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REDEFINED FUZZY CONGRUENCES ON SEMIGROUPS

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Abstract. We redefine a fuzzy congruence, discuss some properties of the fuzzy congruences, find the fuzzy congruence generated by a fuzzy relation on a semigroup, and give some lattice theoretic properties of the fuzzy congruences on semigroups.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([8]). Subsequently, many researchers ([2], [7], [5], [4]) studied fuzzy relations in various contexts. The original definition of a reflexive fuzzy relation μ on a set X was $\mu(x, x) = 1$ for all $x \in X$, which seemed to be too strong. Gupta et al. ([3]) suggested a G-reflexive fuzzy relation by generalizing the definition, defined a fuzzy G-equivalence relation, and developed some properties of that relation. Chon ([1]) defined a generalized fuzzy congruence using the G-reflexive fuzzy relation and characterized that congruence. However the generalized fuzzy congruence turned out not to have some crucial properties (see [1]) such that the congruence on a semigroup is not always generated by a fuzzy relation and the collection of all those congruences is not a complete lattice. In this note, we suggest a new reflexive fuzzy relation as $\mu(x, x) \geq \epsilon > 0$ for all $x \in X$ and

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$\inf_{t\in X} \mu(t,t) \geq \mu(y,z)$ for all $y \neq z \in X$, define a fuzzy congruence, and show that the redefined fuzzy congruence has those crucial properties which the generalized fuzzy congruence does not have. Also our work may be considered as a generalization of the studies which Samhan ([6]) performed based on the original reflexive fuzzy relation.

In section 2 we redefine a fuzzy congruence and review some basic definitions and properties of fuzzy relations which will be used in the next section. In section 3 we discuss some basic properties of the fuzzy congruences, find the fuzzy congruence generated by a fuzzy relation on a semigroup, show that the collection $C(S)$ of all fuzzy congruences on a semigroup S is a complete lattice, and show that if S is a group, then $C_k(S) = \{\mu \in C(S) : \mu(c,c) = k \text{ for all } c \in S\}$ is a modular lattice for $0 < \epsilon \leq k \leq 1$.

2. Preliminaries

We redefine a fuzzy congruence and recall some properties of fuzzy relations which will be used in the next section.

DEFINITION 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy subset* of X. For every $x \in X$, $B(x)$ is called a *membership grade* of x in B. A *fuzzy relation* μ in a set Z is a fuzzy subset of $Z \times Z$.

The original definition of a fuzzy reflexive relation μ in a set X was $\mu(x,x) = 1$ for all $x \in X$. Gupta et al. ([3]) defined a G-reflexive fuzzy relation μ in a set X by $\mu(x,x) > 0$ for all $x \in X$ and $\inf_{t\in X} \mu(t,t) \geq \mu(x,y)$ for all $x, y \in X$ such that $x \neq y$. But the fuzzy congruence defined from the G-fuzzy reflexive relation does not have some crucial properties (see [1]). We redefine the fuzzy congruence for a settlement of these problems.

DEFINITION 2.2. Let μ be a fuzzy relation in a set X. μ is *reflexive* in X if $\mu(x,x) \geq \epsilon > 0$ and $\inf_{t\in X} \mu(t,t) \geq \mu(x,y)$ for all $x, y \in X$ such that $x \neq y$. μ is *symmetric* in X if $\mu(x,y) = \mu(y,x)$ for all x, y in X. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x,y) = \sup_{z\in X} \min(\lambda(x,z), \mu(z,y)).$$

A fuzzy relation μ in X is *transitive* in X if $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called a *fuzzy equivalence relation* if μ is reflexive, symmetric, and transitive.

Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set X. Then it is easy to see that the composition $\circ$ is associative, $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$, and a fuzzy equivalence relation is an idempotent element of $\mathcal{F}_X$.

DEFINITION 2.3. Let μ be a fuzzy relation in a set X. μ is called *fuzzy left (right) compatible* if $\mu(x,y) \leq \mu(zx,zy)$ $(\mu(x,y) \leq \mu(xz,yz))$ for all $x,y,z \in X$. A fuzzy equivalence relation on X is called a *fuzzy left congruence (right congruence)* if it is fuzzy left compatible (right compatible). A fuzzy equivalence relation on X is called a *fuzzy congruence* if it is a fuzzy left and right congruence.

