function of fuzzy set E_t is given by $\mu_{E_t}(e_t(\alpha)) = \alpha$. Hence, $\{E_t\}$ is a well-defined fuzzy time series. Next, let us multiply α to both sides of (16), and integrate both sides with respect to α from 0 to 1. Suppose all the integrals exist. Then, we have the following equation,

$$\int_0^1 \alpha\, x_t(\alpha) d\alpha = \int_0^1 \alpha\, e_t(\alpha) d\alpha + b_1 \int_0^1 \alpha\, e_{t-1}(\alpha) d\alpha + b_2 \int_0^1 \alpha\, e_{t-2}(\alpha) d\alpha + \cdots + b_q \int_0^1 \alpha\, e_{t-q}(\alpha) d\alpha \quad (18)$$

Denote $\tilde{e}_t = \int_0^1 \alpha\, e_t(\alpha) d\alpha \ \forall t$. Then, (18) can be written as

$$\tilde{x}_t = \tilde{e}_t + b_1 \tilde{e}_{t-1} + b_2 \tilde{e}_{t-2} + \cdots + b_q \tilde{e}_{t-q} \quad (19)$$

As $\{\tilde{x}_t\}$ and $\{\tilde{e}_t\}$ are both ordinary time series, (19) can be regarded as a moving average model of fuzzy time series $\{F_t\}$. Note that the moving average model of fuzzy time series is different from that of the conventional time series in that the errors are fuzzy sets in fuzzy time series moving average models. The error fuzzy time series are defined recursively via (17). Note also that the coefficients in (17) are different from those in (18) and (19).

3.1.3 Autoregressive-moving average models

Finally, we consider the autoregressive-moving average (ARMA) models of a fuzzy time series. The key is again to construct a fuzzy time series of the modeling error and use it in the model, which can be the corner stone of the ARMA fuzzy time series models.

For a fuzzy time series $\{F_t\}$, we can construct another fuzzy time series $\{E_t\}$ whose universe of discourse is determined by the following equation provided that a_i and b_j are known $\forall i, j$,

$$\varepsilon_t(\alpha) = x_t(\alpha) - \sum_{i=1}^{p} a_i x_{t-i}(\alpha) - \sum_{j=1}^{q} b_j \varepsilon_{t-j}(\alpha) \tag{20}$$

for t = q+1, $\cdots$, $\varepsilon_t(\alpha) = 0$ for t = 1, 2, $\cdots$, q-1, and $\varepsilon_q(\alpha) = x_q(\alpha)$. Thus, $\{E_t\}$ is a well-defined fuzzy time series and each of the observation is a fuzzy set of the modeling error. Suppose $\{F_t\}$ can be modeled by $\{E_t\}$ in the following manner,

$$x_t(\alpha) = \sum_{i=1}^{p} a_j x_{t-j}(\alpha) + \varepsilon_t(\alpha) + \sum_{j=1}^{q} b_j \varepsilon_{t-j}(\alpha) \tag{21}$$

Then, this is an ARMA(p, q) model with parameters a_i and b_j. Similarly, we can obtain the following equivalent form via integration with respect to α,

$$\begin{aligned}\tilde{x}_t &= a_1\tilde{x}_{t-1} + a_2\tilde{x}_{t-2} + \cdots + a_p\tilde{x}_{t-p} + \tilde{\varepsilon}_t \\ &\quad + b_1\tilde{\varepsilon}_{t-1} + b_2\tilde{\varepsilon}_{t-2} + \cdots + b_q\tilde{\varepsilon}_{t-q}\end{aligned} \tag{22}$$

where $\tilde{\varepsilon}_t = \int_0^1 \alpha\, \varepsilon_t(\alpha) d\alpha \;\forall t$. So, we have derived an ARMA model of fuzzy time series so that the observation value at t can be expressed as a linear combination of the past observations and modeling errors. Note that the coefficients in (20) are different from those in (21) and (22).

3.2 Model Identification and Parameter Estimation

We will discuss the model identification and parameter estimation issues for a fuzzy time series under the new framework in this section.

