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A STUDY ON THE CATEGORY OF NORMAL FUZZY HYPERGROUPS

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Abstract. Although the category $NFHG$ of normal fuzzy hypergroups is not a topos, it forms a pseudo topos. Also we show that there are pseudo power objects in $NFHG$.

1. Introduction

Sun [3] showed that the category $NFHG$ of normal fuzzy hypergroups satisfies all the axiom of topos except for the subobject classifier axiom. So we define a pseudo subobject classifier, pseudo topos and pseudo power object. Also Goldblatt [1] showed that any topos has power objects.

In this paper, we show that $NFHG$ has a pseudo subobject classifier. So $NFHG$ forms a pseudo topos. Also we show that there are pseudo power objects in $NFHG$ which is not a topos.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

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Definition 2.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following;

(T1) $\mathcal{E}$ is finitely complete,
(T2) $\mathcal{E}$ has exponentiation,
(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, the endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A \circ (g \times id) = f$;

$$\begin{array}{ccc} Y \times A & \xrightarrow{f} & B \\ {\scriptstyle g\times id}\downarrow & & \downarrow{\scriptstyle id} \\ B^A \times A & \xrightarrow[ev_A]{} & B \end{array}$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : \mathbf{1} \to \Omega$ such that for any monomorphism $h : D \to C$, there is a unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ which makes the following diagram a pull-back;

$$\begin{array}{ccc} D & \xrightarrow{!} & \mathbf{1} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\ C & \xrightarrow[\chi_h]{} & \Omega \end{array}$$

Example 2.2. Category *Set* is a topos. $\{*\}$ is a terminal object. $\Omega = \{0, 1\}$ and $\top : \{*\} \to \Omega$ with $\top(*) = 1$ is a subobject classifier. If we define

$\chi_h = 1$ if $c = h(d)$ for some $d \in D$,
$\chi_h = 0$ otherwise
then χ_h is a characteristic function of D .

Let H be a nonempty set and $F(H) = [0, 1]^H$ be the set of all fuzzy subset of H and $F^*(H) = F(H) - \{\phi\}$. A fuzzy hyperoperation on H is a mapping $\star : H^2 \to F(H)$ and the couple $(H, \star)$ is called a partial fuzzy hypergroupoid. If the fuzzy hyperoperation $\star$ maps H^2 into $F^*(H)$, then $(H, \star)$ is called a fuzzy hypergroupoid.

Definition 2.3.

(1) A *fuzzy semihypergroup* is a a fuzzy hypergroupoid $(H, \star)$ which satisfies the associative law.
(2) A *fuzzy quasihypergroup* is a a fuzzy hypergroupoid $(H, \star)$ which satisfies the reproductive law.
(3) A *fuzzy hypergroup* is a fuzzy semihypergroup which is also a fuzzy quasihypergroup
(4) A *fuzzy subhypergroup* $(A, \bullet)$ of a fuzzy hypergroup $(B, \bullet)$ is a nonempty subset $A \subseteq B$ such that for any $a \in A$, $a \bullet A = A = A \bullet a$.

Definition 2.4. A fuzzy hypergroup $(H, \star)$ is said to be *normal* if it satisfies the following three conditions;

(1) $(x \star x)(x) = 1$ for all $x \in H$;
(2) $x \star y = x \star x \cup y \star y$ for all $x, y \in H$;
(3) $(x \star x)(z) \geq (x \star x)(y) \wedge (y \star y)(z)$ for all $x, y, z \in H$.

Let $NFHG$ be a category, where objects are normal fuzzy hypergroups and a morphism from $(H, \diamond)$ to $(K, \star)$ is a mapping $f : H \to K$ such that $f(a \diamond b) \subseteq f(a) \star f(b)$.

Definition 2.5. A *pseudo subobject classifier* in a category $\mathcal{E}$ is an object Ω, together with a morphism $\top : \mathbf{1} \to \Omega$ such that for any $(A, \star) \subseteq (B, \star)$ and any inclusion $k : A \to B$, there is a unique morphism $\chi_k : B \to \Omega$ which makes the following diagram a pull-back;

$$\begin{array}{ccc} A & \xrightarrow{\;!\;} & \mathbf{1} \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \top} \\ B & \xrightarrow[\chi_k]{} & \Omega \end{array}$$

Definition 2.6. A *pseudo topos* is a category $\mathcal{E}$ that satisfies the following;

(T1) $\mathcal{E}$ is finitely complete,
(T2) $\mathcal{E}$ has exponentiation,
(T3) $\mathcal{E}$ has a pseudo subobject classifier.

Definition 2.7. A category $\mathcal{E}$ is said to have *pseudo power objects* if to each object A, there are objects $P(A)$ and $E(A)$, and inclusion $e : E(A) \to P(A) \times A$, such that for any object B, and "relation",

$r : R \to B \times A$ there is exactly one morphism $f_r : B \to P(A)$ for which there is a pullback of the form

$$\begin{array}{ccc} R & \longrightarrow & E(A) \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle e} \\ B \times A & \underset{f_r \times i_A}{\longrightarrow} & P(A) \times A \end{array}$$

3. Pseudo Topos $NFHG$ and Pseudo Power Object

THEOREM 3.1. *$NFHG$ has a pseudo subobject classifier.*

Proof. Let $\Omega = \{\top, \bot\}$ and $\diamond : \Omega \times \Omega \to [0,1]^{\Omega}$ defined by
$(\top \diamond \top)(\top) = 1 = (\top \diamond \top)(\bot)$,
$(\bot \diamond \bot)(\top) = 1 = (\bot \diamond \bot)(\bot)$
$(\top \diamond \bot) = (\top \diamond \top) \cup (\bot \diamond \bot)$.
Then $(\Omega, \diamond)$ is a normal fuzzy hypergroup.