DEFINITION 2.4. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x,y) = \mu(y,x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations μ and ν. The following Proposition 2.5, Proposition 2.6, and Proposition 2.7 are due to Samhan ([6]).

PROPOSITION 2.5. *Let μ be a fuzzy relation on a set X. Then $\cup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on X containing μ, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.3 of [6]. □

PROPOSITION 2.6. *Let μ be a fuzzy relation on a set X. If μ is symmetric, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.4 of [6]. □

PROPOSITION 2.7. *If μ is a fuzzy relation on a semigroup S that is fuzzy left and right compatible, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 3.6 of [6]. □

PROPOSITION 2.8. *Let μ and each ν_i be fuzzy relations in a set X for all $i \in I$. Then $\mu \circ (\bigcap_{i\in I} \nu_i) \subseteq \bigcap_{i\in I} (\mu \circ \nu_i)$ and $(\bigcap_{i\in I} \nu_i) \circ \mu \subseteq \bigcap_{i\in I} (\nu_i \circ \mu)$.*

Proof. Straightforward. □

Proposition 2.9. *If μ is a reflexive fuzzy relation on a set X, then $\mu^{n+1}(x,y) \geq \mu^n(x,y)$ for all natural numbers n and all $x, y \in X$.*

Proof. Straightforward. □

3. Redefined fuzzy congruences on semigroups

In this section we develop some basic properties of the fuzzy congruences, find the fuzzy congruence generated by a fuzzy relation on a semigroup, and give some lattice theoretic properties of fuzzy congruences.

Proposition 3.1. *Let μ be a fuzzy relation on a set S. If μ is reflexive, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. Clearly $\mu^1 = \mu$ is reflexive. Suppose that μ^k is reflexive. Then $\mu^{k+1}(x,x) \geq \mu^k(x,x) \geq \epsilon > 0$ for all $x \in S$ by Proposition 2.9. The remaining part of the proof is exactly same as that of Proposition 3.1 in [1]. □

Proposition 3.2. *Let μ and ν be fuzzy congruences in a set X. Then $\mu \cap \nu$ is a fuzzy congruence.*

Proof. It is clear from Proposition 2.8. □

It is easy to see that even though μ and ν are fuzzy congruences, $\mu \cup \nu$ is not necessarily a fuzzy congruence. We find the fuzzy congruence generated by $\mu \cup \nu$ in the following proposition.

Proposition 3.3. *Let μ and ν be fuzzy congruences on a semigroup S. Then the fuzzy congruence generated by $\mu \cup \nu$ in S is $\cup_{n=1}^{\infty}(\mu \cup \nu)^n = (\mu \cup \nu) \cup [(\mu \cup \nu) \circ (\mu \cup \nu)] \cup \ldots$.*

Proof. Clearly $(\mu \cup \nu)(x,x) \geq \epsilon > 0$ for all $x \in S$. The remaining part of the proof is exactly same as that of Proposition 3.3 in [1]. □

We now turn to the characterization of the fuzzy congruence generated by a fuzzy relation on a semigroup.

Definition 3.4. Let μ be a fuzzy relation on a semigroup S and let $S^1 = S \cup \{e\}$, where e is the identity of S. We define the fuzzy relation

μ^* on S as

$$\mu^*(x,y) = \bigcup_{\substack{c,d\in S^1,\\ cad=x,\\ cbd=y}} \mu(a,b) \quad \text{for all } x,y \in S.$$

PROPOSITION 3.5. *Proposition 3.5 Let μ and ν be two fuzzy relations on a semigroup S. Then*

(1) $\mu \subseteq \mu^*$
(2) $(\mu^*)^{-1} = (\mu^{-1})^*$
(3) *If* $\mu \subseteq \nu$, *then* $\mu^* \subseteq \nu^*$
(4) $(\mu \cup \nu)^* = \mu^* \cup \nu^*$
(5) $\mu = \mu^*$ *if and only if* μ *is fuzzy left and right compatible*
(6) $(\mu^*)^* = \mu^*$

Proof. See Proposition 3.5 of [6]. □

The generalized fuzzy congruence in a semigroup is not always generated by a fuzzy relation (see Theorem 3.6 of [1]). We show that the fuzzy congruence on a semigroup, which is newly defined in this note, is always generated by a fuzzy relation.