3.2.1 Model identification

The goal of model identification is to determine the type and the order of a fuzzy time series model. As mentioned above, how to determine the order of the model for a fuzzy time series has been an issue in fuzzy time series modeling. We will present two different methods in this section, each of which has its own strengths and weaknesses.

A. Time series modeling approach

As $\{\tilde{x}_t\}$ is an ordinary time series, all techniques for time series analysis can be used for $\{\tilde{x}_t\}$. For example, the autocorrelation function and the partial autocorrelation function, and even the corresponding power spectra are useful tools for this purpose (Box *et al.*, 1994). To identify the model, we may estimate the autocorrelation function of $\{\tilde{x}_t\}$, and check if there is a cutoff after a finite lag. If there is a cutoff after a finite lag, say q, then $\{\tilde{x}_t\}$ may be modeled by an MA(q) model. If there is no cutoff, then estimate the partial autocorrelation function of $\{\tilde{x}_t\}$ and check if there is a cutoff after a finite lag, say p. If the partial autocorrelation function has a cutoff after p lags, then it may be modeled by an AR(p). If not, then an ARMA model might be needed. Hence, it is relatively simpler to identify an AR(p) or an MA(q) model. But, it is not so easy to identify an ARMA (p, q) model. For this reason, we will utilize the algorithm proposed in Song and Esogbue (2006) to identify an ARMA model for a given time series. Next, we will briefly review the algorithm in Song and Esogbue (2006).

Let us first review the main idea of the algorithm. Suppose $\{\tilde{x}_t\}$ is a stationary time series. Then, $\{\tilde{x}_t\}$ can be modeled in general by the following difference equation:

$$\begin{aligned}\tilde{x}_t &= a_1\tilde{x}_{t-1} + a_2\tilde{x}_{t-2} + \cdots + a_p\tilde{x}_{t-p} + \tilde{\xi}_t \\ &\quad + b_1\tilde{\xi}_{t-1} + b_2\tilde{\xi}_{t-2} + \cdots + b_q\tilde{\xi}_{t-q}\end{aligned} \tag{23}$$

where $\{\tilde{\xi}_t\}$ is an *i.i.d.* modeling error process, p and q are the orders of the model, and this is an ARMA process and denoted as ARMA(p, q). To motivate our algorithm, let us rewrite ARMA(p, q) model (23) in a different form as follows:

$$\tilde{x}_t = a_1\tilde{x}_{t-1} + a_2\tilde{x}_{t-2} + \cdots + a_p\tilde{x}_{t-p} + \tilde{\gamma}_t \tag{24}$$

where

$$\tilde{\gamma}_t = \tilde{\xi}_t + b_1\tilde{\xi}_{t-1} + b_2\tilde{\xi}_{t-2} + \cdots + b_q\tilde{\xi}_{t-q} \tag{25}$$

and $\{\tilde{\gamma}_t\}$ can be regarded as an MA(q) time series. Obviously, (23) is equivalent to (24) and (25) combined. That is, we purposefully decompose an ARMA(p, q) time series into two processes: one is an autoregressive process and the other a moving average process. $\tilde{\gamma}_t$ can be seen as the model error in (24). But, it is not required here that $\tilde{\gamma}_t$ be uncorrelated, as implied by (25). Instead, $\tilde{\gamma}_t$ could be correlated and it is the correlation of $\tilde{\gamma}_t$ that we can draw information from about q based on a chosen value of p.

The algorithm can be explained as follows. Suppose that both (24) and (25) are time series. We pick a value for p of the autoregressive part. Applying an estimation algorithm, we obtain a set of model parameters of $a_1, a_2, \cdots, a_p$. Then, from this AR(p) model we obtain a model residual time series $\{\tilde{\gamma}_t\}$. If this time series is a white noise, then its autocorrelation function has a value of virtually zero for any non-zero lags, and this characteristic is very easy to recognize. In this case, we have identified p correctly and we know that the model is an AR(p). If $\{\tilde{\gamma}_t\}$ is correlated, and if its autocorrelation function has a cutoff after q lags, then $\{\tilde{\gamma}_t\}$ is an MA(q) time series and we need to add an MA(q) part to the model. Otherwise, if its autocorrelation function has tail-off, it means that $\{\tilde{\gamma}_t\}$ is either an autoregressive or a mixed time series. In either case, it suggests that in (24) the p value was not chosen properly. If so, we simply increase p by 1, and repeat the above process.