For any normal fuzzy subhypergroup $(K, \star) \subseteq (H, \star)$ and inclusion $f : K \to H$ defined by $f(k) = k$ for any $k \in K$, we construct a morphism $\chi_f : H \to \Omega$ defined by
$\chi_f(h) = \top$ if $x \in K$
$\chi_f(h) = \bot$ otherwise.

For any $z \in \Omega$, $\chi_f(u \star v)(z) \leq (\chi_f(u) \diamond \chi_f(v))(z) = 1$. So $\chi_f(u \star v) \subseteq \chi_f(u) \diamond \chi_f(v)$. Thus $\chi_f : H \to \Omega$ is a morphism. For any $h : (M, \oplus) \to (H, \star)$ and $! : (M, \oplus) \to (\{*\}, \odot)$ with $\chi_f \circ h = \top \circ\, !$, we have that $\chi_f \circ h = \top \circ\, !$ implies $h(m) \in Im(f)$. That is, $h(m) = f(k)$ for some $k \in K$. So there exists a morphism $g : (M, \oplus) \to (K, \star)$ such that $g(m) = k$ with $h(m) = f(k)$ for all $m \in M$. Clearly, $f \circ g = h$ and such a morphism is unique. □

$$\begin{array}{ccc} K & \overset{!}{\longrightarrow} & \{*\} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ H & \underset{\chi_f}{\longrightarrow} & \Omega \end{array}$$

COROLLARY 3.2. *$NFHG$ is a pseudo topos.*

THEOREM 3.3. *In category $NFHG$, for each object $(A, \oslash)$ there are objects $(P(A), \star)$, $(E(A), \triangle)$ and inclusion $g : (E(A), \triangle) \to (P(A), \star) \times (A, \oslash)$ such that for any object $(B, \oplus)$ and relation $(R, \bigtriangledown)$ from $(A, \oslash)$ to $(B, \oplus)$, there is exactly one morphism $f_r : (B, \oplus) \to (P(A), \star)$ for which there is a pullback of the form*

$$\begin{array}{ccc} (R, \bigtriangledown) & \xrightarrow{\overline{f}} & (E(A), \triangle) \\ r\downarrow & & \downarrow g \\ (B, \oplus) \times (A, \oslash) & \xrightarrow[f_r \times i_A]{} & (P(A), \star) \times (A, \oslash) \end{array}$$

where $((b_1, a_1) \bigtriangledown (b_2, a_2))(r_1, r_2) =((b_1 \oplus b_1)(r_1) \wedge (a_1 \oslash a_1)(r_2)) \vee((b_2 \oplus b_2)(r_1) \wedge (a_2 \oslash a_2)(r_2))$ and $r(b, a) = (b, a)$.

Proof. Let $P(A) = (\Omega, \diamond)^{(A,\oslash)} = \{f : A \to \Omega\}$ where $\star : P(A) \times P(A) \to [0, 1]^{P(A)}$ defined by $(f \star f)(h) = \wedge(f(x) \diamond f(x))h(x)$ and $E(A) = \{< f, a > | f \in P(A), a \in A, f(a) = \top\}$ where $\triangle : E(A) \times E(A) \to [0, 1]^{E(A)}$ defined by $((f, a) \triangle (g, b))(h, c) = ((f \star f)(h) \wedge (a \oslash a)(c)) \vee ((g \star g)(h) \wedge (b \oslash b)(c))$. Then we obtain objects $(P(A), \star)$ and $(E(A), \triangle)$. Consider

$$\begin{array}{ccc} E(A) & \xrightarrow{!} & \{*\} \\ g\downarrow & & \downarrow \top \\ P(A) \times A & \xrightarrow[\chi_g]{} & \Omega \end{array}$$

Let $\chi_g < f, a >= f(a)$, then χ_g is a morphism and $\chi_g \circ g = \top\circ !$. By the property of $(P(A), \star)$ and $(E(A), \triangle)$, Ω is a pseudo subobject classifier of the inclusion $g : E(A) \to P(A) \times A$. So the previous square is a pullback.

Consider

$$\begin{array}{ccccc} R & \xrightarrow{\overline{f}} & E(A) & \xrightarrow{!} & \{*\} \\ r\downarrow & & \downarrow g & & \downarrow \top \\ B \times A & \xrightarrow[f_r \times i_r]{} & P(A) \times A & \xrightarrow[\chi_g]{} & \Omega \end{array}$$

Let $f_r : B \to P(A)$ defined by
$(f_r(b))(a) = (\top\circ !) < b, a >$, if $< b, a >\in R$
$(f_r(b))(a) = \perp$, otherwise

Then $f_r : B \to P(A)$ is a morphism. And Ω is a pseudo subobject classifier of the inclusion $r : R \to B \times A$ with $! : R \to \{*\}$. So the outer square is a pullback. By definition of pullback, there is exactly one morphism $\overline{f} : R \to E(A)$ such that $g \circ \overline{f} = (f_r \times i_r) \circ r$. By pullback Lemma, the left square is a pullback. □

COROLLARY 3.4. *$NFHG$ has pseudo power objects.*

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