THEOREM 3.6. *Let μ be a fuzzy relation on a semigroup S. Then the fuzzy congruence generated by μ is*

$$\begin{cases} \cup_{n=1}^{\infty} [\mu^* \cup (\mu^*)^{-1} \cup \theta^*]^n, & \text{if } \mu(x,y) > 0 \text{ for some } x \neq y \in S \\ \cup_{n=1}^{\infty} (\mu^* \cup \zeta^*)^n, & \text{if } \mu(x,y) = 0 \text{ for all } x \neq y \in S \end{cases}$$

where $\theta(z,z) = \max\,[\sup_{x\neq y\in S} \mu(x,y),\ \epsilon]$ *for all* $z \in S$, $\theta = \theta^{-1}$, $\theta(x,y) \leq \mu(x,y)$ *for all* $x,y \in S$ *with* $x \neq y$, $\zeta(z,z) = \epsilon$ *for all* $z \in S$, $\zeta(x,y) = 0$ *for all* $x \neq y \in S$, *and* μ^*, θ^*, *and* ζ^* *are fuzzy relation on S defined in Definition 3.4.*

Proof. We consider the case that $\mu(x,y) > 0$ for some $x \neq y \in S$. Let $\mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^*$. Then $\mu_1(z,z) \geq \theta^*(z,z) \geq \theta(z,z) \geq \epsilon > 0$ for all $z \in S$. Let $S^1 = S \cup \{e\}$, where e is the identity of S. Since $x \neq y$ implies $a \neq b$ in Definition 3.4, $\mu^*(x,y) \leq \sup_{x\neq y\in S} \mu(x,y) \leq \theta(t,t)$ for all $t \in S$. Since $\theta(x,y) \leq \mu(x,y)$, $\theta^*(x,y) \leq \mu^*(x,y)$ by (3) of Proposition 3.5. That is,

$$\inf_{t\in S} \mu_1(t,t) \geq \inf_{t\in S} \theta^*(t,t) \geq \theta(t,t) \geq \mu^*(x,y) \geq \theta^*(x,y).$$

Since $\inf_{t\in S} \mu_1(t,t) \geq \theta(t,t) \geq \mu^*(y,x)$, $\inf_{t\in S} \mu_1(t,t) \geq (\mu^*)^{-1}(x,y)$. Thus

$$\inf_{t\in S} \mu_1(t,t) \geq \max[\mu^*(x,y),\ (\mu^*)^{-1}(x,y),\ \theta^*(x,y)] = \mu_1(x,y).$$

That is, μ_1 is reflexive. By Proposition 3.1, $\cup_{n=1}^{\infty} \mu_1^n$ is reflexive. Since $\theta = \theta^{-1}$, $\theta^* = (\theta^{-1})^* = (\theta^*)^{-1}$ by (2) of Proposition 3.5, and hence

$$\mu_1(x,y) = \max\ [(\mu^*)^{-1}(y,x), \mu^*(y,x), (\theta^*)^{-1}(x,y)] = \mu_1(y,x).$$

Thus μ_1 is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty} \mu_1^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty} \mu_1^n$ is transitive. Hence $\cup_{n=1}^{\infty} \mu_1^n$ is a fuzzy equivalence relation containing μ. By (2), (4), and (6) of Proposition 3.5,

$$\begin{aligned}\mu_1^* &= (\mu^* \cup (\mu^*)^{-1} \cup \theta^*)^* = (\mu^* \cup (\mu^{-1})^* \cup \theta^*)^* = (\mu^*)^* \cup ((\mu^{-1})^*)^* \cup (\theta^*)^* \\ &= \mu^* \cup (\mu^{-1})^* \cup \theta^* = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu_1.\end{aligned}$$

Thus μ_1 is fuzzy left and right compatible by (5) of Proposition 3.5. By Proposition 2.7, $\cup_{n=1}^{\infty} \mu_1^n$ is fuzzy left and right compatible. Thus $\cup_{n=1}^{\infty} \mu_1^n$ is a fuzzy congruence containing μ. Let ν be a fuzzy congruence containing μ. Then $(\mu \cup \mu^{-1} \cup \theta)(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$. Since $\theta(a,a) = \max\ [\sup_{x\neq y\in S} \mu(x,y),\ \epsilon] \leq \nu(a,a)$ for all $a \in S$, $\max\ [\mu(a,a), \mu^{-1}(a,a), \theta(a,a)] \leq \nu(a,a)$ for all $a \in S$. Thus $\mu \cup \mu^{-1} \cup \theta \subseteq \nu$. By (2), (3), and (4) of Proposition 3.5,

$$\mu_1 = \mu^* \cup (\mu^*)^{-1} \cup \theta^* = \mu^* \cup (\mu^{-1})^* \cup \theta^* = (\mu \cup \mu^{-1} \cup \theta)^* \subseteq \nu^*.$$