To determine the value of p, we will utilize a statistical hypothesis test. As is known, the autocorrelation function values of a white noise are zero for all non-zero lags. For the estimated autocorrelation function of a white noise, the autocorrelation coefficient ρ_l can be regarded as a random variable whose variance can be approximated with the following formula (Box *et al.*, 1994),

$$\sigma = \frac{1}{\sqrt{N}} \tag{26}$$

where N is the sample size of the data used to calculate the autocorrelation function. Thus, for a given significance level τ, if the percentage of autocorrelation coefficients ρ_l of non-zero lags that are outside the confidence interval $(-k\sigma, k\sigma)$ is less than $1-\tau$ where k is chosen properly, then there is a strong reason to believe that the residual process is a white noise. Otherwise, we need to test further whether the residuals are a moving average or an autoregressive time series. The following Algorithm 1 can be used to determine the value of either p or q, depending on the input data of the algorithm. However, in the description, it is assumed to determine the value of q.

Algorithm 1

Step 1. Define a significance level τ, pick a positive number k, and estimate σ using (26). Set $l = 0$. Go to Step 2.

Step 2. If ρ_l is outside the confidence interval ($-k\sigma$, $k\sigma$), set $l = l+1$, and repeat Step 2. Else, go to Step 3.

Step 3. Calculate the percentage ϕ of autocorrelation coefficients ρ_l that are outside of the confidence interval from lag l to the maximum lag. If ϕ is less than or equal to $1-\tau$, then let $q = l$ and stop. Else, q is undetermined and stop.

Once the order of the model has been determined, we need to estimate the parameters in the model. After that, the probability structure of the model errors will be checked, and if needed the above algorithm will be run with a different order values. The above algorithm can be used as a subroutine in the following main algorithm.

Main Algorithm:

Step 1. Use Algorithm 1 to test if the time series is AR or MA. If neither, let $p = 0$, and go to Step 2.

Step 2. Let $p = p+1$. Estimate the parameters of AR(p), and calculate model residuals. Use Algorithm 1 to test if the residuals are an MA time series. If yes, then go to Step 3 with q identified. Else, repeat Step 2.

Step 3. Estimate parameters for the tentative model ARMA(p, q), and calculate model residuals. Then use Algorithm 1 to test if the residuals are a white noise process. If yes, stop. Else, go to Step 2.

The above algorithm has been applied to model some time series published in the literature with very satisfactory results. All models can be built automatically without human interventions (Song and Esogbue, 2006). It is my belief that this algorithm can be used to build an ARMA(p, q) fuzzy time series model without much difficulty, and thus we will leave the details to readers.

B. Sample autocorrelation/partial autocorrelation function approach

This approach has been already employed partially in Song (2003) where the sample autocorrelation function is used. Sample autocorrelation function works well if the time series is an MA, but not if the time series is an AR. So, we will revise the approach here to incorporate partial autocorrelation functions and to utilize the two algorithms of the preceding section.

As shown above, for a given α we can obtain a time series $\{x_t(\alpha)\}$ from a fuzzy time series $\{F_t\}$. This is an ordinary time series and therefore we can calculate the sample autocorrelation function and even the sample partial autocorrelation function of $\{x_t(\alpha)\}$. With Algorithm 1, if the sample autocorrelation function has a cutoff after a finite number of lags, this implies that an MA model might be a proper model for $\{x_t(\alpha)\}$. But, when α changes, we will have a different time series. The sample autocorrelation function of the new time series may or may not have a cutoff after a finite number of lags. To have a better picture of the fuzzy time series, we can calculate the α-contour average sample autocorrelation function of the fuzzy time series, as explained in Section 2.2. Intuitively, if a fuzzy time series can be modeled by an MA(q) model, then for each α, the derived time series $\{x_t(\alpha)\}$ should be modeled approximately by an MA(q) model. Likewise, with Algorithm 1 if the sample partial autocorrelation function of $\{x_t(\alpha)\}$ has a cutoff after a finite number of lags, then it implies that $\{x_t(\alpha)\}$ may be modeled properly by an AR(p) model. If we calculate the α-contour average sample partial autocorrelation function of the fuzzy time series, then it may provide useful modeling information about the fuzzy time series as a whole. For example, if the α-contour average sample partial autocorrelation function has a cutoff after a finite number of lags, then there is a strong belief that the fuzzy time series can be modeled properly by an AR model.

If neither the sample autocorrelation function nor the sample partial autocorrelation function can provide enough information, then further actions are needed. We provide only an outline here as details can be implemented similarly to the main algorithm of the previous section.

Suppose we use the α-contour average sample autocorrelation function. For each α value, we have obtained a time series $\{x_t(\alpha)\}$. Suppose also that the α-contour average sample autocorrelation function does not have a cutoff after a finite number of lags. Then, we can assume that each time series $\{x_t(\alpha)\}$ might be an ARMA series. So, we may assume that the order of the AR part is p, and apply it to time series $\{x_t(\alpha)\}$ for each α value. Then, estimate the parameters for the AR model, and calculate the modeling error time series for each α value. Finally, we calculate the α-contour average sample autocorrelation of the error time series. If the α-contour average sample autocorrelation has a cutoff after a finite number of lags q, we take the model as ARMA (p, q). Then, we estimate the parameters of the model and check the modeling error again. If the α-contour average sample autocorrelation function of the model error has all zeros for all non-zero lags, then the model is believed to be proper. Else, we increase p by 1, and repeat the whole process until a stopping criterion is satisfied. This process is very similar to the one in Section A, and can be generalized to use the other two measures of Section 2.2 to calculate the corresponding sample autocorrelation/partial autocorrelation functions.

However, there is a difficulty in choosing the stopping criterion for this algorithm. This is because here we are working with a family of time series instead of a single one. In this case, we may not be able to see that all the correlation functions have a cutoff after the same finite number of lags at all times. To handle this issue, one approach is to calculate the percentage of the model errors with different α values that have a cutoff after a finite number of lags. If the percentage exceeds a critical value, then we regard the autocorrelation function has a cutoff after a finite number of lags. But, it should be regarded as an open problem how to choose a stopping criterion for this algorithm in general.

3.2.2 Parameter estimation

Once a model is identified, the parameters of the model can be estimated by using the least square method. For an AR(p) model, a_i (i = 1, 2, ⋯, p) will be estimated, and for an MA(q) model, b_j (j = 1, 2, ⋯, q), and for an ARMA(p, q) model, a_i (i = 1, 2, ⋯, p) and b_j (j = 1, 2, ⋯, q) will be estimated. This can be done by using a few different approaches. First, we may use $\{\tilde{x}_t\}$ directly as the data in estimating the parameters. The parameters should be such that the total square errors of the model are minimal, i.e.,

$$\min \sum_t \Big(\tilde{x}_t - a_1\tilde{x}_{t-1} - a_2\tilde{x}_{t-2} - \cdots - a_p\tilde{x}_{t-p} - b_1\tilde{\varepsilon}_{t-1} - b_2\tilde{\varepsilon}_{t-2} - \cdots - b_q\tilde{\varepsilon}_{t-q}\Big)^2 \tag{27}$$

Second, we may select different α values α_i for i = 1 to N and determine the corresponding real numbers for each fuzzy set to obtain $\{x_t(\alpha_i)\}$. This forms an ordinary time series. For each of such time series, apply model (21) and use the least square method to minimize the following objective function,

$$\min \sum_t \sum_{i=1}^{N} \alpha_i \Big(x_t^{\alpha_i} - a_1x_{t-1}^{\alpha_i} - a_2x_{t-2}^{\alpha_i} - \cdots - a_px_{t-p}^{\alpha_i} - b_1\varepsilon_{t-1}^{\alpha_i} - b_2\varepsilon_{t-2}^{\alpha_i} - \cdots - b_q\varepsilon_{t-q}^{\alpha_i}\Big)^2 \tag{28}$$

The second form is to minimize the weighted sum of the squares of model errors. Both (27) and (28) can be used for AR and MA models.