Since ν is fuzzy left and right compatible, $\nu = \nu^*$ by (5) of Proposition 3.5. Thus $\mu_1 \subseteq \nu$. Suppose $\mu_1^k \subseteq \nu$. Then

$$\begin{aligned}\mu_1^{k+1}(b,c) = (\mu_1^k \circ \mu_1)(b,c) &= \sup_{d\in S} \min[\mu_1^k(b,d), \mu_1(d,c)] \\ &\leq \sup_{d\in S} \min\ [\nu(b,d), \nu(d,c)] = (\nu \circ \nu)(b,c)\end{aligned}$$

for all $b, c \in S$. That is, $\mu_1^{k+1} \subseteq (\nu \circ \nu)$. Since ν is transitive, $\mu_1^{k+1} \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for every natural number n. Thus

$$\cup_{n=1}^{\infty} [\mu^* \cup (\mu^*)^{-1} \cup \theta^*]^n = \cup_{n=1}^{\infty} {\mu_1}^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu.$$

We consider the case that $\mu(x,y) = 0$ for all $x \neq y \in S$. Let $\mu_2 = \mu^* \cup \zeta^*$. Then $\mu_2(a,a) \geq \epsilon > 0$ for all $a \in S$. Let $S^1 = S \cup \{e\}$, where e is the identity of S. Since $x \neq y$ implies $a \neq b$ in Definition 3.4,

$\mu^*(x,y) = 0$ and $\zeta^*(x,y) = 0$ from $\mu(x,y) = 0$ and $\zeta(x,y) = 0$. That is, $(\mu^* \cup \zeta^*)(x,y) < \zeta(t,t)$ for all $t \in S$. Thus

$$\inf_{t\in S} \mu_2(t,t) \geq \inf_{t\in S} \zeta^*(t,t) \geq \zeta(t,t) > \max[\mu^*(x,y),\ \zeta^*(x,y)] = \mu_2(x,y).$$

Hence μ_2 is reflexive. By Proposition 3.1, $\cup_{n=1}^{\infty} \mu_2^n$ is reflexive. Since $\mu^*(x,y) = 0$ and $\zeta^*(x,y) = 0$, μ_2 is symmetric. By Proposition 2.6, $\cup_{n=1}^{\infty} \mu_2^n$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty} \mu_2^n$ is transitive. Hence $\cup_{n=1}^{\infty} \mu_2^n$ is a fuzzy equivalence relation containing μ. The proof of the remaining parts is similar to that of the above case. □

We now turn to the lattice theoretic properties of fuzzy congruences. For the collection $\{\mu_j : j \in J\}$ of all generalized fuzzy congruences on a semigroup S with a relation $\lesssim$ defined in Proposition 3.7, it is easy to see that $(\{\mu_j : j \in J\},\ \lesssim)$ is not a complete lattice since $\inf_{j\in J} \mu_j$ does not exist (see [1]). In next proposition, we show that the collection of the redefined fuzzy congruences is a complete lattice.

Proposition 3.7. *Let $C(S)$ be the collection of all fuzzy congruences on a semigroup S. Then $(C(S), \lesssim)$ is a complete lattice, where $\lesssim$ is a relation on the set of all fuzzy congruences on S defined by $\mu \lesssim \nu$ iff $\mu(x,y) \leq \nu(x,y)$ for all $x, y \in S$.*

Proof. Clearly $\lesssim$ is a partial order relation. It is easy to check that the relation σ defined by $\sigma(x,y) = 1$ for all $x, y \in S$ is in $C(S)$ and the relation λ defined by $\lambda(x,y) = \epsilon$ for $x = y$ and $\lambda(x,y) = 0$ for $x \neq y$ is in $C(S)$. Also σ is the greatest element and λ is the least element of $C(S)$ with respect to the ordering $\lesssim$. Let $\{\mu_j\}_{j\in J}$ be a non-empty collection of fuzzy congruences in $C(S)$. Let $\mu(x,y) = \inf_{j\in J} \mu_j(x,y)$ for all $x, y \in S$. Clearly $\mu(x,x) \geq \epsilon$ for all $x \in S$, $\inf_{t\in X} \mu(t,t) \geq \mu(y,z)$ for all $y \neq z \in X$, $\mu = \mu^{-1}$, $\mu(x,y) \leq \mu(zx, zy)$, and $\mu(x,y) \leq \mu(xz, yz)$ for all $x, y, z \in S$. It is easy to see that $\mu \circ \mu \subseteq \mu$ (see Proposition 6.1 of [4]). That is, $\mu \in C(S)$. Since μ is the greatest lower bound of $\{\mu_j\}_{j\in J}$, $(C(S), \lesssim)$ is a complete lattice. □