3.3 Forecasting

Once a model is identified and built, forecasting can be performed using the following formula if we have an AR model,

$$\hat{x}_t(\alpha) = a_1x_{t-1}(\alpha) + a_2x_{t-2}(\alpha) + \cdots + a_px_{t-p}(\alpha) \tag{29}$$

where $\hat{x}_t(\alpha)$ is a real number of the universe of discourse of the fuzzy time series at t and α is a preselected value satisfying $0 < \alpha \le 1$. For different values of α, we may have different $\hat{x}_t(\alpha)$. Thus, we could obtain a fuzzy set as the forecast of the fuzzy time series at time t. In other words, by applying the forecasting model (29) we are able to obtain a fuzzy set as the forecast, which is consistent with the existing literature. We can also defuzzify this fuzzy forecast, if we wish.

For an MA model, we need to first estimate $\hat{e}_t(\alpha)$ using (17), and then use the following to forecast

$$\hat{x}_t(\alpha) = \hat{e}_t(\alpha) + a_1\hat{e}_{t-1}(\alpha) + \cdots + a_q\hat{e}_{t-q}(\alpha) \tag{30}$$

for all α values. In the case where $x_t(\alpha)$ is unknown, $\hat{e}_t(\alpha)$ can be estimated using the average of $\hat{e}_{t-1}(\alpha)$, $\hat{e}_{t-2}(\alpha)$, ⋯.

For an ARMA model, we need also to first estimate the errors $\hat{e}_t(\alpha)$ using (20), and use the following to forecast,

$$\hat{x}_t(\alpha) = a_1 x_{t-1}(\alpha) + \cdots + a_p x_{t-p}(\alpha) + \hat{e}_t(\alpha) + b_1 \hat{e}_{t-1}(\alpha) + \cdots + b_q \hat{e}_{t-q}(\alpha) \quad (31)$$

Again, in the case where $x_t(\alpha)$ is unknown, $\hat{e}_t(\alpha)$ can be estimated using the average of $\hat{e}_{t-1}(\alpha)$, $\hat{e}_{t-2}(\alpha)$, $\cdots$ All the forecasts are fuzzy sets.

3.4 Characteristic Equations and Asymptotic Behaviors

Model (14) defines a dynamic system which describes collectively how a fuzzy time series may evolve with time. The characteristic equation of (14), given below, may provide information about the asymptotic behavior of a fuzzy time series,

$$1 - a_1 z - a_2 z^2 - \cdots - a_p z^p = 0 \quad (32)$$

where z is a complex number. From the dynamic system theory (Nise, 2004), we know that if all the roots of (32) are inside the unit circle on the z-plane, then (14) is asymptotically stable. If one or more roots are outside of the unit circle, then (14) is unstable. From the magnitude and distribution of the roots of the characteristic equation, we may infer the asymptotic behavior, such as whether it will diverge or not as t approaches infinity, about a fuzzy time series, or whether it is bounded, etc. There is potentially a lot to do on this aspect.

3.5 Multivariate Models

Multivariate models were proposed in (Song and Chissom, 1993a), but no applications of the proposed models have been reported in the literature yet. One of the reasons for the lack of application of the proposed multivariate models might be that such models are not easy to implement. Nevertheless, different models of multivariate fuzzy time series have been reported recently (Hsu *et al.*, 2003; Lee *et al.*, 2006). In this section, we consider different multivariate models.

Suppose we have multiple fuzzy time series $\{F_t\}$, $\{H_t\}$, and $\{R_t\}$. It is assumed that the current value of $\{F_t\}$ can be modeled by its own past values and by the current and/or the past values of $\{H_t\}$ and $\{R_t\}$ as well. To model this relationship, we can extend model (13) to have the following

$$\int_0^1 x_t(\alpha)\alpha\, d\alpha = \int_0^1 a_1 x_{t-1}(\alpha)\alpha\, d\alpha + \cdots + \int_0^1 a_p x_{t-p}(\alpha)\alpha\, d\alpha + \int_0^1 b_1 u_{t-1}(\alpha)\alpha\, d\alpha + \cdots + \int_0^1 b_q u_{t-q}(\alpha)\alpha\, d\alpha + \int_0^1 c_1 v_{t-1}(\alpha)\alpha\, d\alpha + \cdots + \int_0^1 c_w v_{t-w}(\alpha)\alpha\, d\alpha + \int_0^1 \varepsilon_t(\alpha)\alpha\, d\alpha \quad (33)$$

or equivalently,

$$\tilde{x}_t = \sum_{i=1}^{p} a_i \tilde{x}_{t-i} + \sum_{j=1}^{q} b_j \tilde{u}_{t-j} + \sum_{k=1}^{w} c_k \tilde{v}_{t-k} + \tilde{\varepsilon}_t \quad (34)$$

where $u_t(\alpha)$ and $v_t(\alpha)$ are the inverse membership functions of $\{H_t\}$ and $\{R_t\}$, respectively, $\tilde{x}_t = \int_0^1 x_t(\alpha)\alpha\, d\alpha$, $\tilde{u}_t = \int_0^1 u_t(\alpha)\alpha\, d\alpha$, $\tilde{v}_t = \int_0^1 v_t(\alpha)\alpha\, d\alpha$, $\tilde{\varepsilon}_t = \int_0^1 \varepsilon_t(\alpha)\alpha\, d\alpha$, p, q, and w are positive integers, determined by applying some identification methods as before. After p, q, and w are determined, the parameters a_i, b_j, and c_k can be estimated with the least square method while the objective function can be of the forms similar to (27) or (28). Model (33) can be generalized to an arbitrary number of fuzzy time series cases.

3.6 Modeling Trends in Fuzzy Time Series

It is possible to find a trend existing in a fuzzy time series. This trend is usually hidden in the fuzzy observations over time. We may say the economy becomes better and better, or my friend is becoming richer and richer. Evidently, there is a trend in the phenomenon described using words. There are some literatures available handling trends in fuzzy time series (Sah and Degtiarev, 2005). In this section, we propose a different approach to modeling trends in fuzzy time series.

Suppose $\{F_t\}$ is a fuzzy time series. Define a fuzzy time series $\{\nabla F_t\}$ as the difference between F_t and F_{t-1} where $\triangledown x_t(\alpha) = x_t(\alpha) - x_{t-1}(\alpha)$ defines the universe of discourse of fuzzy set ∇F_t $\forall \alpha$, t. That is, the universe of discourse of fuzzy set ∇F_t is determined by those of F_t and F_{t-1}. The membership function $\mu_{\nabla F_t}(\triangledown x_t(\alpha))$ is defined by $\mu_{\nabla F_t}(\triangledown x_t) = \alpha$ for a given α. Thus, $\{\nabla F_t\}$ is a well-defined fuzzy time series. Therefore, a proper model can be built for $\{\nabla F_t\}$ using the model building method, for example, introduced in Section 3.2.1. Suppose the value of $\{\nabla F_t\}$ can be forecasted at t+1. Note that $x_t(\alpha) = x_{t-1}(\alpha) + \triangledown x_t(\alpha)$. Hence, if we know $x_t(\alpha)$ and $\nabla \hat{x}_{t+1}(\alpha)$, the forecasted value of $\nabla x_{t+1}(\alpha)$, we can calculate the forecast value of $\{F_t\}$ at t+1, i.e., $\hat{x}_{t+1}(\alpha) = x_t(\alpha) + \triangledown \hat{x}_{t+1}(\alpha)$. If necessary, we may consider repeating this differencing and modeling approach for a few times if the trend in the fuzzy time series is complicated.

Particularly, suppose that $\{\nabla F_t\}$ can be modeled as an AR(p) model. That is, there is an integer $p > 0$ such that

$$\triangledown \tilde{x}_t = a_1 \triangledown \tilde{x}_{t-1} + a_2 \triangledown \tilde{x}_{t-2} + \cdots + a_p \triangledown \tilde{x}_{t-p} + \tilde{\xi}_t \quad (35)$$

where a_i, for i = 1, 2, $\cdots$, p, are determined optimally. If $\triangledown x_{t-1}(\alpha)$, $\triangledown x_{t-2}(\alpha)$, $\cdots$, $\triangledown x_{t-p}(\alpha)$ are known for each α, then the forecast value of $\nabla x_t(\alpha)$ can be obtained by the following formula

$$\nabla\hat{x}_t(\alpha) = a_i\nabla x_{t-1}(\alpha) + a_2\nabla x_{t-2}(\alpha) + \cdots + a_p\nabla x_{t-p}(\alpha). \quad (36)$$