We define a join $\vee$ and a meet $\wedge$ on $C(S)$ by $\mu \vee \nu =< \mu \cup \nu >_c$ and $\mu \wedge \nu = \mu \cap \nu$, where $< \mu \cup \nu >_c$ is the fuzzy congruence generated by $\mu \cup \nu$. It is clear that if $\mu, \nu \in C(S)$, then $\mu \wedge \nu \in C(S)$ and $\mu \vee \nu \in C(S)$ from Proposition 3.2 and Propostion 3.3, respectively. Let

$C_k(S) = \{\mu \in C(S) : \mu(c, c) = k$ for all $c \in S\}$. Then it is easy to see that $(C_k(S), \vee, \wedge)$ is a sublattice of $C(S)$ for $0 < \epsilon \leq k \leq 1$.

Definition 3.8. A lattice $(L, \vee, \wedge)$ is called *modular* if $(x \vee y) \wedge z \leq x \vee (y \wedge z)$ for all $x, y, z \in L$ with $x \leq z$.

Lemma 3.9. *Let μ and ν be fuzzy congruences on a semigroup S such that $\mu(c, c) = \nu(c, c)$ for all $c \in S$. If $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the fuzzy congruence on S generated by $\mu \cup \nu$.*

Proof. $(\mu \circ \nu)(a, a) = \sup_{z \in S} \min [\mu(a, z), \nu(z, a)] \geq \min[\mu(a, a), \nu(a, a)] \geq \epsilon > 0$ for all $a \in S$. The remaining part of the proof is same as that of Lemma 4.3 in [1]. □

Theorem 3.10. *Let S be a semigroup and let H be a sublattice of $(C_k(S), \vee, \wedge)$ such that $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in H$. Then H is a modular lattice for $0 < \epsilon \leq k \leq 1$.*

Proof. Let $\mu, \nu, \rho \in H$ with $\mu \leq \rho$. Let $x, y \in S$. Then it is straightforward (see Theorem 4.5 of [6]) that $(\mu \circ \nu) \wedge \rho \leq \mu \circ (\nu \wedge \rho)$. Since $\mu, \nu \in C_k(S)$, $\mu(c, c) = \nu(c, c) = k$ for all $c \in S$. By Lemma 3.9, $\mu \circ \nu$ is the fuzzy congruence generated by $\mu \cup \nu$. That is, $\mu \vee \nu = \mu \circ \nu$. Thus $(\mu \vee \nu) \wedge \rho \leq \mu \circ (\nu \wedge \rho)$. Since H is a sublattice and $\rho, \nu \in H$, $\nu \wedge \rho \in H$. Since $\mu \in H$ and $\nu \wedge \rho \in H$, $\mu \circ (\nu \wedge \rho) = (\nu \wedge \rho) \circ \mu$. Also $(\nu \wedge \rho)(c, c) = k$ and $\mu(c, c) = k$ for all $c \in S$. By Lemma 3.9, $\mu \circ (\nu \wedge \rho)$ is the fuzzy congruence generated by $\mu \cup (\nu \wedge \rho)$. That is, $\mu \circ (\nu \wedge \rho) = \mu \vee (\nu \wedge \rho)$. Thus $(\mu \vee \nu) \wedge \rho \leq \mu \vee (\nu \wedge \rho)$. Hence H is modular. □

Corollary 3.11. *If S is a group and $0 < \epsilon \leq k \leq 1$, then $(C_k(S), \vee, \wedge)$ is a modular lattice.*

Proof. It is easy to see that if S is a group, then $\mu \circ \nu = \nu \circ \mu$ for all $\mu, \nu \in C_k(S)$ (see Proposition 4.3 of [6]). By Theorem 3.10, $(C_k(S), \vee, \wedge)$ is modular. □

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