Hence, if $x_{t-1}(\alpha)$ is also known, then the forecast of $x_t(\alpha)$ can be determined by

$$\hat{x}_t(\alpha) = x_{t-1}(\alpha) + \nabla\hat{x}_{t-1}(\alpha) \quad \forall\alpha \quad (37)$$

where $0 < \alpha \leq 1$. Evidently, (35) can be replaced by an MA or an ARMA model to estimate $\nabla\tilde{x}_t$.

3.7 Modeling Seasonalities in Fuzzy Time Series

Seasonalities in fuzzy time series, or modeled with fuzzy set theory, have been studied in the literature (Chang, 1997; Song, 1999; Tseng and Tzeng, 2002). In this section, we propose a new approach to modeling seasonal fuzzy time series.

According to the definition of seasonal fuzzy time series (Song, 1999), a fuzzy time series $\{F_t\}$ is seasonal if there is a minimum positive integer S such that $F_{t+S} = F_t$ for all t. From this definition, for a given α value, then it is natural to obtain the following relation

$$x_{t+S}(\alpha) = x_t(\alpha) \quad (38)$$

for any t. Thus, the sequence $\{x_t(\alpha)\}$ is an ordinary seasonal time series. Hence, we can build a model for a seasonal fuzzy time series. One approach is to use fuzzy logic relations as models and construct a model for each seasonal index (Song, 1999). Numerical results indicate that this is a practical and viable approach. Nevertheless, we will explore a different approach in this section under the new framework.

Suppose S is the periodicity of a seasonal fuzzy time series $\{F_t\}$ and suppose for a given α value, $\{x_t(\alpha)\}$ is a seasonal time series. Then, $\{x_t(\alpha)\}$ can be modeled as follows,

$$\Phi(B^S)x_t(\alpha) = \Theta(B^S)a_t(\alpha) \quad (39)$$

where B^S is the differencing operator defined as $x_t(\alpha) - x_{t-S}(\alpha)$, $a_t(\alpha)$ is the model error, and $\Phi(B^S)$ and $\Theta(B^S)$ are polynomials of B^S. (39) relates $x_t(\alpha)$ and $x_{t-S}(\alpha)$, $x_{t-2S}(\alpha)$, etc., which are a multiple of S lags apart from each other. If $\{a_t(\alpha)\}$ is correlated, then it can be modeled as follows.

$$\phi(B)a_t(\alpha) = \theta(B)\varepsilon_t(\alpha) \quad (40)$$

where $\{\varepsilon_t\}$ is a white noise process. Combining (39) and (40) will yield

$$\phi'(B)\Phi'(B^S)x_t(\alpha) = \theta'(B)\Theta'(B^S)\varepsilon_t(\alpha) \quad (41)$$

where $\phi'(B)$, $\Phi'(B^S)$, $\theta'(B)$, and $\Theta'(B^S)$ are polynomials of certain orders. Detailed algorithms can be found in Song and Esogbue (2008) on how to identify (41). For simplicity, we may take the following forms $\phi'(B)\Phi'(B^S) = \sum_{i=1}^{p}\left(a_i B^i + b_i B^{i+S}\right)$, and $\theta'(B)\Theta'(B^S) = \sum_{i=1}^{q}\left(c_i B^i + d_i B^{i+S}\right)$, albeit other forms exist. Then, we will have the following difference equation

$$x_t(\alpha) = \sum_{i=1}^{p} a_i x_{t-i}(\alpha) + \sum_{i=1}^{p} b_i x_{t-i-S}(\alpha) + \sum_{j=0}^{q} c_j\varepsilon_{t-j}(\alpha) + \sum_{j=1}^{q} d_j\varepsilon_{t-j-S}(\alpha) \quad (42)$$

where a_i, b_i, c_j, and d_j are coefficients to be estimated and $c_0 = 1$. To calculate the modeling error fuzzy time series $\{E_t\}$, use the following recursive definitions

$$\varepsilon_t(\alpha) = 0 \quad (43)$$

for $t = 0, 1, 2, \cdots, 2S\text{-}1$ assuming $S \geq p, q$, and

$$\varepsilon_t(\alpha) = x_t(\alpha) - \sum_{i=1}^{p} a_i x_{t-i}(\alpha) - \sum_{i=1}^{p} b_i x_{t-i-S}(\alpha) - \sum_{j=0}^{q} c_j\varepsilon_{t-j}(\alpha) - \sum_{j=1}^{q} d_j\varepsilon_{t-j-S}(\alpha) \quad (44)$$

$\forall t \geq 2S$ where the coefficients are different from those in (42).

Now, multiply α to both sides of (42), and integrate both sides with respect to α, we will have the following

$$\int_0^1 \alpha\, x_t(\alpha)d\alpha = \sum_{i=1}^{p} a_i\int_0^1 \alpha x_{t-i}(\alpha)d\alpha + \sum_{i=1}^{p} b_i\int_0^1 \alpha\, x_{t-i-S}(\alpha)d\alpha + \sum_{j=1}^{q} c_j\int_0^1 \alpha\,\varepsilon_{t-j}(\alpha)d\alpha + \sum_{j=1}^{q} d_j\int_0^1 \alpha\,\varepsilon_{t-j-S}(\alpha)d\alpha$$

Or equivalently,

$$\tilde{x}_t = \sum_{i=1}^{p} a_i\tilde{x}_{t-i} + \sum_{i=1}^{p} b_i\tilde{x}_{t-i-S} + \sum_{j=1}^{q} c_i\tilde{\varepsilon}_{t-i} + \sum_{j=1}^{q} d_i\tilde{\varepsilon}_{t-i-S} \quad (45)$$

where $\tilde{x}_{t-i} = \int_0^1 \alpha\, x_{t-i}(\alpha)d\alpha$, $\tilde{\varepsilon}_{t-i} = \int_0^1 \alpha\,\varepsilon_{t-i}(\alpha)d\alpha$ and $\tilde{\varepsilon}_{t-i-S} = \int_0^1 \alpha\,\varepsilon_{t-i-S}(\alpha)d\alpha$. Model (45) can be built using the algorithm presented in Section 3.2.1 with some minor changes. However, there should be no difficulties in doing so. When there exists a local trend in the fuzzy time series in addition to seasonality, the method in Section 3.6 can be used to build a proper seasonal model.

4. CONCLUSIONS

In this paper, a new framework of autoregressive fuzzy time series models is outlined and issues are discussed on how to build a new fuzzy time series model, how to forecast a fuzzy time series, how to model trends and seasonalities, and how to model multivariate fuzzy time series. The motivation for developing such a new framework is to overcome the three major difficulties or drawbacks in applying the existing fuzzy time series models, i.e., the fuzzy relation models, and the rule-based models. With the existing models, it is hard to determine the type, or the order of the model in advance, it is hard to implement a high-order model (except for rule-based models), and it is hard to analyze a fuzzy time series via models. It is believed that these three major difficulties or drawbacks can be ameliorated or resolved to a large extent with the new framework. It must be admitted that all the results presented here are still preliminary, and more effort is needed to make the framework complete. Particularly, there may be some theoretical issues involved that need a better understanding. These constitute research subjects in the near future.

It can be seen that the main idea is to derive an ordinary time series using the α-level point x_{α}, and model all the α-level points collectively using a weighted sum of the data. As a result, model identification and building algorithms widely used in time series analysis can be borrowed with some modifications or enhancements (Box *et al*., 1994).

However, to build a new fuzzy time series model proposed in this paper, a large amount of historic data is required to ensure the validity of the model. When only a handful data are available, it is difficult to apply the modeling technique to build a valid new autoregressive fuzzy time series model. In this case, it is suggested to resort to the existing fuzzy time series models, such as the fuzzy relation models or the rule-based models. Applications of the new models are quite straightforward, and will be reported elsewhere.

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